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Stability of Solutions to Some Evolution Problems

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Abstract. Large time behavior of solutions to abstract differential equations is studied. The corresponding evolution problem is:

$$\dot{u} = A(t)u + F(t, u) + b(t), \quad t \geq 0; \quad u(0) = u_0. \qquad (*)$$

Here $\dot{u} := \frac{du}{dt}$, $u = u(t) \in H$, H is a Hilbert space, $t \in \mathbb{R}_+ := [0, \infty)$, $A(t)$ is a linear dissipative operator: $\mathrm{Re}(A(t)u, u) \leq -\gamma(t)(u, u)$, $\gamma(t) \geq 0$, $F(t, u)$ is a nonlinear operator, $\|F(t, u)\| \leq c_0\|u\|^p$, $p > 1$, c_0, p are constants, $\|b(t)\| \leq \beta(t)$, $\beta(t) \geq 0$ is a continuous function.

Sufficient conditions are given for the solution $u(t)$ to problem (*) to exist for all $t \geq 0$, to be bounded uniformly on $\mathbb{R}_+$, and a bound on $\|u(t)\|$ is given. This bound implies the relation $\lim_{t\to\infty} \|u(t)\| = 0$ under suitable conditions on $\gamma(t)$ and $\beta(t)$.

The basic technical tool in this work is the following nonlinear inequality:

$$\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t), \; t \geq 0; \quad g(0) = g_0,$$

which holds on any interval $[0, T)$ on which $g(t) \geq 0$ exists and has bounded derivative from the right, $\dot{g}(t) := \lim_{s\to+0} \frac{g(t+s)-g(t)}{s}$. It is assumed that $\gamma(t)$, and $\beta(t)$ are real-valued, continuous functions of t, defined on $\mathbb{R}_+ := [0, \infty)$, the function $\alpha(t, g)$ is defined for all $t \in \mathbb{R}_+$, locally Lipschitz with respect to g uniformly with respect to t on any compact subsets $[0, T]$, $T < \infty$.If there exists a function $\mu(t) > 0$, $\mu(t) \in C^1(\mathbb{R}_+)$, such that

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \leq \frac{1}{\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \forall t \geq 0; \quad \mu(0)g(0) \leq 1,$$

then $g(t)$ exists on all of $\mathbb{R}_+$, that is $T = \infty$, and the following estimate holds:

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0.$$

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1 Introduction

Consider an abstract nonlinear evolution problem

$$\dot{u} = A(t)u + F(t,u) + b(t), \qquad \dot{u} := \frac{du}{dt}, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $u(t)$ is a function with values in a Hilbert space H, $A(t)$ is a linear bounded dissipative operator in H, which satisfies inequality

$$\mathrm{Re}(A(t)u,u) \le -\gamma(t)\|u\|^2, \qquad t \ge 0; \qquad \forall u \in H, \tag{3}$$

where $F(t,u)$ is a nonlinear map in H,

$$\|F(t,u)\| \le c_0 \|u(t)\|^p, \qquad p > 1, \tag{4}$$

$$\|b(t)\| \le \beta(t), \tag{5}$$

$\gamma(t) > 0$ and $\beta(t) \ge 0$ are continuous function, defined on all of $\mathbb{R}_+ := [0,\infty)$, $c_0 > 0$ and $p > 1$ are constants.

Recall that a linear operator A in a Hilbert space is called dissipative if $\mathrm{Re}(Au,u) \le 0$ for all $u \in D(A)$, where $D(A)$ is the domain of definition of A. Dissipative operators are important because they describe systems in which energy is dissipating, for example, due to friction or other physical reasons. Passive nonlinear networks can be described by equation (1) with a dissipative linear operator $A(t)$, see [14], [15], Chapter 3, and [16].

Let $\sigma := \sigma(A(t))$ denote the spectrum of the linear operator $A(t)$, $\Pi := \{z : Rez < 0\}$, $\ell := \{z : Rez = 0\}$, and $\rho(\sigma,\ell)$ denote the distance between sets σ and ℓ. We assume that

$$\sigma \subset \Pi, \tag{6}$$

but we allow $\lim_{t\to\infty} \rho(\sigma,\ell) = 0$. This is the basic *novel* point in our theory. The usual assumption in stability theory (see, e.g., [1]) is $\sup_{z\in\sigma} Rez \le -\gamma_0$, where $\gamma_0 = const > 0$. For example, if $A(t) = A^*(t)$, where A^* is the adjoint operator, and if the spectrum of $A(t)$ consists of eigenvalues $\lambda_j(t)$, $0 \ge \lambda_j(t) \ge \lambda_{j+1}(t)$, then, we allow $\lim_{t\to\infty} \lambda_1(t) = 0$. This is in contrast with the usual theory, where the assumption is $\lambda_1(t) \le -\gamma_0$, $\gamma_0 > 0$ is a constant, is used.

Moreover, our results cover the case, apparently not considered earlier in the literature, when $\mathrm{Re}(A(t)u,u) \le \gamma(t)$ with $\gamma(t) > 0$, $\lim_{t\to\infty} \gamma(t) = 0$. This means that the spectrum of $A(t)$ may be located in the half-plane $\mathrm{Re}z \le \gamma(t)$, where $\gamma(t) > 0$, but $\lim_{t\to\infty} \gamma(t) = 0$.

Our goal is to give sufficient conditions for the existence and uniqueness of the solution to problem (1)-(2) for all $t \ge 0$, that is, for global existence of $u(t)$, for boundedness of $\sup_{t\ge0} \|u(t)\| < \infty$, or to the relation $\lim_{t\to\infty} \|u(t)\| = 0$.

If $b(t) = 0$ in (1), then $u(t) = 0$ solves equation (1) and $u(0) = 0$. This equation is called zero solution to (1) with $b(t) = 0$.

Recall that the zero solution to equation (1) with $b(t) = 0$ is called Lyapunov stable if for any $\epsilon > 0$, however small, one can find a $\delta = \delta(\epsilon) > 0$, such that if $\|u_0\| \le \delta$, then the solution to Cauchy problem (1)-(2) satisfies the estimate $\sup_{t\ge 0} \|u(t)\| \le \epsilon$. If, in addition, $\lim_{t\to\infty} \|u(t)\| = 0$, then the zero solution to equation (6) is called asymptotically stable in the Lyapunov sense.

If $b(t) \not\equiv 0$, then one says that (1)-(2) is the problem with persistently acting perturbations. The zero solution is called Lyapunov stable for problem (1)-(2) with persistently acting perturbations if for any $\epsilon > 0$, however small, one can find a $\delta = \delta(\epsilon) > 0$, such that if $\|u_0\| \le \delta$, and $\sup_{t\ge 0} \|b(t)\| \le \delta$, then the solution to Cauchy problem (1)-(2) satisfies the estimate $\sup_{t\ge 0} \|u(t)\| \le \epsilon$.

The approach, developed in this work, consists of reducing the stability problems to some nonlinear differential inequality and estimating the solutions to this inequality.

In Section 2 the formulation and a proof of two theorems, containing the result concerning this inequality and its discrete analog, are given. In Section 3 some results concerning Lyapunov stability of zero solution to equation (1) are obtained. In Section 4 we derive stability results in the case when $\gamma(t) > 0$. This means that the linear operator $A(t)$ in (1) may have spectrum in the half-plane $\mathrm{Re} z > 0$.

The results of this paper are based on the works [6]- [15]. They are closely related to the Dynamical Systems Method (DSM), see [10], [7], [8], [11].

In the theory of chaos one of the reasons for the chaotic behavior of a solution to an evolution problem to appear is the lack of stability of solutions to this problem ([2], [3]). The results presented in Section 3 can be considered as sufficient conditions for chaotic behavior not to appear in the evolution system described by problem (1)-(2).

2 Differential inequality

In this Section a self-contained proof is given of an estimate for solutions of a nonlinear inequality

$$\dot{g}(t) \le -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t),\ t \ge 0;\ g(0) = g_0; \quad \dot{g} := \frac{dg}{dt}. \tag{7}$$

In Section 3 some of the many possible applications of this estimate (estimate (11)) are demonstrated.

It is not assumed a priori that solutions $g(t)$ to inequality (7) are defined on all of $\mathbb{R}_+$, that is, that these solutions exist globally. In Theorem 1 we give sufficient conditions for the global existence of $g(t)$. Moreover, under these conditions a bound on $g(t)$ is given, see estimate (11) in Theorem 1. This bound yields the relation $\lim_{t\to\infty} g(t) = 0$ if $\lim_{t\to\infty} \mu(t) = \infty$ in (11).

Let us formulate our assumptions.

Assumption A_1). We assume that the function $g(t) \ge 0$ is defined on some interval $[0, T)$, has a bounded derivative $\dot{g}(t) := \lim_{s\to +0} \frac{g(t+s)-g(t)}{s}$ from the right at any point of this interval, and $g(t)$ satisfies inequality (7) at all t at which $g(t)$ is defined. The functions $\gamma(t)$, and $\beta(t)$, are real-valued, defined on all of $\mathbb{R}_+$ and continuous there. The function $\alpha(t, g) \ge 0$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ and locally Lipschitz with respect to g. This means that

$$|\alpha(t, g) - \alpha(t, h)| \le L(T, M)|g - h|, \tag{8}$$

if $t \in [0,T]$, $|g| \leq M$ and $|h| \leq M$, $M = const > 0$, where $L(T,M) > 0$ is a constant independent of g, h, and t.
Assumption A_2). There exists a $C^1(\mathbb{R}_+)$ function $\mu(t) > 0$, such that

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \leq \frac{1}{\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \forall t \geq 0, \tag{9}$$

and

$$\mu(0)g(0) \leq 1. \tag{10}$$

Theorem 1. *If Assumptions A_1) and A_2) hold, then any solution $g(t) \geq 0$ to inequality* (7) *exists on all of $\mathbb{R}_+$, i.e., $T = \infty$, and satisfies the following estimate:*

$$0 \leq g(t) \leq \frac{1}{\mu(t)} \quad \forall t \in \mathbb{R}_+. \tag{11}$$

Remark 1. *If $\lim_{t\to\infty} \mu(t) = \infty$, then $\lim_{t\to\infty} g(t) = 0$.*

Proof of Theorem 1. Let us rewrite inequality for μ

$$-\gamma(t)\mu^{-1}(t) + \alpha(t, \mu^{-1}(t)) + \beta(t) \leq \frac{d\mu^{-1}(t)}{dt}. \tag{12}$$

Let $\phi(t)$ solve the following Cauchy problem:

$$\dot{\phi}(t) = -\gamma(t)\phi(t) + \alpha(t, \phi(t)) + \beta(t), \quad t \geq 0, \quad \phi(0) = \phi_0. \tag{13}$$

The assumption that $\alpha(t,g)$ is locally Lipschitz guarantees local existence and uniqueness of the solution $\phi(t)$ to problem (13). From the known comparison result (see, e.g., [4], Theorem III.4.1) it follows that

$$\phi(t) \leq \mu^{-1}(t) \qquad \forall t \geq 0, \tag{14}$$

provided that $\phi(0) \leq \mu^{-1}(0)$, where $\phi(t)$ is the unique solution to problem (14). Let us take $\phi(0) = g(0)$. Then $\phi(0) \leq \mu^{-1}(0)$ by the assumption, and an inequality, similar to (14), implies that

$$g(t) \leq \phi(t) \qquad t \in [0,T). \tag{15}$$

Inequalities $\phi(0) \leq \mu^{-1}(0)$, (14), and (15) imply

$$g(t) \leq \phi(t) \leq \mu^{-1}(t), \qquad t \in [0,T). \tag{16}$$

By the assumption, the function $\mu(t)$ is defined for all $t \geq 0$ and is bounded on any compact subinterval of the set $[0,\infty)$. Consequently, the functions $\phi(t)$ and $g(t) \geq 0$ are defined for all $t \geq 0$, and estimate (11) is established.

Theorem 1 is proved. □

Let us formulate and prove a discrete version of Theorem 1.

Theorem 2. *Assume that $g_n \geq 0$, $\alpha(n, g_n) \geq 0$,*

$$g_{n+1} \leq (1 - h_n\gamma_n)g_n + h_n\alpha(n, g_n) + h_n\beta_n; \quad h_n > 0, \quad 0 < h_n\gamma_n < 1, \tag{17}$$

and $\alpha(n, g_n) \geq \alpha(n, p_n)$ *if* $g_n \geq p_n$. *If there exists a sequence* $\mu_n > 0$ *such that*

$$\alpha(n, \frac{1}{\mu_n}) + \beta_n \leq \frac{1}{\mu_n}(\gamma_n - \frac{\mu_{n+1} - \mu_n}{h_n \mu_n}), \tag{18}$$

and

$$g_0 \leq \frac{1}{\mu_0}, \tag{19}$$

then

$$0 \leq g_n \leq \frac{1}{\mu_n}, \qquad \forall n \geq 0. \tag{20}$$

Proof. For $n = 0$ inequality (20) holds because of (19). Assume that it holds for all $n \leq m$ and let us check that then it holds for $n = m + 1$. If this is done, Theorem 2 is proved.

Using the inductive assumption, one gets:

$$g_{m+1} \leq (1 - h_m \gamma_m)\frac{1}{\mu_m} + h_m \alpha(m, \frac{1}{\mu_m}) + h_m \beta_m.$$

This and inequality (18) imply:

$$\begin{aligned} g_{m+1} &\leq (1 - h_m \gamma_m)\frac{1}{\mu_m} + h_m \frac{1}{\mu_m}(\gamma_m - \frac{\mu_{m+1} - \mu_m}{h_m \mu_m}) \\ &= \mu_m^{-1} - \frac{\mu_{m+1} - \mu_m}{\mu_m^2} \leq \mu_{m+1}^{-1}. \end{aligned}$$

The last inequality is obvious since it can be written as

$$-(\mu_m - \mu_{m+1})^2 \leq 0.$$

Theorem 2 is proved.

Theorem 2 was formulated in [5] and proved in [6]. We included for completeness a proof, which is shorter than the one in [6].

3 Stability results 1

In this Section we develop a method for a study of stability of solutions to the evolution problems described by the Cauchy problem (1)-(2) for abstract differential equations with a dissipative bounded linear operator $A(t)$ and a nonlinearity $F(t, u)$ satisfying inequality (4). Condition (4) means that for sufficiently small $\|u(t)\|$ the nonlinearity is of the higher order of smallness than $\|u(t)\|$. We also study the large time behavior of the solution to problem (1)-(2) with persistently acting perturbations $b(t)$.

In this paper we assume that $A(t)$ is a bounded linear dissipative operator, but our methods are valid also for unbounded linear dissipative operators $A(t)$, for which one can prove global existence of the solution to problem (1)-(2). We do not go into further detail in this paper.

Let us formulate the first stability result.

Theorem 3. *Assume that* $Re(Au, u) \leq -k\|u\|^2$ $\forall u \in H$, $k = const > 0$, *and inequality* (3) *holds with* $\gamma(t) = k$. *Then the solution to problem* (1)-(2) *with* $b(t) = 0$ *satisfies an esimate* $\|u(t)\| = O(e^{-(k-\epsilon)t})$ *as* $t \to \infty$. *Here* $0 < \epsilon < k$ *can be chosen arbitrarily small if* $\|u_0\|$ *is sufficiently small.*

This theorem implies asymptotic stability in the sense of Lyapunov of the zero solution to equation (1) with $b(t) = 0$. Our proof of Theorem 3 is new and very short.

Proof of Theorem 3. Multiply equation (1) (in which $b(t) = 0$ is assumed) by u, denote $g = g(t) := \|u(t)\|$, take the real part, and use assumption (3) with $\gamma(t) = k > 0$, to get

$$g\dot{g} \leq -kg^2 + c_0 g^{p+1}, \qquad p > 1. \tag{21}$$

If $g(t) > 0$ then the derivative $\dot{g}$ does exist, and

$$\dot{g}(t) = Re\left(\dot{u}(t), \frac{u(t)}{\|u(t)\|}\right),$$

as one can check. If $g(t) = 0$ on an open subset of $\mathbb{R}_+$, then the derivative $\dot{g}$ does exist on this subset and $\dot{g}(t) = 0$ on this subset. If $g(t) = 0$ but in in any neighborhood $(t - \delta, t + \delta)$ there are points at which g does not vanish, then by $\dot{g}$ we understand the derivative from the right, that is,

$$\dot{g}(t) := \lim_{s \to +0} \frac{g(t+s) - g(t)}{s} = \lim_{s \to +0} \frac{g(t+s)}{s}.$$

This limit does exist and is equal to $\|\dot{u}(t)\|$. Indeed, the function $u(t)$ is continuously differentiable, so

$$\lim_{s \to +0} \frac{\|u(t+s)\|}{s} = \lim_{s \to +0} \frac{\|s\dot{u}(t) + o(s)\|}{s} = \|\dot{u}(t)\|.$$

The assumption about the existence of the bounded derivative $\dot{g}(t)$ from the right in Theorem 3 was made because the function $\|u(t)\|$ does not have, in general, the derivative in the usual sense at the points t at which $\|u(t)\| = 0$, no matter how smooth the function $u(t)$ is at the point τ. Indeed,

$$\lim_{s \to -0} \frac{\|u(t+s)\|}{s} = \lim_{s \to -0} \frac{\|s\dot{u}(t) + o(s)\|}{s} = -\|\dot{u}(t)\|,$$

because $\lim_{s \to -0} \frac{|s|}{s} = -1$. Consequently, the right and left derivatives of $\|u(t)\|$ at the point t at which $\|u(t)\| = 0$ do exist, but are different. Therefore, the derivative of $\|u(t)\|$ at the point t at which $\|u(t)\| = 0$ does not exist in the usual sense.

However, as we have proved above, the derivative $\dot{g}(t)$ from the right does exist always, provided that $u(t)$ is continuously differentiable at the point t.

Since $g \geq 0$, inequality (21) yields inequality (7) with $\gamma(t) = k > 0$, $\beta(t) = 0$, and $\alpha(t, g) = c_0 g^p$, $p > 1$. Inequality (9) takes the form

$$\frac{c_0}{\mu^p(t)} \leq \frac{1}{\mu(t)}\left(k - \frac{\dot{\mu}(t)}{\mu(t)}\right), \qquad \forall t \geq 0. \tag{22}$$

Let

$$\mu(t) = \lambda e^{bt}, \qquad \lambda, b = const > 0. \tag{23}$$

We choose the constants λ and b later. Inequality (9), with μ defined in (23), takes the form

$$\frac{c_0}{\lambda^{p-1} e^{(p-1)bt}} + b \le k, \qquad \forall t \ge 0. \tag{24}$$

This inequality holds if it holds at $t = 0$, that is, if

$$\frac{c_0}{\lambda^{p-1}} + b \le k. \tag{25}$$

Let $\epsilon > 0$ be arbitrary small number. Choose $b = k - \epsilon > 0$. Then (25) holds if

$$\lambda \ge \big(\frac{c_0}{\epsilon}\big)^{\frac{1}{p-1}}. \tag{26}$$

Condition (10) holds if

$$\|u_0\| = g(0) \le \frac{1}{\lambda}. \tag{27}$$

We choose λ and b so that inequalities (26) and (27) hold. This is always possible if $b < k$ and $\|u_0\|$ is sufficiently small.

By Theorem 1, if inequalities (25)-(27) hold, then one gets estimate (11):

$$0 \le g(t) = \|u(t)\| \le \frac{e^{-(k-\epsilon)t}}{\lambda}, \qquad \forall t \ge 0. \tag{28}$$

Theorem 3 is proved. □

Remark 3. *One can formulate the result differently. Namely, choose* $\lambda = \|u_0\|^{-1}$. *Then inequality* (27) *holds, and becomes an equality. Substitute this* λ *into* (25) *and get*

$$c_0\|u_0\|^{p-1} + b \le k.$$

Since the choice of the constant $b > 0$ *is at our disposal, this inequality can always be satisfied if* $c_0\|u_0\|^{p-1} < k$. *Therefore, condition*

$$c_0\|u_0\|^{p-1} < k$$

is a sufficient condition for the estimate

$$\|u(t)\| \le \|u_0\| e^{-(k - c_0\|u_0\|^{p-1})t},$$

to hold (assuming that $c_0\|u_0\|^{p-1} < k$*).*

Let us formulate the second stability result.

Theorem 4. *Assume that inequalities* (3)-(5) *hold and*

$$\gamma(t) = \frac{c_1}{(1+t)^{q_1}}, \quad q_1 \le 1; \quad c_1, q_1 = const > 0. \tag{29}$$

Suppose that $\epsilon \in (0, c_1)$ *is an arbitrary small fixed number,*

$$\lambda \ge \Big(\frac{c_0}{\epsilon}\Big)^{1/(p-1)} \qquad \text{and} \quad \|u(0)\| \le \frac{1}{\lambda}.$$

Then the unique solution to (1)-(2) *with* $b(t) = 0$ *exists on all of* $\mathbb{R}_+$ *and*

$$0 \leq \|u(t)\| \leq \frac{1}{\lambda(1+t)^{c_1-\epsilon}}, \qquad \forall t \geq 0. \tag{30}$$

Theorem 4 gives the size of the initial data, namely, $\|u(0)\| \leq \frac{1}{\lambda}$, for which estimate (30) holds. For a fixed nonlinearity $F(t,u)$, that is, for a fixed constant c_0 from assumption (4), the maximal size of $\|u(0)\|$ is determined by the minimal size of λ.

The minimal size of λ is determined by the inequality $\lambda \geq \left(\frac{c_0}{\epsilon}\right)^{1/(p-1)}$, that is, by the maximal size of $\epsilon \in (0, c_1)$. If $\epsilon < c_1$ and $c_1 - \epsilon$ is very small, then $\lambda > \lambda_{min} := \left(\frac{c_0}{c_1}\right)^{1/(p-1)}$ and λ can be chosen very close to λ_{min}.

Proof of Theorem 4. Let

$$\mu(t) = \lambda(1+t)^{\nu}, \qquad \lambda, \nu = const > 0. \tag{31}$$

We will choose the constants λ and ν later. Inequality (9) (with $\beta(t) = 0$) holds if

$$\frac{c_0}{\lambda^{p-1}(1+t)^{(p-1)\nu}} + \frac{\nu}{1+t} \leq \frac{c_1}{(1+t)^{q_1}}, \qquad \forall t \geq 0. \tag{32}$$

If

$$q_1 \leq 1 \qquad \textit{and} \qquad (p-1)\nu \geq q_1, \tag{33}$$

then inequality (32) holds if

$$\frac{c_0}{\lambda^{p-1}} + \nu \leq c_1. \tag{34}$$

Let $\epsilon > 0$ be an arbitrary small number. Choose

$$\nu = c_1 - \epsilon. \tag{35}$$

Then inequality (34) holds if inequality (26) holds. Inequality (10) holds because we have assumed in Theorem 4 that $\|u(0)\| \leq \frac{1}{\lambda}$. Combining inequalities (26), (27) and (11), one obtains the desired estimate:

$$0 \leq \|u(t)\| = g(t) \leq \frac{1}{\lambda(1+t)^{c_1-\epsilon}}, \qquad \forall t \geq 0. \tag{36}$$

Condition (26) holds for any fixed small $\epsilon > 0$ if λ is sufficiently large. Condition (27) holds for any fixed large λ if $\|u_0\|$ is sufficiently small.

Theorem 4 is proved. □

Let us formulate a stability result in which we assume that $b(t) \not\equiv 0$. The function $b(t)$ has physical meaning of persistently acting perturbations.

Theorem 5. *Let* $b(t) \not\equiv 0$, *conditions* (3)- (5) *and* (29) *hold, and*

$$\beta(t) \leq \frac{c_2}{(1+t)^{q_2}}, \tag{37}$$

where $c_2 > 0$ *and* $q_2 > 0$ *are constants. Assume that*

$$q_1 \leq \min\{1, q_2 - \nu, \nu(p-1)\}, \qquad \|u(0)\| \leq \lambda_0^{-1}, \tag{38}$$

where $\lambda_0 > 0$ *is a constant defined in* (45)*, and*

$$c_2^{1-\frac{1}{p}} c_0^{\frac{1}{p}} (p-1)^{\frac{1}{p}} \frac{p}{p-1} + \nu \leq c_1. \tag{39}$$

Then problem (1)-(2) *has a unique global solution* $u(t)$*, and the following estimate holds:*

$$\|u(t)\| \leq \frac{1}{\lambda_0 (1+t)^{\nu}}, \qquad \forall t \geq 0. \tag{40}$$

Proof of Theorem 5. Let $g(t) := \|u(t)\|$. As in the proof of Theorem 4, multiply (1) by u, take the real part, use the assumptions of Theorem 5, and get the inequality:

$$\dot{g} \leq -\frac{c_1}{(1+t)^{q_1}} g + c_0 g^p + \frac{c_2}{(1+t)^{q_2}}. \tag{41}$$

Choose $\mu(t)$ by formula (31). Apply Theorem 1 to inequality (41). Condition (9) takes now the form

$$\frac{c_0}{\lambda^{p-1}(1+t)^{(p-1)\nu}} + \frac{\lambda c_2}{(1+t)^{q_2-\nu}} + \frac{\nu}{1+t} \leq \frac{c_1}{(1+t)^{q_1}} \quad \forall t \geq 0. \tag{42}$$

If assumption (38) holds, then inequality (42) holds provided that it holds for $t = 0$, that is, provided that

$$\frac{c_0}{\lambda^{p-1}} + \lambda c_2 + \nu \leq c_1. \tag{43}$$

Condition (10) holds if

$$g(0) \leq \frac{1}{\lambda}. \tag{44}$$

The function $h(\lambda) := \frac{c_0}{\lambda^{p-1}} + \lambda c_2$ attains its global minimum in the interval $[0, \infty)$ at the value

$$\lambda = \lambda_0 := \left(\frac{(p-1)c_0}{c_2} \right)^{1/p}, \tag{45}$$

and this minimum is equal to

$$h_{min} = c_0^{\frac{1}{p}} c_2^{1-\frac{1}{p}} (p-1)^{\frac{1}{p}} \frac{p}{p-1}.$$

Thus, substituting $\lambda = \lambda_0$ in formula (43), one concludes that inequality (43) holds if the following inequality holds:

$$c_0^{\frac{1}{p}} c_2^{1-\frac{1}{p}} (p-1)^{\frac{1}{p}} \frac{p}{p-1} + \nu \leq c_1, \tag{46}$$

while inequality (44) holds if

$$\|u(0)\| \leq \frac{1}{\lambda_0}. \tag{47}$$

Therefore, by Theorem 1, if conditions (46)-(47) hold, then estimate (11) yields

$$\|u(t)\| \le \frac{1}{\lambda_0(1+t)^\nu}, \qquad \forall t \ge 0, \tag{48}$$

where λ_0 is defined in (45).

Theorem 5 is proved. □

4 Stability results 2

Let us assume that $\mathrm{Re}(A(t)u,u) \le \gamma(t)\|u\|^2$, where $\gamma(t) > 0$. This corresponds to the case when the linear operator $A(t)$ may have spectrum in the right half-plane $\mathrm{Re}z > 0$. Our goal is to derive under this assumption sufficient conditions on $\gamma(t)$, $\alpha(t,g)$, and $\beta(t)$, under which the solution to problem (1) is bounded as $t \to \infty$, and stable. We want to demonstrate new methodology, based on Theorem 1. By this reason we restrict ourselves to a derivation of the simplest results under simplifying assumptions. However, our derivation illustrates the method applicable in many other problems.

Our assumptions in this Section are:

$$\beta(t) = 0, \quad \gamma(t) = c_1(1+t)^{-m_1}, \quad \alpha(t,g) = c_2(1+t)^{-m_2}g^p,\ p > 1.$$

Let us choose

$$\mu(t) = d + \lambda(1+t)^{-n}.$$

The constants c_j, m_j, λ, d, n, are assumed positive.

We want to show that a suitable choice of these parameters allows one to check that basic inequality (9) for μ is satisfied, and, therefore, to obtain inequality (11) for $g(t)$. This inequality allows one to derive global boundedness of the solution to (1), and the Lyapunov stability of the zero solution to (1) (with $u_0 = 0$). Note that under our assumptions $\dot\mu < 0$, $\lim_{t\to\infty}\mu(t) = d$. We choose $\lambda = d$. Then $(2d)^{-1} \le \mu^{-1}(t) \le d^{-1}$ for all $t \ge 0$. The basic inequality (9) takes the form

$$c_1(1+t)^{-m_1} + c_2(1+t)^{-m_2}[d+\lambda(1+t)^{-n}]^{-p+1} \le n\lambda(1+t)^{-n-1}[d+\lambda(1+t)^{-n}]^{-1}, \tag{49}$$

and

$$g_0(d+\lambda) \le 1. \tag{50}$$

Since we have chosen $\lambda = d$, condition (50) is satisfied if

$$d = (2g_0)^{-1}. \tag{51}$$

Choose n so that

$$n+1 \le \min\{m_1, m_2\}. \tag{52}$$

Then (49) holds if

$$c_1 + c_2 d^{-p+1} \le n\lambda d^{-1}. \tag{53}$$

Inequality (53) is satisfied if c_1 and c_2 are sufficiently small. Let us formulate our result, which folows from Theorem 1.

Theorem 6. *If inequalities* (53) *and* (52) *hold, then*

$$0 \le g(t) \le [d + \lambda(1+t)^{-n}]^{-1} \le d^{-1}, \qquad \forall t \ge 0. \tag{54}$$

Estimate (54) proves global boundedness of the solution $u(t)$, and implies Lyapunov stability of the zero solution to problem (1) with $b(t) = 0$ and $u_0 = 0$.

Indeed, by the definition of Lyapunov stability of the zero solution, one should check that for an arbitrary small fixed $\epsilon > 0$ estimate $\sup_{t\ge 0} \|u(t)\| \le \epsilon$ holds provided that $\|u(0)\|$ is sufficiently small. Let $\|u(0)\| = g_0 = \delta$. Then estimate (54) yields $\sup_{t\ge 0} \|u(t)\| \le d^{-1}$, and (51) implies $\sup_{t\ge 0} \|u(t)\| \le 2\delta$. So, $\epsilon = 2\delta$, and the Lyapunov stability is proved. □

References

1.Yu. L. Daleckii and M. G. Krein, *Stability of solutions of differential equations in Banach spaces*, Amer. Math. Soc., Providence, RI, 1974.

2.B. Davies, *Exploring chaos*, Perseus Books, Reading, Massachusetts, 1999.

3.R. L. Devaney, *An introduction to chaotic dynamical systems*, Addison-Wesley, Reading, Massachusetts, 1989.

4.P. Hartman, *Ordinary differential equations*, J. Wiley, New York, 1964.

5.N.S. Hoang and A. G. Ramm, DSM of Newton-type for solving operator equations $F(u) = f$ with minimal smoothness assumptions on F, *International Journ. Comp.Sci. and Math.* (IJCSM), 3, N1/2, (2010), 3-55.

6.N. S. Hoang and A. G. Ramm, A nonlinear inequality and applications, *Nonlinear Analysis: Theory, Methods and Appl.,* 71, (2009), 2744-2752.

7.N. S. Hoang and A. G. Ramm, The Dynamical Systems Method for solving nonlinear equations with monotone operators, Asian Europ. Math. Journ., 3, N1, (2010), 57-105.

8.N. S. Hoang and A. G. Ramm, DSM of Newton-type for solving operator equations $F(u) = f$ with minimal smoothness assumptions on F, International Journ. Comp.Sci. and Math. (IJCSM), 3, N1/2, (2010), 3-55.

9.A. G. Ramm, *Dynamical systems method for solving operator equations*, Elsevier, Amsterdam, 2007.

10.A. G. Ramm, Dynamical systems method (DSM) and nonlinear problems, in the book: Spectral Theory and Nonlinear Analysis, World Scientific Publishers, Singapore, 2005, 201-228. (ed J. Lopez-Gomez).

11.A. G. Ramm, How large is the class of operator equations solvable by a DSM Newton-type method ? Appl. Math. Lett, 24, N6, (2011), 860-865.

12.A. G. Ramm, A nonlinear inequality and evolution problems, *Journ. of Inequalities and Special Functions,* 1, N1, (2010), 1-9.

13.A. G. Ramm, Asymptotic stability of solutions to abstract differential equations, *Journ. of Abstract Diff. Equations, (JADEA)*, 1, N1, (2010), 27-34.

14.A. G. Ramm, Stationary regimes in passive nonlinear networks, in the book "*Nonlinear Electromagnetics*", Editor P. Uslenghi, Acad. Press, New York, 1980, pp. 263-302.

15.A. G. Ramm, *Theory and applications of some new classes of integral equations*, Springer-Verlag, New York, 1980.

16.R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.