ANALYTICAL AND NUMERICAL SOLUTIONS OF SOME NON-NEWTONIAN ENTRANCE REGION FLOW PROBLEMS

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CHAPTER 1

INTRODUCTION

When a viscous fluid enters a conduit (a circular pipe or a flat duct) from a very large reservoir through a well-rounded entrance, the assumed flat velocity profile at the entry will gradually change as fluid moves down the conduit because the walls of the conduit tend to retard the flow. Eventually, the velocity profile will develop to a form which remains unchanged with respect to the direction of flow. The flow in the region of changing velocity profile is so-called hydrodynamic entrance region flow.

Applying the concepts of boundary layer theory, the boundary layer form at the walls, beginning with zero thickness at the entry, and increase continuously in the down stream direction. At a certain distance from the entrance, the boundary layers become so thick that they meet each other at the center line of the conduit (excluding the cases of fluids with a yield stress). During the formation of the boundary layers in the entrance region, the flat velocity profile near the walls changes to a curved one due to the viscous drag at the fluid-solid interface, and the velocity distribution outside the boundary layer remains uniform. Since the total flow rate is constant along the conduit for steady-state flow, the fluid in the center core of the flow is accelerated in consequence of the altered velocity profile in the boundary layers. Because of increasing kinetic energy and higher viscous friction the pressure drop in the entrance region is considerably larger than that in the fully developed region where the velocity profile is invariant with respect to the direction of flow.

Originally the study of entrance region flow was of interest in connection with the correction of capillary viscometric data. It became of practical importance in engineering design because an exact description of the velocity and pressure fields in the entrance region would provide a basis for analysis and understanding of the energy and mass transport in the entrance region of a flow system, especially a compact one.

The literature on hydrodynamic entrance region flow before 1965 is well surveyed and classified by Fan and Hwang (11). Attempts to examine the hydrodynamic entrance flow theoretically have been made by many researchers for the last hundred years. They tried to obtain the velocity profile, the pressure drop and the value of the entrance length at which the fully developed flow is attained. The earliest effort in this field was to predict the total pressure drop from the reservoir to the fully developed region by Hagenback (13) in 1860. The second effort was due to Boussinesq (3) in 1890 and 1891. Although the boundary layer theory was not yet formulated at that time he analyzed this problem by neglecting the small term in the Navier Stokes equation and assuming that pressure could be considered constant across the cross section.

After Prandtl's classic paper published in 1904, the boundary layer concept has been used to reduce the difficulties in solving the non-linear partial differential equation. Although the equations of motion are much simplified by the approximation based on the boundary layer model, we are still unable to obtain their exact analytical solutions. Therefore, various approximation methods have been employed to solve the entrance region flow problem based on the boundary layer model.

The basic momentum integral method was devised by Schiller (31) in 1922 for Newtonian flow in a circular pipe. The flow is divided into two portions, the boundary layer near the wall with the assumed parabolic velocity profile and the potential flow in the center core. The boundary layer equation of motion is integrated across the boundary layer and then solved analytically or numerically. Later the method was modified and different fluid models employed (2, 4, 12, 30, 38).

The second approximation method is to linearize the boundary layer equation by assuming the inertial terms equal to some function of the coordinate in the direction of flow. This method was developed first by Langhaar (21) in 1940 and by Targ (37) for Newtonian fluids with different functions in 1951. To date it has been applied to Newtonian fluids only.

The third category of solutions, which was first used by Schlichting (32) in 1934 for a Newtonian flat duct flow, is obtained by matching the boundary layer solution, valid near the entry, with the solution for approaching fully developed flow. The same technique was employed by Collins and Schowalter (7, 8) to power law non-Newtonian fluids both in a channel and a circular pipe.

In another approach finite difference equations obtained from the continuity and momentum equations are solved numerically on a digital computer. For a Newtonian fluid this method was used by Dodoria and Osterle (1) in 1961 and Hwang and Fan (15). The same approach was applied to the Powell-Eyring fluid model by Hornbeck (14) and to the power law fluid by Christiansen and Lemmon (6) in 1965.

The variational method was originally given in a paper by Uematsu et al. (40) in 1955 to treat the Newtonian entrance region flow in a circular pipe. In 1961 Tomita (39) extended it to power law fluids and the same approach was applied to Bingham fluids by Michiyochi et al. (29) in 1966. In this method, instead of solving the equation of motion, the velocity field is determined so that some functional of velocities is an extremum.

The theoretical study on some hydrodynamic entrance region flows of

Newtonian fluids was reviewed and summarized by Dealy (9) and Li (25). This report treats non-Newtonian entrance region flow, especially that associated with power law and Bingham fluids.

Chapter 2 provides the classification of non-Newtonian fluids, their shear stress expressions, (or the rheological equations of state) and flow behaviors in the fully developed region of pipes and channels. Since the entrance region flow will eventually approach the fully developed one, exact information concerning the fully developed flow is necessary.

Chapter 3 presents a preliminary study of the governing equations. The general form of the governing equations is first derived in terms of shear stress. The various assumptions, approximations, and shear stress expressions in terms of velocity gradients and rheological parameters can be used to obtain simplified governing equations for each non-Newtonian entrance region flow problem. The boundary layer equations of power law and Bingham fluids are derived in the later part of this chapter.

Some published work on the entrance region flow of power law fluids is classified and introduced in detail in Chapter 4. The approach and the results of each method are compared and discussed.

The basic momentum integral method and the Campbell-Slattery method are used to analyze the entrance region flow of Bingham fluids in Chapter 5. The numerical results are compared with those of Michiyochi's variational method which is the only published paper on the Bingham entrance region flow.

CHAPTER 2

NON-NEWTONIAN FLOW

In this chapter we shall describe various models of fluids in general, and mathematical models of non-Newtonian fluids in particular. The fully developed laminar flow of non-Newtonian fluids in pipes and channels is also discussed in the later part of this chapter.

2.1 CLASSIFICATION OF FLUIDS

In the field of fluid dynamics it is customary to classify fluids into two main categories, i.e., perfect (or ideal) fluids and real fluids. The imaginary perfect fluids which are frictionless and incompressible offer much simplification to theoretical study of the flow of fluid. On the other hand, all of the real fluids experience a certain degree of internal resistance to a change in shape, or tangential (shear) stresses between two contact layers of the fluids. The existence of tangential stresses which may depend upon the extent of deformation or duration of applied shear, or upon both, characterizes the special properties of each fluid.

For theoretical analysis, rheologists subdivide the real fluids into two general groups, namely, Newtonian fluids, and non-Newtonian fluids according to their flow behaviors (26, 27). To further simplify the study of each fluid, non-Newtonian fluids are again subdivided. It should be noted that any of the real fluids may belong to one of the rheologically subdivided groups of fluids but sometimes it may ahve complex properties combining two or more fluid groups.

2.1-1 Newtonian Fluids

Newtonian fluids are those which behave according to Newton's law of viscosity which states that shear stress, $\underline{\underline{\tau}}$, is directly proportional to shear rate in the laminar flow region at constant temperature. For incompressible fluids it can be expressed in the form of (42)

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\Delta}} \tag{1}$$

In this expression μ is the viscosity of the fluid and $\underline{\underline{\Delta}}$ is the symmetrical "rate of deformation tensor" with cartesian components

$$\Delta_{ij} = \left(\frac{\partial v_i}{\partial x_j}\right) + \left(\frac{\partial v_j}{\partial x_i}\right) \tag{2}$$

in which $\partial v_i / \partial x_j$ is the velocity gradient of v_i in the j direction. In cylindrical coordinates (r, θ, z), Δ_{ij} can be obtained from Eq. (2) by coordinate transformation. The coefficient of viscosity μ depends on the local temperature and pressure but it is independent of the shear stress or the rate of deformation tensor.

Consider a fully developed flow of a Newtonian fluid between the two parallel plates as shown in Fig. 1. Then Eq. (1) may reduce to a simple form

$$\tau_{yx} = -\mu \frac{du}{dy} \tag{3}$$

where τ_{yx} is the shear stress acting on the layer perpendicular to the y axis and in the direction parallel to the x coordinate and u is the velocity in the x direction.

Graphically Eq. (3) can be shown on arithmetic or logarithmic coordinates as given in Figs. 2a and 2b. All gases and all homogeneous liquids or solutions of low molecular weight materials (i.e., non-polymeric) obey this Newtonian flow behavior.

2.1-2 Non-Newtonian Fluids

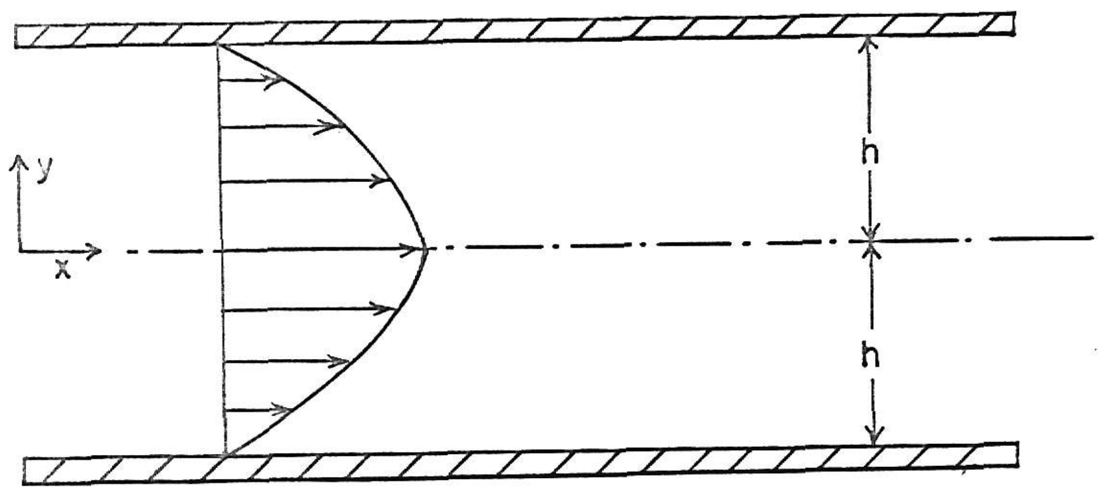
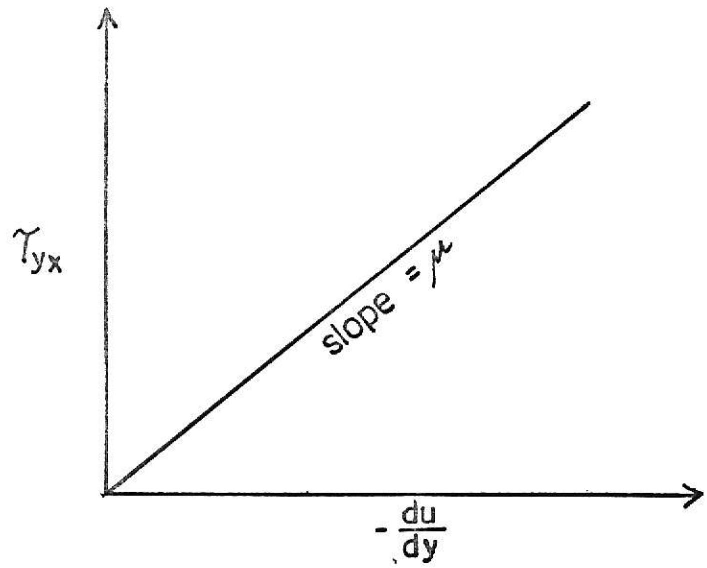
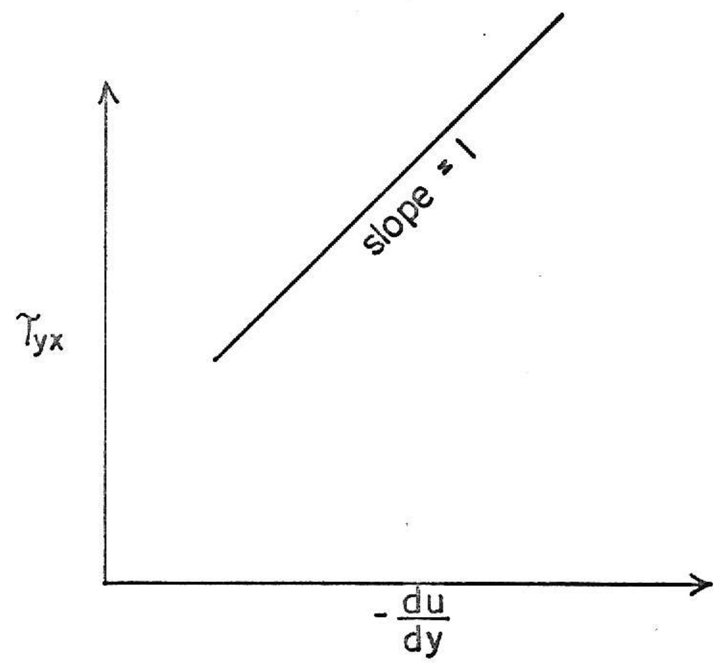


Fig. 1. Laminar channel flow of the Newtonian fluid .



a. Arithmetic scale



b. Logarithmic scale

Fig. 2. Shearing stress of the Newtonian fluid as a function of shear rate.

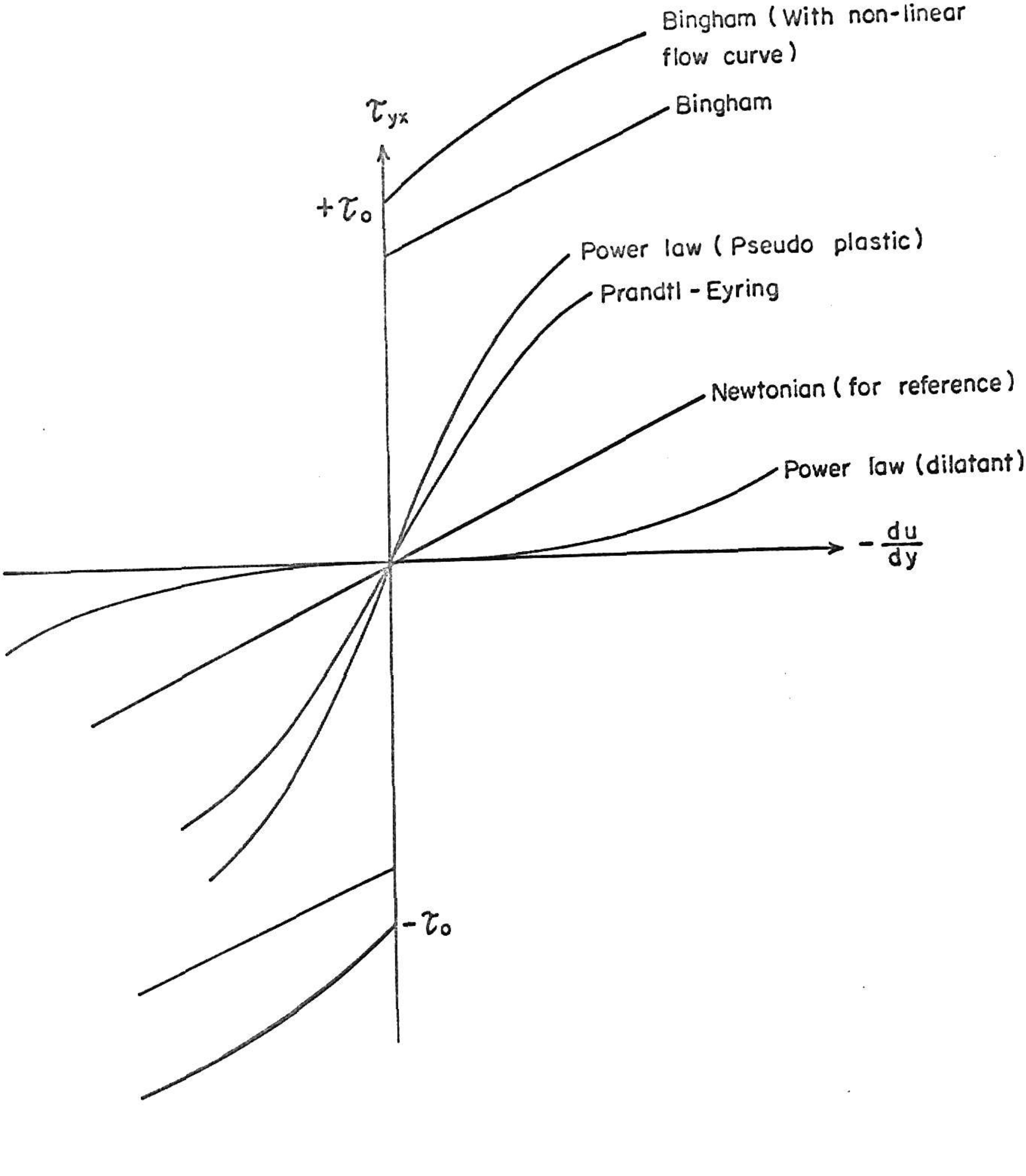


Fig. 3. Flow curves for time-independent two-parameter non-Newtonian fluids.

Non-Newtonian fluids are those which do not yield the Newtonian flow curve, i.e., a straight line passing through the origin as shown in Fig. 2a. Non-Newtonian fluids are usually divided into three general groups (26):

(1) Fluids with properties independent of time or duration of shear stress (i.e., steady state).

(2) Fluids with properties dependent upon duration of shear stress (unsteady state).

(3) Fluids which have characteristics of both solids and liquids and exhibit partial elastic recovery after applied stress is released, the so-called visco-elastic materials.

In a mathematical sense, non-Newtonian fluids are those which do not obey the relationship between τ_{yx} and $-du/dy$ by Eq. (3) at constant pressure and temperature. For time-independent fluids which are also called non-Newtonian viscous fluids, shear rate ia a function of shear stress only, i.e.,

$$-\frac{du}{dy} = f(\tau_{yx}) \qquad (4)$$

The time-independent fluids obey the following relationship

$$-\frac{du}{dy} = f(\tau_{yx}, t) \qquad (5)$$

in which the duration of shear stress, t, is included. In addition to τ_{yx}, the shear rate of visco-elastic fluids is also a function of the shear strain γ,

$$-\frac{du}{dy} = f(\tau_{yx}, \gamma) \qquad (6)$$

More complete descriptions of time-dependent and visco-elastic fluids are available elsewhere (26, 27, 28).

2.2 TIME-INDEPENDENT NON-NEWTONIAN FLUIDS

Time-independent non-Newtonian fluids are sometimes called non-Newtonian viscous fluids. There is another term, "purely viscous fluid", to represent both Newtonian and time-independent non-Newtonian fluids.

The formulation of shear stress for time-independent non-Newtonian fluids in terms of shear rate is, in general, a complex problem and needs a well-developed theoretical basis (42). Let us consider the simple types of time-independent non-Newtonian fluids. The relation between $\underline{\underline{\tau}}$ and $\underline{\underline{\Delta}}$ can be expressed by a generalized form (42)

$$\underline{\underline{\tau}} = - \eta \underline{\underline{\Delta}} \tag{7}$$

where the non-Newtonian viscosity η is a scalar function of $\underline{\underline{\Delta}}$ (or $\underline{\underline{\tau}}$) and also a function of temperature and pressure. If the flow situation is the same as described in Fig. 1, i.e., fully developed flow, Eq. (7) can be expressed by

$$\tau_{yx} = -\eta \frac{du}{dy} \tag{8}$$

in which η is a positive value and a function of τ_{yx} or $-du/dy$. When η is equal to constant μ, the expression in Eq. (8) reduces to Newton's law of viscosity.

2.2-1 The Constitutive Equations

Various empirical models using two or more adjustable parameters and obeying the tensor transformation have been proposed (42) in order to approximate the behaviors of various fluids. They are described below:

The Ostwald-de Waele Model (or Power Law Model):

$$\tau_{yx} = - m \left|\frac{du}{dy}\right|^{n-1} \frac{du}{dy} \tag{9}$$

The non-Newtonian viscosity η can be expressed as

$$\eta = m \left|\frac{du}{dy}\right|^{n-1}$$

This two-parameter model which is also called the power law model is simplest and most useful for theoretical calculations. For n less than unity, it represents pseudoplastic behavior where by definition decreases with the increasing shear rate in some range of τ_{yx}. For n larger than unity, increases with increasing shear rate and the fluid is called dilatant. For n equal to unity, it reduces to a Newtonian fluid with $\mu = m$. It is also noted that n is not a real constant for all possible ranges of shear rate. It may vary, but for some range of shear stress it may be regarded as constant. The flow curves of the power law model are shown in Fig. 3.

The Bingham Model:

$$\tau_{yx} = - \left(\mu_o + \frac{\tau_o}{\left|\frac{du}{dy}\right|}\right) \frac{du}{dy} \qquad \text{if } |\tau_{yx}| > \tau_o \tag{10}$$

$$\frac{du}{dy} = 0 \qquad \text{if } |\tau_{yx}| < \tau_o \tag{11}$$

Many substances such as blood, waste water, suspension of a catalyst in liquid, paste, fuels, and drilling muds follow this two-parameter model which has a flow behavior such that it remains rigid when the applied shear stress is less than the yield stress, τ_o. When the shear stress exceeds the yield stress, the shear stress of a Bingham fluid is proportional to shear rate as shown in Fig. 3. Three other models with the yield stress and nonlinear relation between τ_{yx} and - du/dy have also been proposed (46).

The Prandtl-Eyring Model:

$$\tau_{yx} = - \left[\frac{A \sinh^{-1}\left(\frac{1}{B}\frac{du}{dy}\right)}{\frac{du}{dy}}\right]\frac{du}{dy} \tag{12}$$

This two-parameter semi-theoretical equation was derived by Eyring based on the molecular theory of liquids. At a finite value of τ_{yx}, the Prandtl-Eyring fluid has pseudoplastic behavior. It reduces asymptotically to a Newtonian fluid with $\mu = A/B$ as τ_{yx} approaches zero.

The Satterby Model (or Generalized Eyring Model):

$$\tau_{yx} = - \eta_0 \left[\frac{\sinh^{-1}\left(\beta \frac{du}{dy}\right)}{\frac{du}{dy}}\right]^{\alpha}\frac{du}{dy} \tag{13}$$

This model contains three adjustable constants α, β, and η_0. If α is set equal to unity, this reduces to the Prandtl-Eyring model.

The Ellis Model:

$$\tau_{yx} = - \frac{\eta_0}{1 + \left|\frac{\tau_{yx}}{\tau_{\frac{1}{2}}}\right|^{\alpha-1}} \frac{du}{dy} \tag{14}$$

or

$$\tau_{yx} = - \frac{1}{\psi_0 + \psi_1 |\tau_{yx}|^{\alpha-1}} \frac{du}{dy}$$

This is a very useful model with three adjustable positive parameters. $\tau_{\frac{1}{2}}$ is the shear stress when the non-Newtonian viscosity η is equal to $\eta_0/2$. If α is larger than unity this model reduces to Newton's law for small τ_{yx}; if α is less than unity, it also reduces to Newton's law for large τ_{yx}. In addition, the Ellis model becomes Newton's law for $\psi_1 = 0$ and the power law for $\psi_o = 0$.

The Meter-Bud Model:

$$\tau_{yx} = - \frac{\eta_o}{\dfrac{1 + |\tau_{yx}/\tau_m|^{-1}}{1 + |\tau_{yx}/\tau_m|^{-1}(\eta_\infty/\eta_o)}} \frac{du}{dy}$$

Here η_o is the lower limiting viscosity which is equal to the non-Newtonian viscosity at the lower limit of τ_{yx}, whereas η_∞ is the upper limiting viscosity. τ_m is the shear stress when the non-Newtonian viscosity η is equal to the arithmetic average of η_o and η_∞.

The Reiner-Philippoff Model:

$$\tau_{yx} = - (\mu_\infty + \frac{\mu_o - \mu_\infty}{1 + (\frac{\tau_{yx}}{\tau_s})^2}) \frac{du}{dy}$$

This three-constant model may be reduced to Newton's law both at very high and very low shear rate with $\mu = \mu_\infty$ and $\mu = \mu_o$ respectively.

2.2-2 Laminar Flow in a Fully Developed Region

In this section we consider the fully developed flow of time-independent non-Newtonian fluids in a circular pipe and between flat parallel plates. By fully developed we mean that the velocity profile is independent of the direction of flow. As a result, the velocity component perpendicular to the flow direction is zero for the case of steady state flow (16).

In the course of derivation of the velocity profile, pressure drop, and volume rate of flow, it is assumed that the flow is laminar, isothermal, steady, and no external force. The gravitational force may be neglected if the fluids flow in a horizontal conduit.

A. Flow in a circular tubo

Consider the flow of a fluid in a horizontal circular pipo of constant radius R shown in Fig. 4. Making a momentum balance over the contrl volume we have

$$\pi r^2 P_1 - \pi r^2 P_2 - 2\ rL\tau_{rz} = 0 \tag{15}$$

where P_1 and P_2 are pressures acting over the cross-sections 1 and 2, and τ_{yz} is the shear stress acting on the contrl volume by the adjacent fluid layer. Therefore, we obtain

$$\tau_{rz} = \frac{|\Delta P|}{2L}\ r \tag{16}$$

and

$$d\tau_{rz} = \frac{|\Delta P|}{2L}\ dr \tag{17}$$

where

$$\Delta P = P_2 - P_1$$

and absolute sign is used here since P_2 is less than P_1.

At the wall,

$$(\tau_{rz})_{r=R} = \tau_w = \frac{|\Delta P|}{2L}\ R \tag{18}$$

Equation (16) indicates that the shear stress is directly propotional to the radius r. The volume rate of flow Q is obtained by integration of the velocity distrbution:

$$\begin{aligned} Q &= 2\pi \int_0^R v_z r\ dr \\ &= 2\pi\left(v_z r^2/2 - \int r^2/2\ dv_z \right)_{r=0}^{r=R} \end{aligned} \tag{19}$$

The condition of no slip between the fluid and the wall eliminates the first term of Eq. (19),

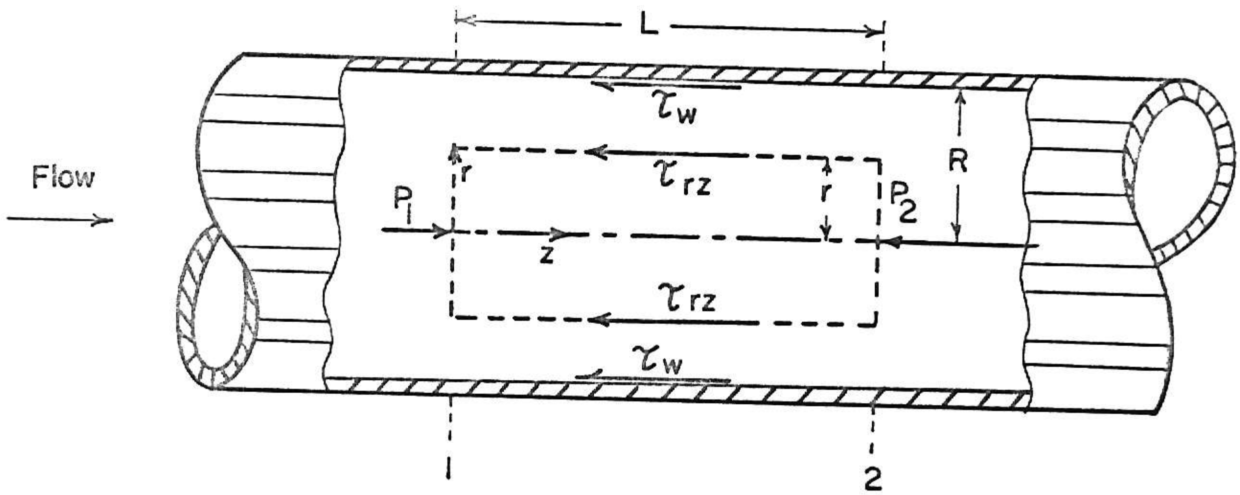


Fig. 4 . Fully-developed laminar flow in a circular pipe .

$$Q = -\pi \left(\int r^2 \, dv_z \right)_{r=0}^{r=R} \tag{20}$$

Many two-phase suspensions show a characteristic slippage at the wall surface (17); hence, the so-called wall effects should be considered in Eq. (19).

As can be seen from the form of many rheological equations of state the time-independent non-Newtonian fluids have the flow equation

$$-\frac{dv_z}{dr} = f(\tau_{rz}) \tag{21}$$

which can be substituted in to Eq. (20) to obtain

$$Q = \pi \int_0^R r^2 \, f(\tau_{rz}) \, dr \tag{22}$$

Combining Eqs. (16), (17), and (18) gives

$$\tau_{rz} = \tau_w \frac{r}{R} \qquad \text{or} \qquad r^2 = \frac{R^2}{\tau_w^{\,2}} \tau_{rz}^2 \tag{23}$$

and

$$dr = \frac{R}{\tau_w} \, d\,\tau_{rz} \tag{24}$$

Substituting these two relations into Eq. (22) we obtain

$$Q = \frac{\pi R^3}{\tau_w^{\,3}} \int_0^{\tau_w} \tau_{rz}^{\,2} \, f(\tau_{rz}) \, d\,\tau_{rz} \tag{25}$$

The average velocity across the cross section can be obtained by dividing the volume rate of flow by the cross sectional area R^2,

$$\bar{u} = \frac{Q}{\pi R^2} = \frac{R}{\tau_w^{\,3}} \int_0^{\tau_w} \tau_{rz}^{\,2} \, f(\tau_{rz}) \, d\,\tau_{rz} \tag{26}$$

The velocity profile can be obtained by substituting the expression of τ_{rz} in

Eq. (23) into Eq. (21) and integrating it with respect to r,

$$v_z = -\int f(\tau_w \frac{r}{R})\, dr \qquad (27)$$

Eqs. (16), (25), (26), and (27) are the generalized expressions to be used to obtain the shear stress distribution, the relation between volume rate of flow and pressure drop, the average velocity and velocity distribution when the appropriate functions of $f(\tau_{rz})$ of time- independent non-Newtonian fluids are inserted in these equations. The derivation of these expressions are shown in Appendix I and the results are tabulated in Tables 1 through 3.

B. Flow in a Channel

Consider the steady flow of a fluid between two horizontal pqrallel plates separated by a distance 2h as shown in Fig. 5. The plates are stationary and extend infinitely to eliminate the end effects.

Following the method used in the preceding section the general expressions of the distribution of shear stress at the walls, volume rate of flow, and velocity distribution can be expressed as follows:

$$\tau_{yx} = \frac{|\Delta P|}{L} y = \frac{\tau_w}{h} y \qquad (28)$$

$$w = \frac{|\Delta P|}{L} h \qquad (29)$$

$$Q = \frac{2Wh^2}{\tau_w^2} \int_0^{\tau_w} \tau_{yx} f(\tau_{yx})\, d\tau_{yx} \qquad (30)$$

and

$$v_x = - f(\frac{\tau_w}{h} y)\, dy \qquad (31)$$

The expressions of τ_{yx}, τ_w, Q, V_x, the average velocity $\bar{u}$ and center-line

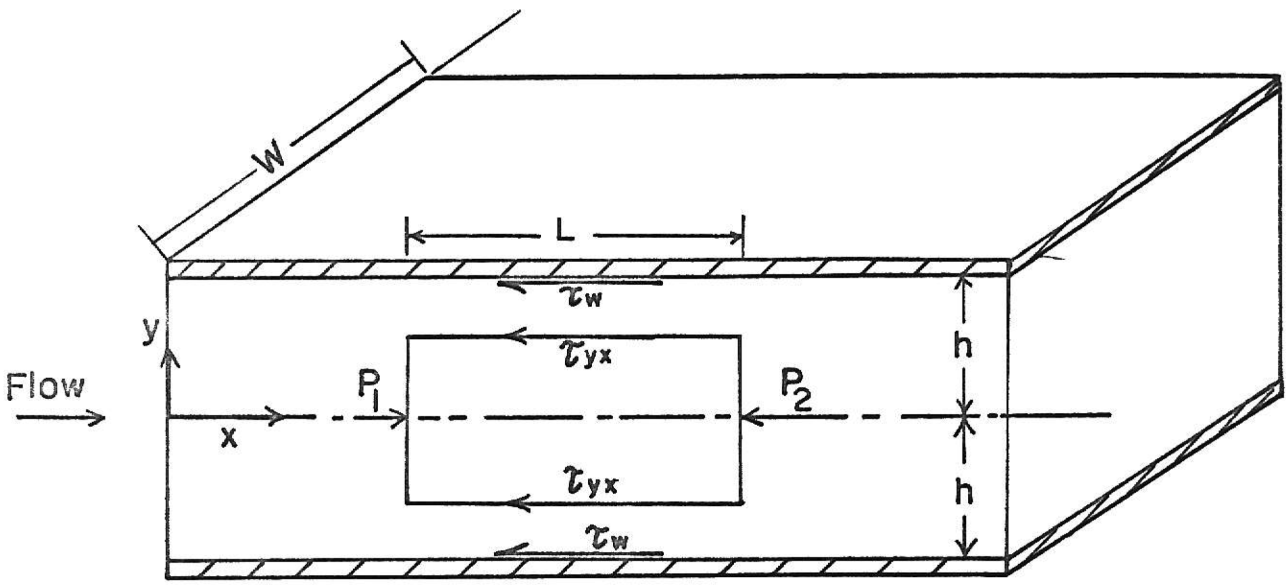


Fig. 5. Fully developed laminar flow between stationary flat parallel plates.

velocity U for some time-independent non-Newtonian fluids are derived in Appendix I and the results are shown in Tables 2 and 3.

For illustration, some fully developed velocity profiles of the power law and Bingham fluid are shown in Figs. 6 and 7 and pressure drops in Figs. 8 and 9.

The fully developed laminar flow of the power law, Bingham, and Ellis fluids in annuli has been reviewed extensively by Skelland (46) and will not be repeated here.

Table 1. Shear rate function, volume rate and velocity profile of laminar flow in fully developed region of circular pipes.

Fluid model	$f(\tau_{rz})$	Volume rate, Q	Velocity distribution, v_z
Newtonian	$\frac{1}{\mu}\tau_{rz}$	$\frac{\pi R^3 \tau_w}{4}$	$\frac{\tau_w R}{2}\left[1 - \left(\frac{r}{R}\right)^2\right]$
Power law	$\left(\frac{\tau_{rz}}{m}\right)^{1/n}$	$\frac{n\pi R^3}{3n+1}\left(\frac{\tau_w}{m}\right)^{1/n}$	$\frac{nR}{n+1}\left(\frac{\tau_w}{m}\right)^{1/n}\left[1 - \left(\frac{r}{R}\right)^{n+1/n}\right]$
Bingham	$\frac{1}{\mu_o}(\tau_{rz} - \tau_o)$	$\frac{\pi R^3 \tau_w}{4\mu_o}\left[1 - \frac{4}{3}\frac{\tau_o}{\tau_w} + \frac{1}{3}\left(\frac{\tau_o}{\tau_w}\right)^4\right]$	$\frac{\tau_w R}{2\mu_o}\left[1 - \left(\frac{r}{R}\right)^2\right] - \frac{\tau_o R}{\mu_o}\left(1 - \frac{r}{R}\right)$ for $r_o \leq r \leq R$
Ellis	$\frac{\tau_{rz}}{\eta_o}\left[1 + \left(\frac{\tau_{rz}}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$	$\frac{\pi R^3 \tau_w}{4_o}\left[1 + \frac{4}{\alpha+3}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$	$\frac{\tau_w R}{2\eta_o}\left\{\left[1 - \left(\frac{r}{R}\right)^2\right] + \frac{2}{\alpha+1}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\left[1 - \left(\frac{r}{R}\right)^{\alpha+1}\right]\right\}$

Table 2. Shear rate function, volume rate and velocity profile of laminar flow in fully developed region of flat plates.

Fluid model	$f(\tau_{yx})$	Volume rate, Q	Velocity distribution, v_x
Newtonian	$\frac{1}{\mu}\tau_{yx}$	$\frac{2Wh^2\tau_w}{3\,L}$	$\frac{\tau_w h}{2}\left[1 - (\frac{y}{h})^2\right]$
Power law	$(\frac{\tau_{yx}}{m})^{1/n}$	$\frac{2nWh^2}{2n+1}\,(\frac{\tau_w}{m})^{1/n}$	$\frac{nh}{n+1}(\frac{\tau_w}{m})^{1/n}\left[1 - (\frac{y}{h})^{n+1/n}\right]$
Bingham	$\frac{1}{\mu_o}(\tau_{yx} - \tau_o)$	$\frac{2Wh^2\tau_w}{3\mu_o}\left[1-\frac{3}{2}(\frac{\tau_o}{\tau_w}) + \frac{1}{2}(\frac{\tau_o}{\tau_w})^3\right]$	$\frac{\tau_w h}{\mu_o}\left[1 - (\frac{y}{h})^2\right] + \frac{\tau_o h}{\mu_o}(1 - \frac{y}{h})$ for $y_o \leq y \leq h$
Ellis	$\frac{\tau_{yx}}{\eta_o}\,1 + (\frac{\tau_{yx}}{\tau_{\frac{1}{2}}})^{\alpha-1}$	$\frac{2Wh^2\tau_w}{3\eta_o}\left[1 + \frac{3}{\alpha+2}(\frac{\tau_w}{\tau_{\frac{1}{2}}})^{\alpha-1}\right]$	$\frac{\tau_w h}{2\eta_o}\left\{\left[1 - (\frac{y}{h})^2\right] + (\frac{\tau_w}{\tau_{\frac{1}{2}}})^{\alpha-1}\frac{2}{\alpha+1}\left[1 - (\frac{y}{h})^{\alpha+1}\right]\right\}$

Table 3. Average and center velocities of laminar flow in fully developed region.

	Fluid model	Average velocity $\bar{u}$	Center velocity U
Circular tube	Newtonian	$\frac{\lvert\Delta P\rvert R^2}{8\mu L}$	$\frac{\lvert\Delta P\rvert R^2}{4\mu L}$
	Power law	$\frac{nR}{3n+1}\left(\frac{\lvert\Delta P\rvert R}{2mL}\right)^{1/n}$	$\frac{nR}{n+1}\left(\frac{\lvert\Delta P\rvert R}{2mL}\right)^{1/n}$
	Bingham	$\frac{\lvert\Delta P\rvert R^2}{8\mu_0 L}\left[1 - \frac{4}{3}\frac{2L\tau_0}{\lvert\Delta P\rvert R} + \frac{1}{3}\left(\frac{2L\tau_0}{\lvert\Delta P\rvert R}\right)^4\right]$	$\frac{\lvert\Delta P\rvert R^2}{4\mu_0 L}\left(1 - \frac{2L\tau_0}{\lvert\Delta P\rvert R}\right)^2$
	Ellis	$\frac{\lvert\Delta P\rvert R^2}{8\eta_0}\left[1 + \frac{4}{\alpha+3}\left(\frac{\lvert\Delta P\rvert R}{2\tau_{\frac{1}{2}} L}\right)^{\alpha-1}\right]$	$\frac{\lvert\Delta P\rvert R^2}{4\eta_0 L}\left[1 + \frac{2}{\alpha+1}\left(\frac{\lvert\Delta P\rvert R}{2\tau_{\frac{1}{2}} L}\right)^{\alpha-1}\right]$
Flat plates	Newtonian	$\frac{\lvert\Delta P\rvert h^2}{3\mu L}$	$\frac{\lvert\Delta P\rvert h^2}{2\mu L}$
	Power law	$\frac{nh}{2n+1}\left(\frac{\lvert\Delta P\rvert h}{mL}\right)^{1/n}$	$\frac{nh}{n+1}\left(\frac{\lvert\Delta P\rvert h}{mL}\right)^{1/n}$
	Bingham	$\frac{\lvert\Delta P\rvert h^2}{3\mu_0 L}\left[1 - \frac{3}{2}\frac{\tau_0 L}{\lvert\Delta P\rvert h} + \frac{1}{2}\left(\frac{\tau_0 L}{\lvert\Delta P\rvert h}\right)^3\right]$	$\frac{\lvert\Delta P\rvert h^2}{2\mu_0 L}\left(1 - \frac{\tau_0 L}{\lvert\Delta P\rvert h}\right)^2$
	Ellis	$\frac{\lvert\Delta P\rvert h^2}{3\eta_0 L}\left[1 + \frac{3}{\alpha+2}\left(\frac{\lvert\Delta P\rvert h}{\tau_{\frac{1}{2}} h}\right)^{\alpha-1}\right]$	$\frac{\lvert\Delta P\rvert h^2}{2\eta_0 L}\left[1 + \frac{2}{\alpha+1}\left(\frac{\lvert\Delta P\rvert h}{\tau_{\frac{1}{2}} h}\right)^{\alpha-1}\right]$

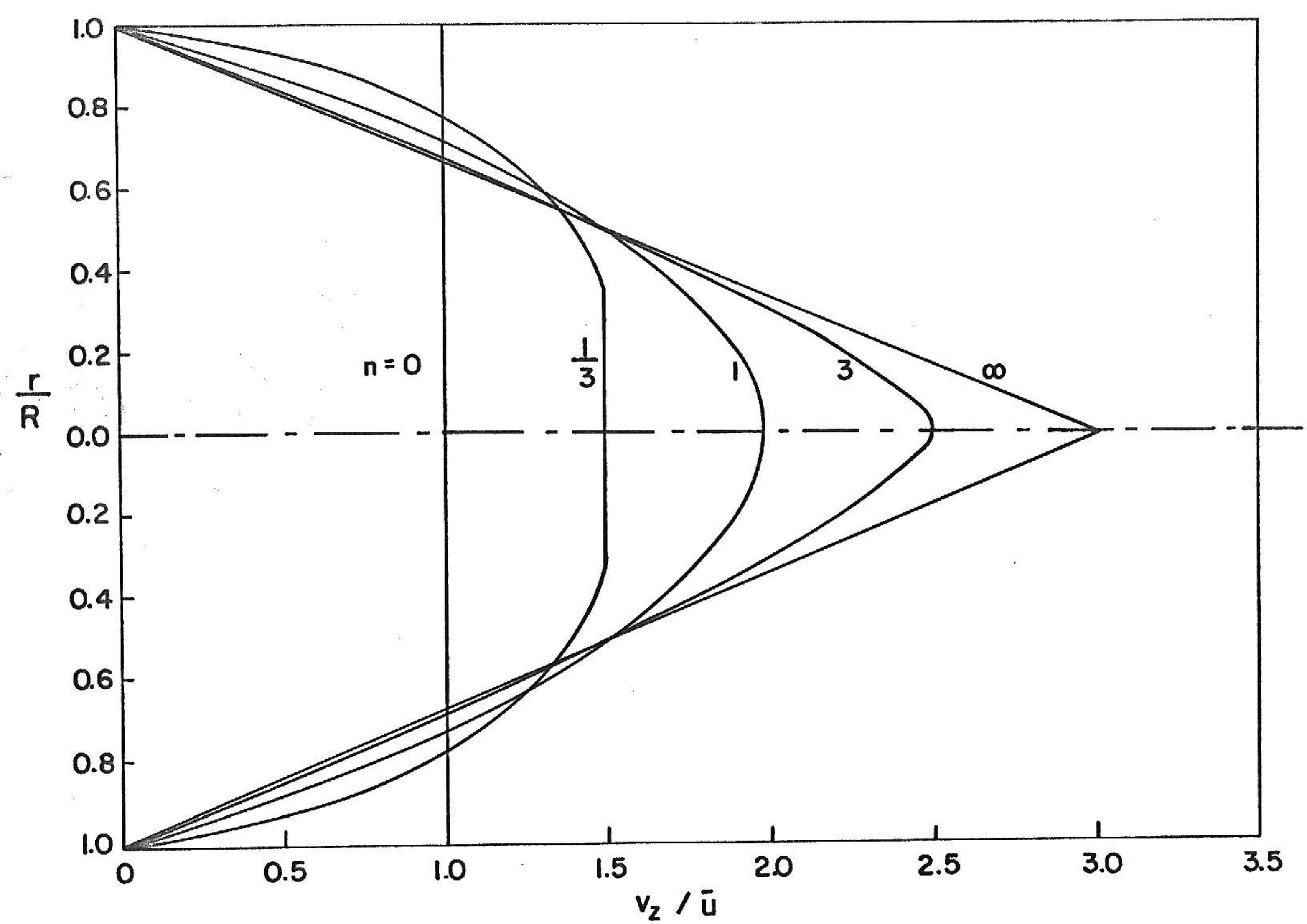


Fig. 6 . Fully developed velocity profile of the power law fluid in pipes.

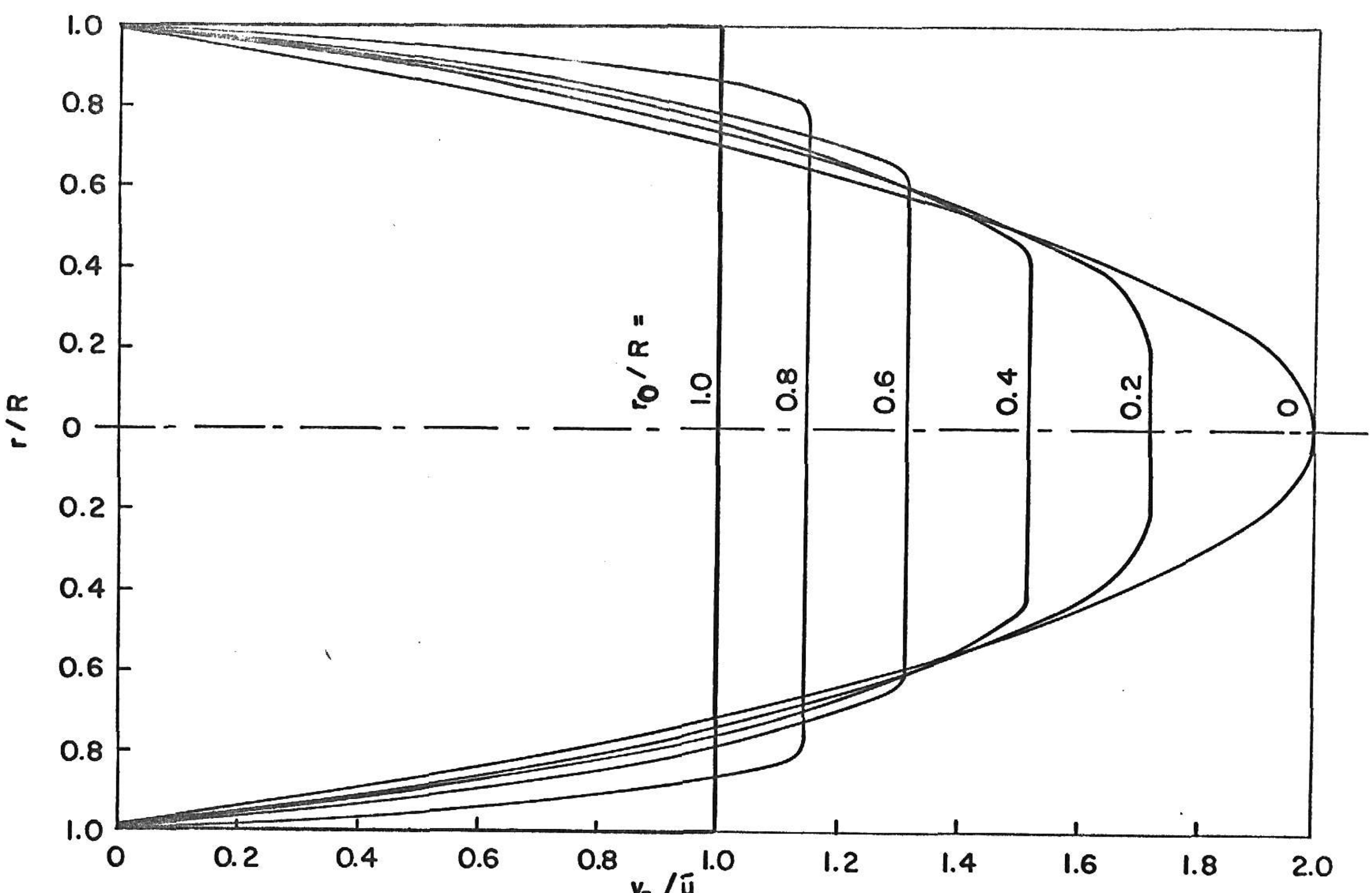


Fig. 7. Fully developed velocity profiles of the Bingham fluid in pipes.

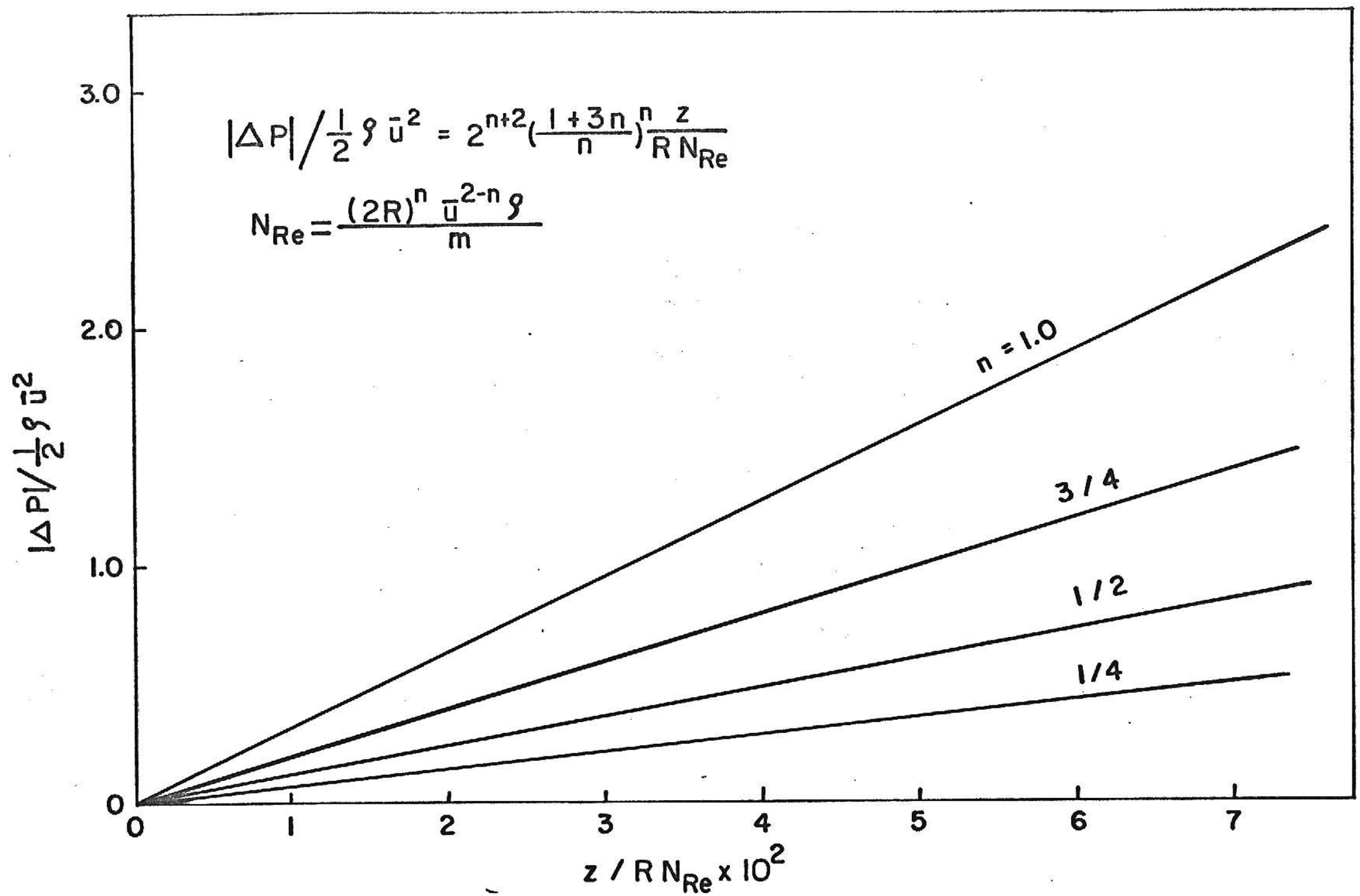


Fig. 8. Fully developed pressure drop versus tube length for the power law fluid.

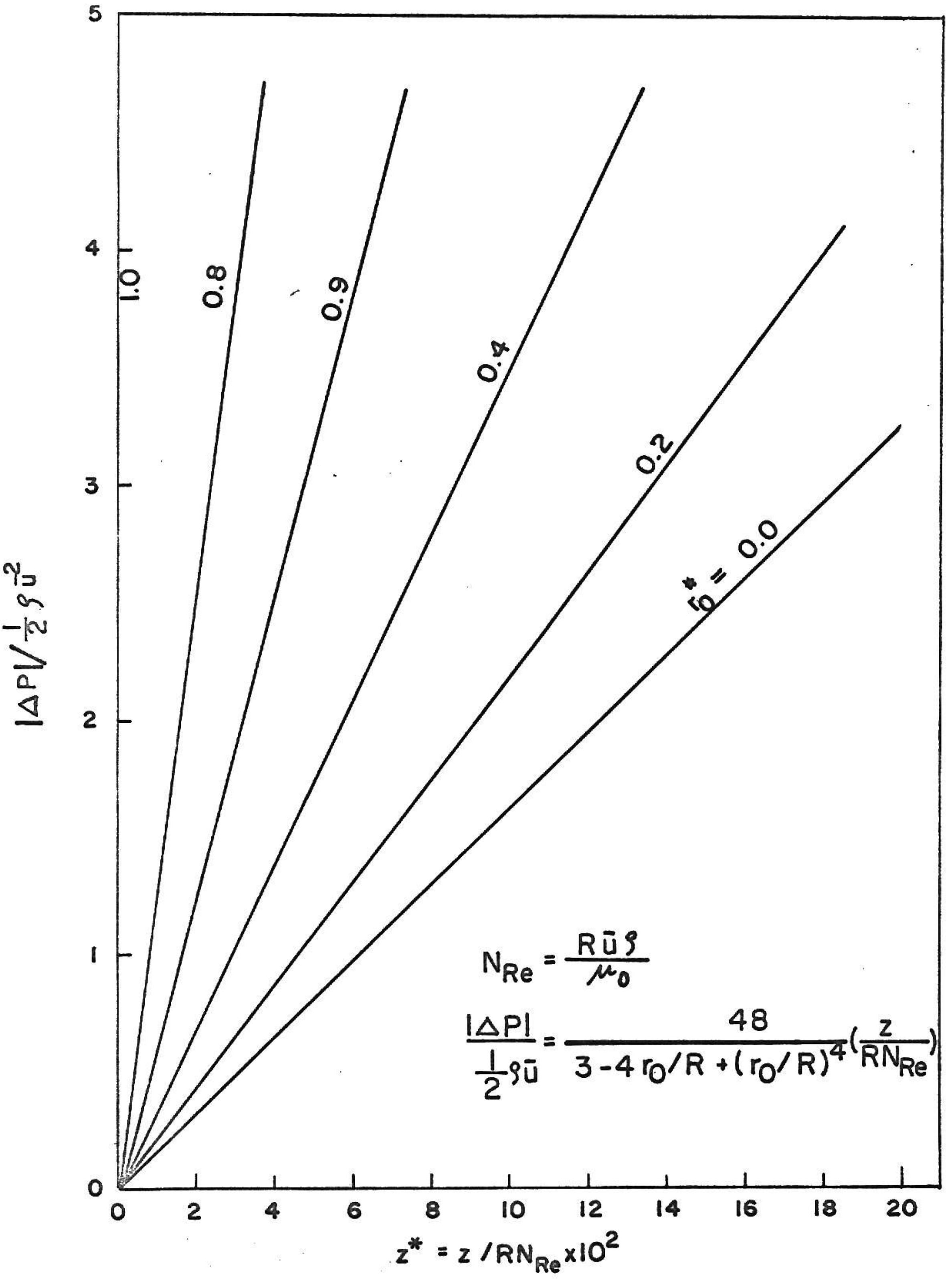


Fig. 9. Fully-developed pressure drop versus tube length for the Bingham fluid.

CHAPTER 3

GOVERNING EQUATIONS OF NON-NEWTONIAN ENTRANCE REGION FLOW

In this chapter the general form of governing equations of non-Newtonian flow inside a conduit is first derived in terms of shear stresses. In order to obtain the solutions of the velocity field and the pressure field from three governing equations, the appropriate expressions for the shear stresses in terms of velocity gradients and fluid properties are introduced and inserted in the governing equations. Then, several assumptions are made to simplify the entrance region flow problem. The resulting differential equations are still very complicated due to their nonlinearity. Therefore, the equations are further simplified using the boundary layer concept. The boundary layer equations are to be solved with appropriate boundary conditions by different approximate methods which will be presented in the following chapters.

3.1. GOVERNING EQUATIONS

Consider a fluid entering a circular tube or a channel as shown in Figs. 10 and 11. The equations of continuity and motion for isothermal entrance region flow are respectively (42)

$$\frac{D\rho}{Dt} = -\rho(\underline{\nabla}\cdot\underline{V}) \tag{1}$$

and

$$\rho\frac{D\underline{V}}{Dt} = -\underline{\nabla}P - (\underline{\nabla}\cdot\underline{\underline{\tau}}) \tag{2}$$

Here the external forces are neglected, and $\frac{D}{Dt}$ is the substantial derivative,

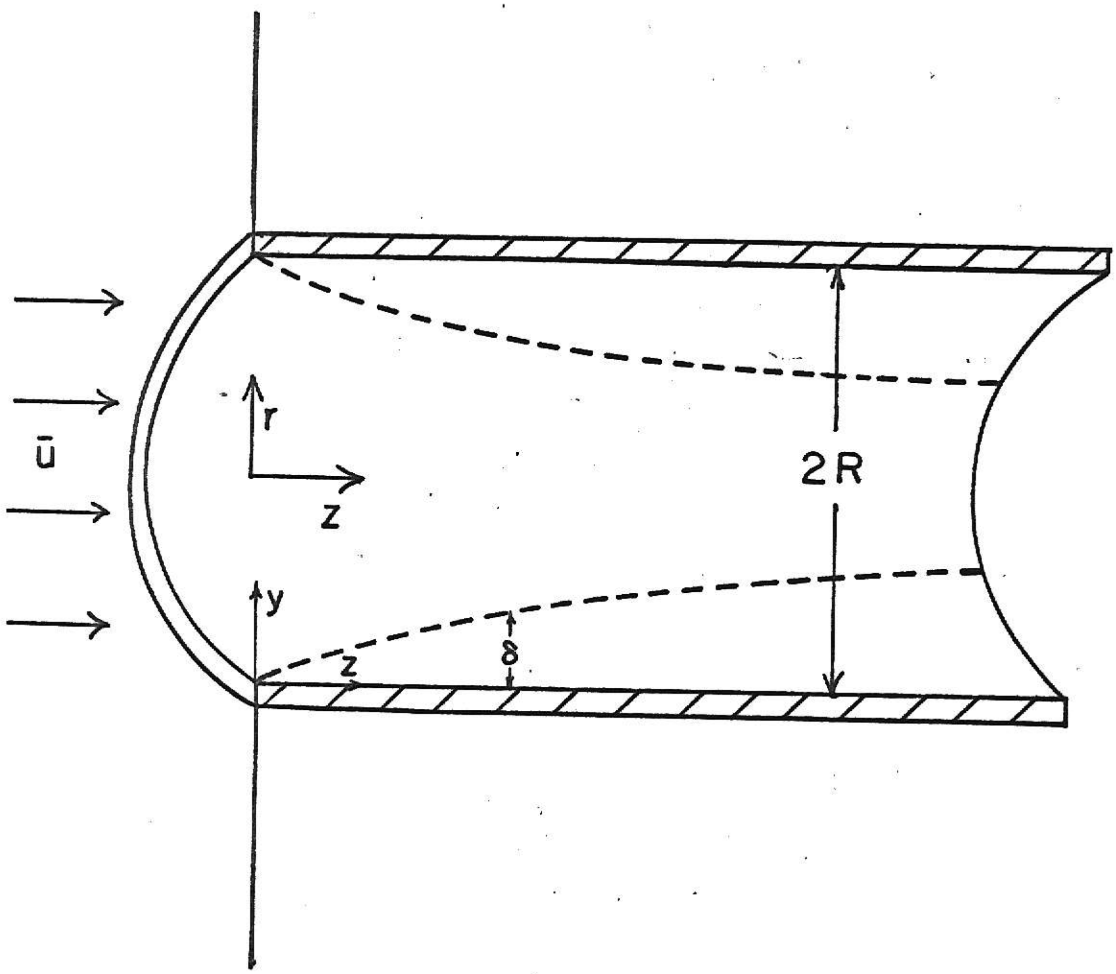


Fig. 10. Coordinate system of the entrance flow in a pipe.

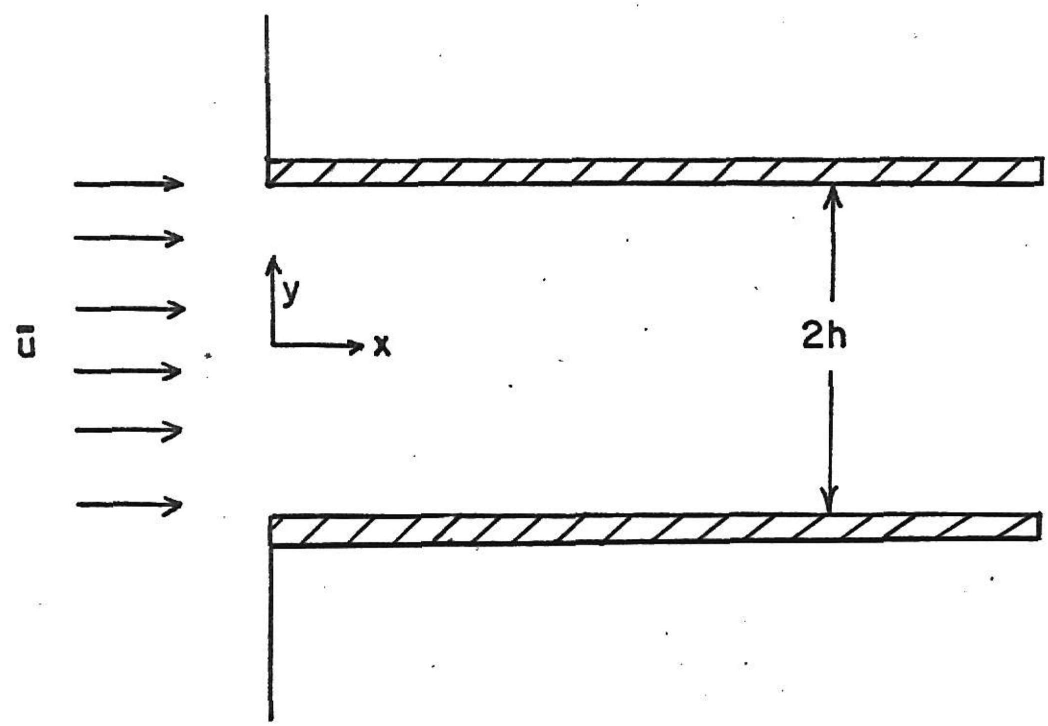


Fig. 11. Coordinate system of the entrance flow in a channel.

ρ is density of the fluid, $\underline{\nabla}$ is the gradient operator, $\underline{V}$ is the velocity vector.

In order to obtain velocity distributions from the equations of continuity and motion, we need expressions for various stresses in terms of velocity gradients and fluid properties. For two-dimensional fully developed flow, these expressions were given in Section 2.2. For incompressible Newtonian fluids the rheological equation of state can be written in the general form

$$\tau_{ij} = -\mu \Delta_{ij} \qquad (3)$$

in which the component of the rate of deformation tensor Δ_{ij} has been defined in Chapter 2.

For incompressible non-Newtonian fluids the empirical expressions in complex geometries are (42):

Power law model

$$\underline{\underline{\tau}} = - \; m \; |\phi|^{n-1} \; \underline{\underline{\Delta}} \qquad (4)$$

Bingham model

$$\underline{\underline{\tau}} = -\left(\mu_o + \frac{\tau_o}{|\phi|}\right)\underline{\underline{\Delta}} \quad \text{for } \frac{1}{2}\left(\underline{\underline{\tau}}:\underline{\underline{\tau}}\right) > \tau_0^2 \qquad (5)$$

$$\underline{\underline{\Delta}} = 0, \qquad \text{for } \frac{1}{2}\left(\underline{\underline{\tau}}:\underline{\underline{\tau}}\right) < \tau_o^2 \qquad (6)$$

where ϕ can be expressed in rectangular and cylindrical coordinates as follows (42):

Rectangular coordinates:

$$\phi^2 = \frac{1}{2}\left(\underline{\underline{\Delta}}:\underline{\underline{\Delta}}\right) = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right]$$

$$+ \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)^2 + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 \quad (7)$$

Cylindrical coordinates:

$$\phi^2 = \frac{1}{2}\left(\underline{\underline{\Delta}} : \underline{\underline{\Delta}}\right) = 2\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2$$

$$+ \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 \quad (8)$$

In obtaining solutions of the equations of motion for entrance region flow, the following assumptions have been made to simplify the problem by investigators in this field.

(1) The velocity distributions are independent of time, i.e., a steady state flow.

(2) The flow is laminar.

(3) The fluids are assumed to have constant physical properties, i.e., the flow is incompressible and rheological parameters are constant.

(4) The flow is isothermal and there is no mass transfer in the flow system.

(5) The flow is two dimensional in case of flat ducts or channels and axisymmetric in case of circular tubes or cylindrical annuli.

(6) The velocity at the tube entrance is uniform. Since the solutions of the equations of motion applicable to the flow from a large reservoir to a conduit entrance with a square cut are not available (5), this assumption is necessary.

The governing equations, Eqs. (1) and (2) can be greatly simplified under the conditions of (1) through (5).

Rectangular coordinates:

continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{9}$$

motion:

(x-component) $$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) \tag{10}$$

(y-component) $$\rho\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) \tag{11}$$

Cylindrical coordinates:

continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \tag{12}$$

motion:

(r-component) $$\rho\left(v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z}\right) \tag{13}$$

(z-component) $$\rho\left(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}\right) \tag{14}$$

The expression of   becomes

Rectangular coordinates:

$$\phi^2 = \frac{1}{2}(\underline{\underline{\Delta}}:\underline{\underline{\Delta}}) = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2\right] + \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)^2 \tag{15}$$

Cylindrical coordinates:

$$\phi^2 = \frac{1}{2}(\underline{\underline{\Delta}}:\underline{\underline{\Delta}}) = 2\left(\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \left(\frac{\partial v_z}{\partial r}\right)^2 \tag{16}$$

The component of the rate of deformation tensor $\underline{\underline{\Delta}}$ also become

Rectangular coordinates

$$\Delta_{xx} = 2\frac{\partial v_x}{\partial x} \tag{17}$$

$$\Delta_{yy} = 2\frac{\partial v_y}{\partial y} \tag{18}$$

$$\Delta_{xy} = \Delta_{yx} = \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{19}$$

Cylindrical coordinates

$$\Delta_{rr} = 2\frac{\partial v_r}{\partial r} \tag{20}$$

$$\Delta_{zz} = 2\frac{\partial v_z}{\partial z} \tag{21}$$

$$\Delta_{zr} = \Delta_{rz} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \tag{22}$$

Making use of the above expressions for Δ_{ij} we have the following expressions of shear stress in terms of velocity gradients and fluid parameters.

Power law model:

Rectangular coordinates:

$$\tau_{xx} = -2m\,|\phi|^{n-1}\frac{\partial v_x}{\partial x} \tag{23}$$

$$\tau_{yy} = -2m\,|\phi|^{n-1}\frac{\partial v_y}{\partial y} \tag{24}$$

$$\tau_{xy} = \tau_{yx} = -m\,|\phi|^{n-1}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \tag{25}$$

Cylindrical coordinates:

$$\tau_{rr} = -2m\,|\phi|^{n-1}\frac{\partial v_r}{\partial r} \tag{26}$$

$$\tau_{zz} = -2m\,|\phi|^{n-1}\frac{\partial v_z}{\partial z} \tag{27}$$

$$\tau_{rz} = \tau_{zr} = -m\,|\phi|^{n-1}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \tag{28}$$

Bingham model:

Rectangular coordinates:

$$\tau_{xx} = -2(\mu_o + \frac{\tau_o}{|\phi|})\frac{\partial v_x}{\partial x} \tag{29}$$

$$\tau_{yy} = -2(\mu_o + \frac{\tau_o}{|\phi|})\frac{v_y}{y} \tag{30}$$

$$\tau_{yx} = \tau_{xy} = -(\mu_o + \frac{\tau_o}{|\phi|})(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}) \tag{31}$$

Cylindrical coordinates:

$$\tau_{rr} = -2(\mu_o + \frac{\tau_o}{|\phi|})(\frac{\partial v_r}{\partial r}) \tag{32}$$

$$\tau_{zz} = -2(\mu_o + \frac{\tau_o}{|\phi|})(\frac{\partial v_z}{\partial z}) \tag{33}$$

$$\tau_{rz} = \tau_{zr} = -(\mu_o + \frac{\tau_o}{|\phi|})(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}) \tag{34}$$

The desired governing equations under the assumptions of (1) through (5) can be obtained if the corresponding expressions of shear stresses in Eqs. (23) through (34) are substituted into Eqs. (10),(11), (13), and (14).

The boundary conditions which have often been used in solving the differential equations of motion are

(1) The velocity components are zero at the solid wall.

(2) At the center line of the conduit in general and in the region beyond the edge of the boundary layer (or in the plug flow region for a fluid with a yield stress), the velocity derivatives in the direction perpendicular to the flow direction are zero.

(3) The velocity at the conduit entrance is uniform.

(4) Far from the entry, the velocity profile becomes fully developed.

In general, the attempt to find exact solutions of Eqs. (10), (11), (13) and (14) is confronted with great mathematical difficulties, primarily due to the nonlinearity of the convective terms. The superposition principle which applied so well in linear partial differential equations, is thus excluded in this case. Except for certain particular flow systems in which non-linear terms vanish automatically or potential flow outside of the boundary layer is restricted to a certain parttern, exact solutions are very difficult to obtain.

To obtain an approximate solution of the original governing differential equations (equations of motion) of the entrance region flow, the boundary layer model has been developed to simplify the original differential equations. The boundary layer model has been successfully used to solve many other fluid flow problems. However, the resulting governing equations for entrance region flow based on the boundary layer concept are usually not sufficiently simple for exact analytical solutions. Thus various approximation methods of solution have been employed to solve the entrance region flow problem. These methods will be discussed in detail for non-Newtonian fluids in the later chapters.

3.2. DERIVATION OF THE NON-NEWTONIAN BOUNDARY LAYER EQUTIONS

The boundary layer equations for a Newtonian fluid were derived by Schlichting (45) based on Prantl's boundary layer model and a consideration of approximate orders of magnitude. The mathematical simplification achieved in this derivation is considerable. The equation of motion in the direction perpendicular to flow direction is completely dropped out and the viscous force in the flow direction is also dropped out as will be shown later. Schowalter (34) applied the same order of magnitude argument to obtain the boundary layer equation for the power law fluid. Michiyochi et al. (29) also used the same technique to derive the Bingham boundary layer equations which are different from the results obtained here. This will be discussed in Section 3, Chapter 5.

We shall derive the boundary layer equations for the power law and Bingham fluids using an approach different from that discussed above. It is, however, still based on the fundamental concept of the boundary layer theory. The results for power law fluids are identical to those obtained in Schowalter's work on steady two-dimensional flow of the power law fluid and the boundary layer equations for the Bingham fluid obtained in this section have never appeared in the literature. To show the approach used in this section, Newtonian fluids will be considered first.

3.2-1. Newtonian Fluids

Consider the two dimensional flow shown in Fig. 12. Under the basic assumptions of the boundary layer model, the thickness of the boundary layer δ_L is very small compared with some characteristic length L of the solid body adjoint to the boundary layer. In other words, the flow is at a sufficiently large Reynolds number so that the viscous forces are of the same order of magnitude as the inertia forces in the immediate neighborhood of the surface of the body. Since

$$\frac{\text{Inertial force}}{\text{Viscous force}} = \frac{\rho v_x \partial v_x/\partial x}{\mu \partial^2 v_x/\partial y^2} \sim \frac{\rho U_\infty^2/L}{\mu U_\infty/\delta_L^2} = \left(\frac{\rho UL}{\mu}\right)\left(\frac{\delta_L}{L}\right)^2 = N_{Re}\left(\frac{\delta_L}{L}\right)^2$$

we must have

$$\delta_L \sim L/N_{Re}^{1/2} \tag{35}$$

where U_∞ is some characteristic velocity and

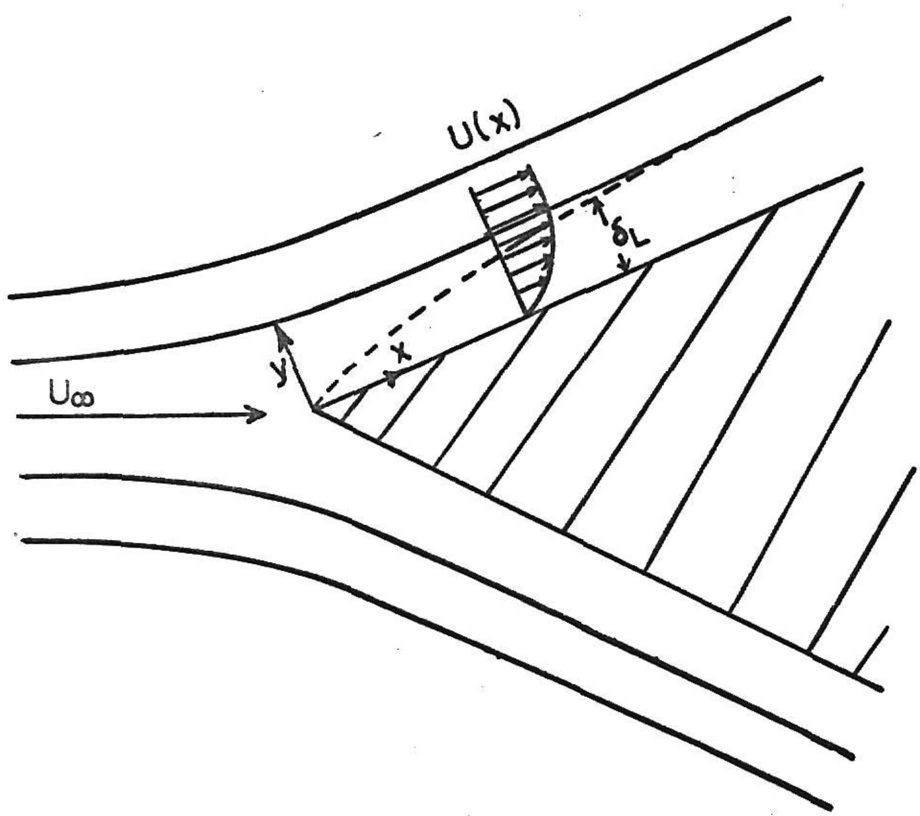


Fig. 12. Boundary layer flow along a wall.

$$N_{Re} = \frac{\rho U_\infty L}{\mu}$$

In addition, it is also considerd that the order of magnitude of each term in the two-dimensional continuity equation (Eq. (9)) is the same; that is,

$$\frac{\partial v_x/\partial x}{\partial v_y/\partial y} \sim \frac{U_\infty/L}{v_y/\delta_L} \sim 1 \tag{36}$$

Hence the velocity component v_y perpendicular to the solid boundary is of

$$v_y \sim U_\infty \delta_L/L = U_\infty/N_{Re}^{1/2} \tag{37}$$

Utilizing the results obtained above, the following dimensionless quantities can be introduced.

$$X = x/L, \quad Y = y/\delta_L = N_{Re}^{1/2} y/L$$

$$U = v_x/U_\infty, \quad V = N_{Re}\ v_y/U_\infty \tag{38}$$

$$P = p/\rho U_\infty^2$$

It follows from Eqs. (35) and (37) that the dimensionless quantities, Y, and V are of the order of unity.

The equations of continuity and motion in x and y components under the assumptions (1) through (5) given in Section 3.1 can be written in the following form:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{39}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{40}$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \tag{41}$$

Substituting the dimensionless quantities in Eq. (38) into Eqs. (39), (40), and (41), we obtain after rearrangement

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{42}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{N_{Re}}\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \tag{43}$$

$$\frac{1}{N_{Re}}\left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) = -\frac{\partial P}{\partial Y} + \frac{1}{N_{Re}^2}\frac{\partial^2 V}{\partial X^2} + \frac{1}{N_{Re}}\frac{\partial^2 V}{\partial Y^2} \tag{44}$$

The assumption that all the dimensionless terms are of the same order of magnitude for a large Reynolds number N_{Re}, gives rise to the limiting expressions of the equations of continuity and motion as follows (44):

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{45}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \tag{46}$$

$$0 = -\frac{\partial P}{\partial Y} \tag{47}$$

Equations (45) through (47) are the Newtonian boundary layer equations in dimentionless form under the assumptions (1) through (5) given in Section 3.1. The corresponding dimensional boundary layer equations are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{48}$$

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{p}{x} + \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2} \tag{49}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{50}$$

Applying the same approach we obtain the boundary layer equations in cylindrical coordinates in the form of

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \tag{51}$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) \tag{52}$$

$$0 = - \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{53}$$

3.2-2. Power Law Fluids

For a power law fluid the same method can be employed if the following form of the Reynolds number is used.

$$N_{Re} = \frac{L^n U_\infty^{2-n} \rho}{m} \tag{54}$$

where n and m are parameters of the power law fluid model defined in Chapter 2. If the condition that the inertia forces are the same order of magnitude as the viscous forces in the boundary layer is satisfied, we can write

$$\frac{\text{Inertial force}}{\text{Viscous force}} = \frac{\rho v_x \partial v_x / \partial x}{m \frac{\partial}{\partial y}\left(\phi^{\,n-1} \frac{\partial v_x}{\partial y}\right)} \sim \frac{\rho U_\infty / L}{m U_\infty^n / \delta_L^{n+1}} = N_{Re}\left(\frac{\delta_L}{L}\right)^{n+1} \sim 1$$

or

$$\delta_L = \frac{L}{N_{Re}^{\frac{1}{n+1}}} \tag{55}$$

Consequently, the condition that each term in the continuity equation has the same order of magnitude leads to the conclusion,

$$v_y \sim \frac{U_\infty}{L}\delta_L = \frac{U_\infty}{N_{Re}^{1/n+1}} \tag{56}$$

The dimensionless quantities Y and V can then be defined as

$$Y = \frac{y}{\delta_L} = N_{Re}^{1/n+1}\, y/L, \qquad V = N_{Re}^{1/n+1}\, v_y/U_\infty \tag{57}$$

Introducing the corresponding dimensionless quantities for the power law fluid into Eq. (15) yields

$$\phi = \frac{U_\infty}{L}\left\{2\left[\left(\frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2\right] + \left(\frac{1}{N_{Re}^{1/n+1}} \frac{\partial V}{\partial X} + N_{Re}^{1/n+1}\frac{\partial U}{\partial Y}\right)^2\right\}^{\frac{1}{2}} \tag{58}$$

When N_{Re} is very large, Eq. (58) becomes

$$\phi = \frac{U_\infty}{L} N_{Re}^{1/n+1} \frac{\partial U}{\partial Y} \tag{59}$$

Equations (10) and (11) now can be rewritten in dimensionless form using the quantities U, X, and P in Eq. (38), Y and V in Eq. (57) and shear stress expressions in Eqs. (23), (24) and (25).

x-component:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{N_{Re}^{2/n+1}}\left\{2\frac{\partial}{\partial X}\left(\left|\frac{\partial U}{\partial Y}\right|^{n-1}\frac{\partial U}{\partial X}\right) + \frac{\partial}{\partial Y}\left[\left|\frac{\partial U}{\partial Y}\right|^{n-1}\left(\frac{\partial V}{\partial X} + N_{Re}^{2/n+1}\frac{\partial U}{\partial Y}\right)\right]\right\} \tag{60}$$

y-component:

$$\frac{1}{N_{Re}^{2/n+1}}\left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) = -\frac{\partial P}{\partial Y} + \frac{1}{N_{Re}^{2/n+1}}\left\{\frac{\partial}{\partial X}\left[\left|\frac{\partial U}{\partial Y}\right|^{n-1}\left(\frac{1}{N_{Re}^{2/n+1}}\frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y}\right)\right] + 2\frac{\partial}{\partial Y}\left(\left|\frac{\partial U}{\partial Y}\right|^{n-1}\frac{\partial V}{\partial Y}\right)\right\} \tag{61}$$

when N_{Re} becomes very large, we have

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{\partial}{\partial Y}\left[\left|\frac{\partial U}{\partial Y}\right|^{n-1}\left(\frac{\partial U}{\partial Y}\right)\right] \tag{62}$$

$$0 = -\frac{\partial P}{\partial Y} \tag{63}$$

Equations (62) and (63) are the dimensionless boundary layer equations for the power law fluid. In dimensional form these can be written as

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{m}{\rho}\frac{\partial}{\partial y}\left[\left|\frac{\partial v_x}{\partial y}\right|^{n-1}\left(\frac{\partial v_x}{\partial y}\right)\right] \tag{64}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{65}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{66}$$

In cylindrical coordinates the boundary layer equations become

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0 \tag{67}$$

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{m}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left[r\left|\frac{\partial v_z}{\partial r}\right|^{n-1}\left(\frac{\partial v_z}{\partial r}\right)\right] \tag{68}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{69}$$

The two-dimensional boundary layer equations, Eqs. (64), (65), and (66), are identical to those obtained by Schowalter (34) which have been widely accepted. It should be noted that the applicability of Eqs. (64) through (68) is determined by the criterion that $N_{Re} \sim (L/\delta_L)^{n+1} \gg 1$.

3.2-3. Bingham Fluids

Procedures for deriving the boundary layer equations for the Bingham fluid are essentially the same as those given in sections 3.2-1 and 3.2-2. Let us consider a Bingham fluid entering a circular tube from a large reservoir. The equations of motion and continuity in cylindrical coordinates under the assumptions (1) through (5) in section 3.1 have been derived in the same section. The Reynolds number for the Bingham model is similar to that for the Newtonian fluid except μ_o is used in place of μ as follows

$$N_{Re} = \frac{L U_\infty \rho}{\mu_o} \tag{70}$$

Applying the same approach described before we first introduce the following non-dimensional quantities

$$Z = z/L, \qquad R = N_{Re}^{\ 1/2} r/L$$
$$U = v_z/U_\infty, \quad V = N_{Re}^{\ 1/2} v_r/U_\infty \tag{71}$$
$$P = p/\rho U_\infty^2$$

The quantity ϕ becomes

$$\phi = \frac{U_\infty}{L}\left\{2\left[\left(\frac{\partial V}{\partial R}\right)^2 + \left(\frac{V}{R}\right)^2 + \left(\frac{\partial U}{\partial Z}\right)^2\right] + \frac{1}{N_{Re}}\left(\frac{\partial V}{\partial Z}\right)^2 + 2\left(\frac{\partial V}{\partial Z}\right)\left(\frac{\partial U}{\partial R}\right) + N_{Re}\left(\frac{\partial U}{\partial R}\right)^2\right\}^{1/2} \tag{72}$$

When N_{Re} is very large this reduces to

$$\phi = \frac{U_\infty}{L} N_{Re}^{\ 1/2} \frac{\partial U}{\partial R} \tag{73}$$

Substituting the dimensionless quantities into Eqs. (13) and (14) we obtain

r-component:

$$\frac{1}{N_{Re}}\left(V\frac{\partial V}{\partial R} + U\frac{\partial V}{\partial Z}\right) = -\frac{\partial P}{\partial R} + \frac{1}{N_{Re}}\left\{\frac{2}{R}\frac{\partial}{\partial R}\left[R\left(1 + \frac{\tau_o^*}{N_{Re}^{\ 1/2}}\frac{1}{\left|\frac{\partial U}{\partial R}\right|}\right)\frac{\partial V}{\partial R}\right]\right.$$
$$\left. + \frac{\partial}{\partial Z}\left[\left(1 + \frac{\tau_o^*}{N_{Re}^{\ 1/2}}\frac{1}{\left|\frac{\partial U}{\partial R}\right|}\right)\left(\frac{1}{N_{Re}}\frac{\partial V}{\partial Z} + \frac{\partial U}{\partial R}\right)\right]\right\} \tag{74}$$

z-component:

$$V\frac{\partial U}{\partial R} + U\frac{\partial U}{\partial Z} = -\frac{\partial P}{\partial Z} + \frac{1}{N_{Re}}\left\{\frac{1}{R}\frac{\partial}{\partial R}\left(1 + \frac{\tau_o^*}{N_{Re}^{\ 1/2}\left|\frac{\partial U}{\partial R}\right|}\right)\left(\frac{\partial V}{\partial Z} + N_{Re}\frac{\partial U}{\partial R}\right)\right.$$
$$\left. + 2\frac{\partial}{\partial Z}\left[\left(1 + \frac{\tau_o^*}{N_{Re}^{\ 1/2}\left|\frac{\partial U}{\partial R}\right|}\right)\frac{\partial U}{\partial Z}\right]\right\} \tag{75}$$

For a large N_{Re}, Eqs. (74) and (75) becomes

$$0 = -\frac{\partial P}{\partial R} \tag{76}$$

$$V\frac{\partial U}{\partial R} + U\frac{\partial U}{\partial Z} = -\frac{\partial P}{\partial Z} + \frac{1}{R}\frac{\partial}{\partial R}\left[\left(1 + \frac{\tau_o^*}{N_{Re}^{\ 1/2}\left|\frac{\partial U}{\partial R}\right|}\right)\frac{\partial U}{\partial R}\right] \tag{77}$$

The corresponding dimensional equations are

$$\frac{\partial p}{\partial y} = 0 \tag{78}$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{r\rho}\frac{\partial}{\partial r}\left[-r(\mu_o + \frac{\tau_o}{\left|\frac{\partial v_z}{\partial r}\right|})\frac{\partial v_z}{\partial r}\right] \tag{79}$$

The equation of continuity is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{80}$$

Since

$$\tau_{rz} \sim -(\mu_o + \frac{\tau_o}{\left|\frac{\partial v_z}{\partial r}\right|})\frac{\partial v_z}{\partial r} \tag{81}$$

Eq. (79) becomes

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{r\rho}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{82}$$

Let us consider the case in the rectangular coordinates. Applying the same treatment as shown above, we will come up with the following boundary layer equations in dimensional form:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}(\mu_o \frac{\partial v_x}{\partial y} + \tau_o \frac{\frac{\partial v_x}{\partial y}}{\left|\frac{\partial v_x}{\partial y}\right|}) \tag{83}$$

$$\frac{\partial p}{\partial y} = 0 \tag{84}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{85}$$

It should be noted that the applicability of the boundary layer equations for the Bingham fluids derived above depends on the conditions

$$N_{Re} \sim (L/\delta_L)^2 >> 1$$

and

$$\tau_o^* / N_{Re}^{1/2} \sim 1$$

CHARPTER 4

ENTRANCE REGION FLOW OF POWER LAW FLUIDS

4.1 INTRODUCTION

This chapter presents the various methods available in literature for solving the governing equations, based on the boundary layer model, of the entrance region flow in a circular pipe using the cylindrical coordinate system as shown in Fig. 10. A power law fluid with uniform velocity $\bar{u}$ enters the pipe at z =0 and flows horizontally in the z direction. Under the given assumptions and with the known boundary conditions (see Section 3.1), we wish to obtain the velocity and pressure fields in the entrance region and the entrance length required to attain the fully developed flow.

Four major methods have been developed, namely, the momentum integral method (2, 30), the variational method (39), the matching method (7, 8) and the finite differnce method (6). In the following sections these methods will be introduced and their results compared.

4.2 MOMENTUM INTEGRAL METHOD

The basic momentum integral method for the entrance region flow was developed by Schiller (31) for Newtonian flow in a circular pipe. He assumed a parabolic velocity profile

$$v_z = U(z)\ (2\,\frac{y}{\delta} - \frac{y^2}{\delta^2}\ ),\qquad 0 \leq y \leq \delta$$

in the boundary layer with a thickness δ (z) and a uniform velocity profile

$$v_z = U(z)$$

outside the boundary layer ($y \geq \delta$). Here v_z is the velocity in the flow direction, U is the center core velocity and y is the radial coordinate measured from the wall, or

$$y = R - r$$

Those assumed forms of the velocity profile are used in the momentum integral equation to obtain the relationship of U or to z. Note that the assumed form of the velocity profile should satisfy all the boundary conditions shown in Section 3.1.

The basic momentum integral method (31) was first extended to the power law fluid by Bogue (2) using the cubic boundary layer velocity profile. Later Pawlek and Tien (30) used the same approach but with a fourth degree polynomial as the boundary layer velocity profile. The arbitrary constants occuring in the assumed profile are determined by the constancy of volume rate of flow and kinetic energy at the entrance length calculated from the assumed velocity profile or from the fully developed one. Because of its simplicity and typical procedure of the method, Pawlek and Tien's work will be introduced in detail in this section.

The equations of continuity and motion are

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial z}{\partial z} = 0 \qquad (1)$$

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{dp}{dz} + \frac{m}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\left|\frac{\partial v_z}{\partial r}\right|^{n-1}\frac{\partial v_z}{\partial r}\right) \qquad (2)$$

According to the boundary layer assumption, the flow outside the boundary layer can be assumed as

$$-\frac{1}{\rho}\frac{dp}{dz} = U\frac{dU}{dz} \qquad (3)$$

where U is the center core velocity and is a function of z only. It follows from Eq. (1) that the velocity in the r direction can be written in the form

$$v_r = -\frac{1}{r}\int_R^r \frac{\partial v_z}{\partial z}\, rdr \qquad (4)$$

Substituting the expressions of Eqs. (3) and (4) into Eq. (2) we have

$$-\frac{1}{r}\int_R^r \frac{\partial v_z}{\partial z} r dr \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = U\frac{dU}{dz} + \frac{m}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r \left|\frac{\partial v_z}{\partial r}\right|^{n-1} \frac{\partial v_z}{\partial r}\right) \quad (5)$$

Multiplying Eq. (5) throughout by rdr and integrating it from the wall ($r=R$) to the edge of boundary layer ($r = R-\delta$) we have

$$\frac{dU}{dz}\int_R^{R-\delta}(U-v_z)rdr + \frac{d}{dz}\int_R^{R-\delta} v_z(U-v_z)rdr$$

$$= \frac{mR}{\rho}\left|\left(\frac{\partial v_z}{\partial r}\right)_{r=R}\right|^{n-1}\left(\frac{\partial v_z}{\partial r}\right)_{r=R} \quad (6)$$

where δ is the boundary layer thickness.

The velocity profile in the boundary layer is assumed to be of a fourth degree polynomial,

$$\frac{v_z}{U} = C_1\left(\frac{y}{\delta}\right) + C_2\left(\frac{y}{\delta}\right)^2 + C_3\left(\frac{y}{\delta}\right)^3 + C_4\left(\frac{y}{\delta}\right)^4 \quad (7)$$

where

$$y = R - r \quad (8)$$

The coefficients C_1, C_2, C_3 and C_4 can be determined from the following boundary conditions,

$$v_z = 0, \qquad \text{at } y = 0 \quad (9)$$

$$v_z = U, \qquad \text{at } y = \delta \quad (10)$$

$$\frac{\partial v_z}{\partial y} = 0, \qquad \text{at } y = \delta, \quad (11)$$

and the compatibility conditions that the volume rate of flow and kinetic energy at entrance length must be constant whether it is calculated from the assumed velocity profile or from the fully developed velocity profile,

$$\frac{v_z}{U} = 1 - \left(1 - \frac{y}{R}\right)^{n+1/n} \quad (12)$$

The compatibility conditions can be written respectively as

$$\int_0^R [C_1(1-\frac{r}{R}) + C_2(1-\frac{r}{R})^2 + C_3(1-\frac{r}{R})^3 + C_4(1-\frac{r}{R})^4] r\,dr$$

$$= \int_0^R [1-(\frac{r}{R})^{\frac{n+1}{n}}] r\,dr \qquad (13)$$

and

$$\int_0^R [C_1(1-\frac{r}{R}) + C_2(1-\frac{r}{R})^2 + C_3(1-\frac{r}{R})^3 + C_4(1-\frac{r}{R})^4]^2 r\,dr$$

$$= \int_0^R [1-(\frac{r}{R})^{\frac{n+1}{n}}]^2 r\,dr \qquad (14)$$

Since Eq. (9) is automatically satisfied by Eq. (7), the coefficients C_i's of Eq. (7) are determined by Eqs. (10), (11), (13) and (14). Combining these equations we obtain the following four simultaneous equations

$$C_1 + C_2 + C_3 + C_4 = 1 \qquad (15)$$

$$C_1 + 2C_2 + 3C_3 + 4C_4 = 0 \qquad (16)$$

$$\frac{C_1}{3} + \frac{C_2}{6} + \frac{C_3}{10} + \frac{C_4}{15} = 1 - \frac{2n}{1+3n} \qquad (17)$$

$$\frac{C_1^2}{12} + \frac{C_2^2}{30} + \frac{C_3^2}{56} + \frac{C_4^2}{90} + \frac{C_1C_3}{15} + \frac{C_1C_4}{21} + \frac{C_2C_3}{21} + \frac{C_2C_4}{28} + \frac{C_3C_4}{36}$$

$$= \frac{1}{2} - \frac{2n}{3n+1} + \frac{n}{4n+2} \qquad (18)$$

Solving for C_i's for various values of the power law parameter n 1 from Eqs. (15) through (18) yields the results shown in Table 4. Note that for n=1, the velocity profile reduces to the parabolic form used in the basic momentum integral method.

The integration in Eq. (6) can be performed if the assumed velocity profile (Eq. (7)) is used. Eq. (6) can be integrated in the form

Table 4

The Coefficients C_i for Assumed Velocity Profile

n	C_1	C_2	C_3	C_4
1.00000	2.00000	-1.00000	0	0
0.80000	2.23813	-1.30832	-0.09776	0.16795
0.75000	2.32052	-1.44875	-0.06406	0.19229
0.60000	2.65803	-2.14727	0.32047	0.16778
0.50000	3.00000	-3.00000	1.00000	0
0.40000	3.50758	-4.44699	2.37124	-0.43183
0.33333	4.00000	-6.00000	4.00000	-1.00000
0.20000	5.79192	-12.45958	11.54342	-3.87575
0.10655	10.38180	-32.77036	38.38518	-15.00000

$$\frac{dU}{dz}\left(K_1 UR\delta + K_2 U\delta^2\right) + \frac{d}{dz}\left(K_3 U^2 R\delta - K_4 U^2\delta^2\right) = \frac{mRC_1^n\, U^n}{\delta^n} \tag{19}$$

where

$$K_1 = 1 - \frac{C_1}{2} - \frac{C_2}{3} - \frac{C_3}{4} - \frac{C_4}{5} \tag{20}$$

$$K_2 = -\frac{1}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \frac{C_4}{6} \tag{21}$$

$$K_3 = \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_1^2}{3} - \frac{C_2^2}{5} - \frac{C_3^2}{7} - \frac{C_4^2}{9}$$

$$- \frac{C_1C_2}{2} - \frac{2C_1C_3}{5} - \frac{C_1C_4}{3} - \frac{C_2C_3}{2} - \frac{2C_2C_4}{7} - \frac{C_3C_4}{4} \tag{22}$$

$$K_4 = \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \frac{C_4}{6} - \frac{C_1^2}{4} - \frac{C_2^2}{6} - \frac{C_3^2}{8} - \frac{C_4^2}{10}$$

$$- \frac{2C_1C_2}{5} - \frac{C_1C_3}{3} - \frac{2C_1C_4}{7} - \frac{2C_2C_3}{7} - \frac{C_2C_4}{4} - \frac{2C_3C_4}{9}$$

If the following dimensionless quantities,

$$U^* = U/\bar{u}, \quad \delta^* = \delta/R, \quad z^* = z/RN_{Re}, \quad N_{Re} = (2R)^n \bar{u}^{2-n}\rho/m$$

are used, Eq. (19) becomes

$$\frac{dU^*}{dz^*}(K_1U^*\delta^* + K_2U^*\delta^{*2}) + \frac{d}{dz^*}(K_3U^{*2}\delta^* - K_4U^{*2}\delta^{*2}) = C_1^nU^{*n}/\delta^{*n} \tag{23}$$

This differential equation determines the relationship between the center core velocity U^*, boundary layer thickness δ^*, and tube distance from the entry z^*. For a steady flow, U^* and δ^* have a certain relationship which can be derived as follows. From the macroscopic mass balance at any section in the pipe, we have the relation

$$(R-\delta)^2 U + \int_{R-\delta}^{R} 2\pi v_z r\,dr = \pi R^2 \bar{u} \tag{24}$$

When the velocity profile Eq. (17) is used in Eq. (24) to carry out the integration, Eq. (24) turns out to be

$$K_2\delta^2 + K_1 R\delta = \frac{1}{2}R^2(1 - \frac{1}{U^*}) \tag{25}$$

or

$$K_2\delta^{*2} + K_1\delta^* = \frac{1}{2}(1 - \frac{1}{U^*}) \tag{26}$$

Solving Eq. (26) for δ^* in terms of U^* we obtain

$$\delta^* = \frac{K_1}{2K_2}\left[-1 + \sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}\ \right] \tag{27}$$

Equation (27) determines the relation between the dimensionless boundary layer thickness δ^* and dimensionless center core velocity U^*. Eliminating the dimensionless boundary layer thickness δ^* between Eqs. (23) and (27) yields

$$\left\{\frac{-K_1\left[1 - \sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}\ \right]}{2U^* C_1 K_2}\right\}^n \left\{U^*\left(\frac{1}{2} - \frac{K_1K_3}{K_2} - \frac{K_4K_1^2}{K_2^2} - \frac{K_4}{K_2}\right)\right.$$
$$+ \frac{K_4}{2K_2} + \frac{1}{2} - \frac{K_1U^*}{K_2}\sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}\left(K_3 + \frac{K_4K_1}{K_2}\right)$$
$$\left. + \frac{1}{2\sqrt{1 + \frac{2K_2}{K_1^2}(1 - \frac{1}{U^*})}}\left(\frac{K_4}{K_1} + \frac{K_4}{K_2}\right)\right\} dU^* = dz^* \tag{28}$$

This can be numerically integrated with the initial condition $U^* = 1$, at $z^*=0$, i.e., the uniform velocity at the entry. The computation terminates when U^* reaches the value of fully developed flow

$$U_e^* = \frac{1+3n}{1+n} \tag{29}$$

Combining Eq. (27) and the solution of Eq. (28) gives the relationship between δ^* and z^*. The computational results from Eq. (28) for n = 1/4, 1/2 and 3/4 are plotted in Fig. 13. The entrance length z_e^* as a function of the parameter n is shown in Fig. 14, the original plot.

The pressure drop, $|\Delta p|$, in the entrance region can be approximated by performing the integration of Eq. (3) from the entry to any cross section in the entrance region.

$$|\Delta p^*| = \frac{1}{2}\rho(\,U^2 - \bar{u}^2\,) \tag{30}$$

or

$$|\Delta p^*| = \frac{|\Delta p|}{\frac{1}{2}\rho\bar{u}^2} = U^{*2} - 1 \tag{31}$$

Since U^* is related to z^*, the pressure drop can be calculated using the numerical results of Eq. (28) as shown in Fig. 15.

The pressure drop from the tube inlet to any cross section in the fully developed region z^* is customarily defined as

$$|\Delta p^*| = |\Delta p_e^*| + |\Delta p_f^*| \tag{32}$$

where $|\Delta p_e^*|$ is the total pressure drop in the entrance region and $|\Delta p_f^*|$ is the pressure drop in the fully developed portion of the tube, $\Delta z_f^* = z^* - z_e^*$,

$$|\Delta p_f^*| = 2^{n+2}\left(\frac{1+3n}{n}\right)^n \Delta z_f^* \tag{33}$$

Eq. (32) can be written as

$$|\Delta p^*| = U_e^{*2} - 1 + 2^{n+2}\left(\frac{1+3n}{n}\right)^n \Delta z_f^* \tag{34}$$

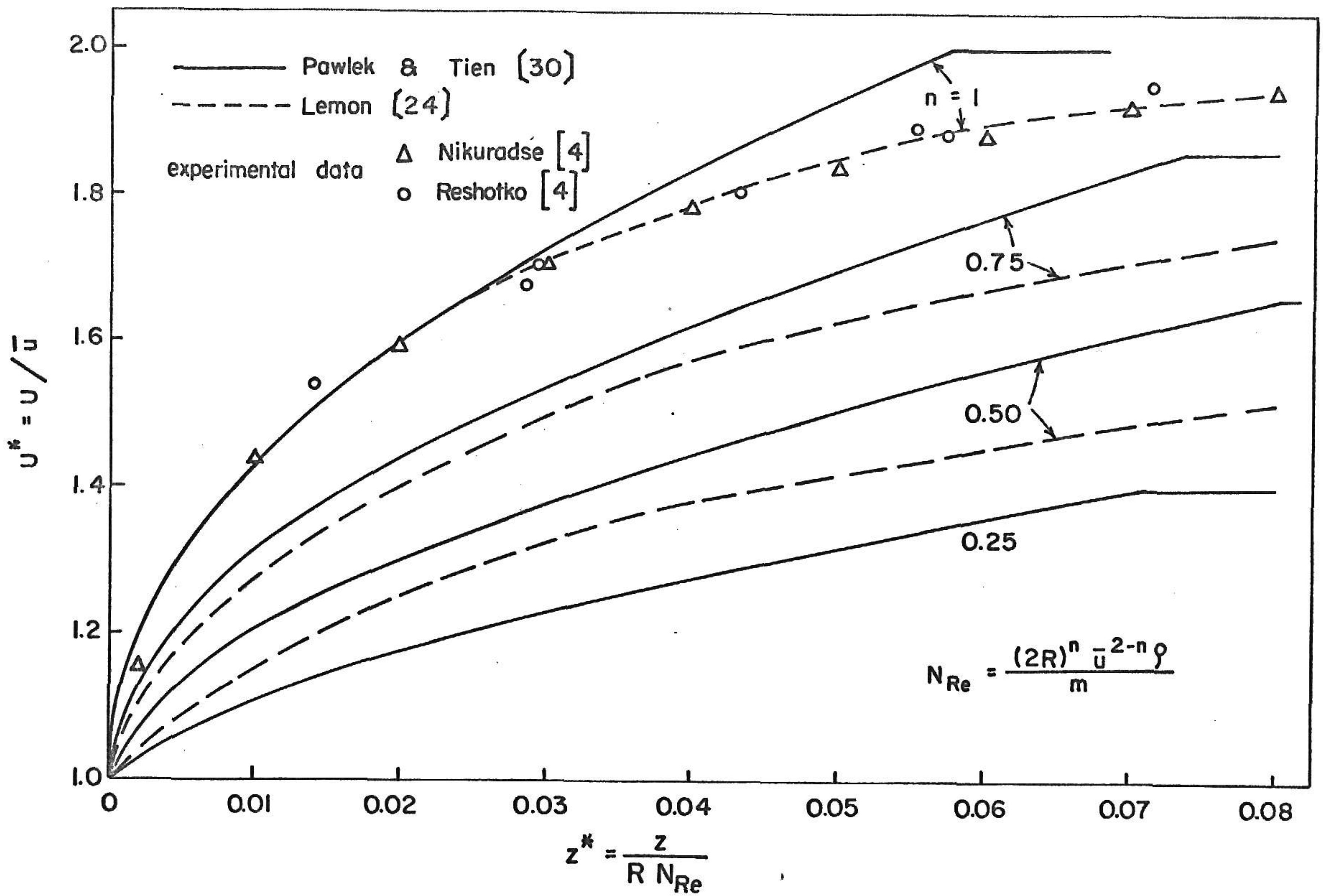


Fig. 13. Center velocity of the power law fluid as a function of tube length.

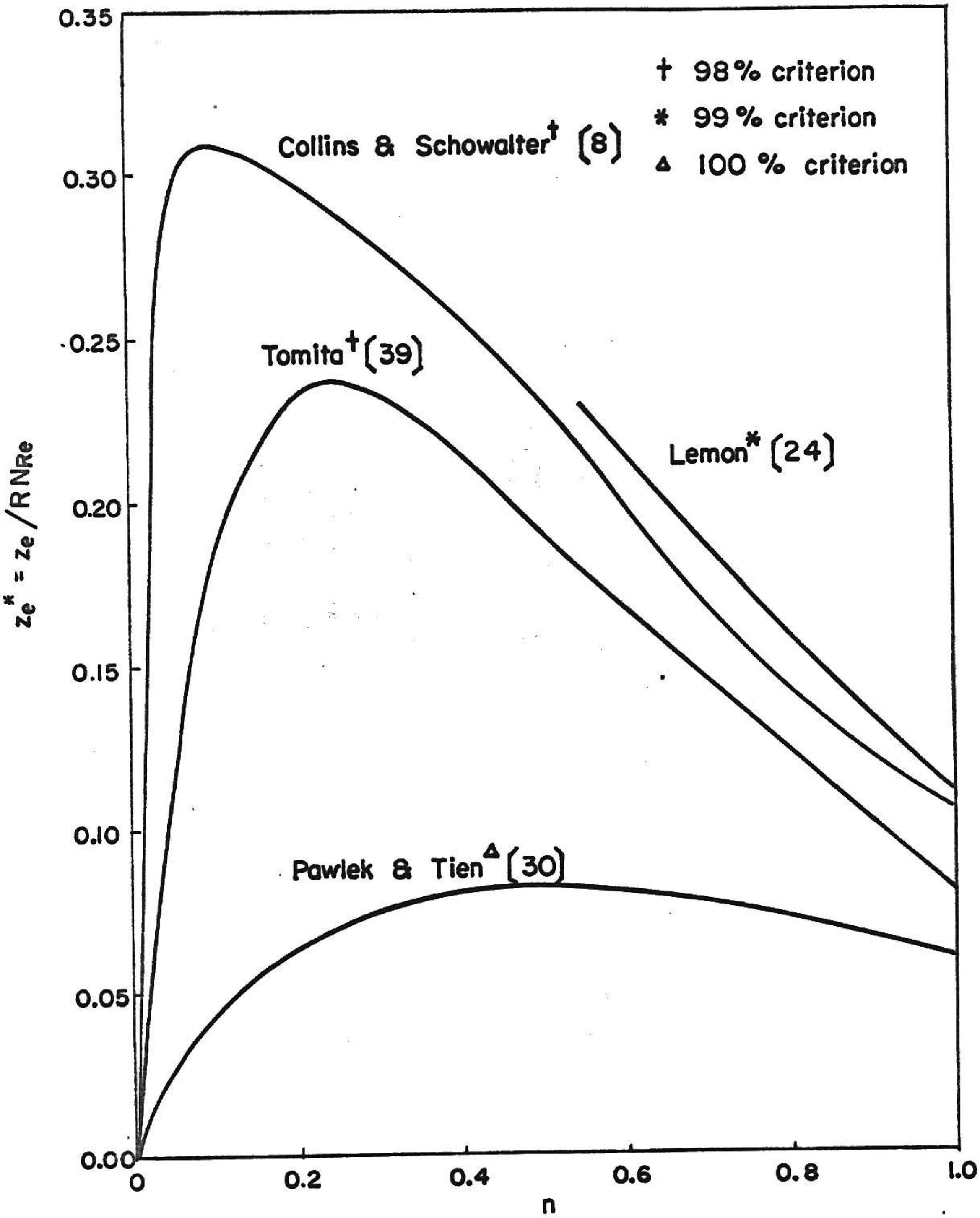


Fig. 14. Circular pipe entrance length as a function of flow behavior parameter.

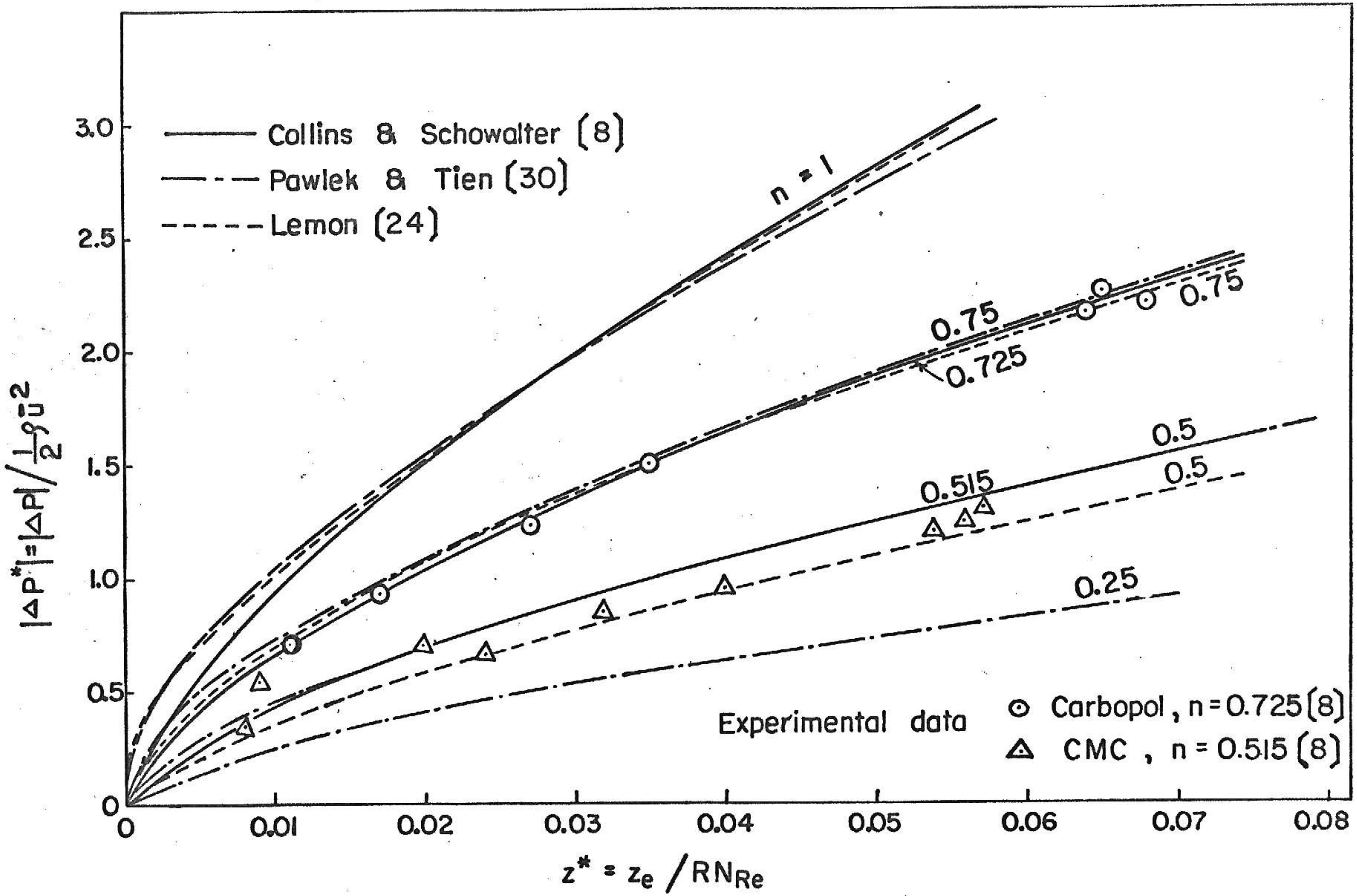


Fig. 15. Pressure drop of the power law fluid in entrance region of a pipe as a function of pipe length.

$$C = U_e^{*2} - 1 - 2^{n+2}\left(\frac{1+3n}{n}\right)^n z_e^* \qquad (36)$$

The correction factors are plotted against n in Fig. 16.

Table 5 presents the numerical values of entrance length, fully developed center velocity and boundary layer thickness, pressure drop, and correction factor. The pressure drop and correction factor are calculated here based on the numerical results of Eq. (28) given by Pawlek and Tien (30). Hence Figs. 15 and 16 do not appear in their work. Also note that Fig. 14 does not appear in their work (30).

Results and Discussion

The center line velocity obtained by the present method is plotted against pipe length z^* in Fig. 13 and is also compared with experimental data (for the Newtonian fluid only) and the results from the finite difference method by Lemmon (24) which is presented in Section 4.5. This figure indicates that the center line velocity from the present work does not approach the fully developed value asymptotically and is somewhat higher that obtained from the finite difference method for the same values of n. On comparing the results from this work with the experimental data for the Newtonian fluid, the agreement is good only in the region near the entry

$$z^* < 0.020 \ \left(z^* = \frac{z}{RN_{Re}}, \quad N_{Re} = \frac{(2R)\bar{u}\rho}{\mu}\right) \qquad (37)$$

The entrance length shown in Fig. 14 for different values of n is considerably shorter compared with those from the matching method, variational method and finite difference method which will be presented in the later sections. This is because the center line velocity obtained by the momentum integral method does not approach the fully developed values asymptotically.

The pressure drop curves shown in Fig. 15 indicate that the results

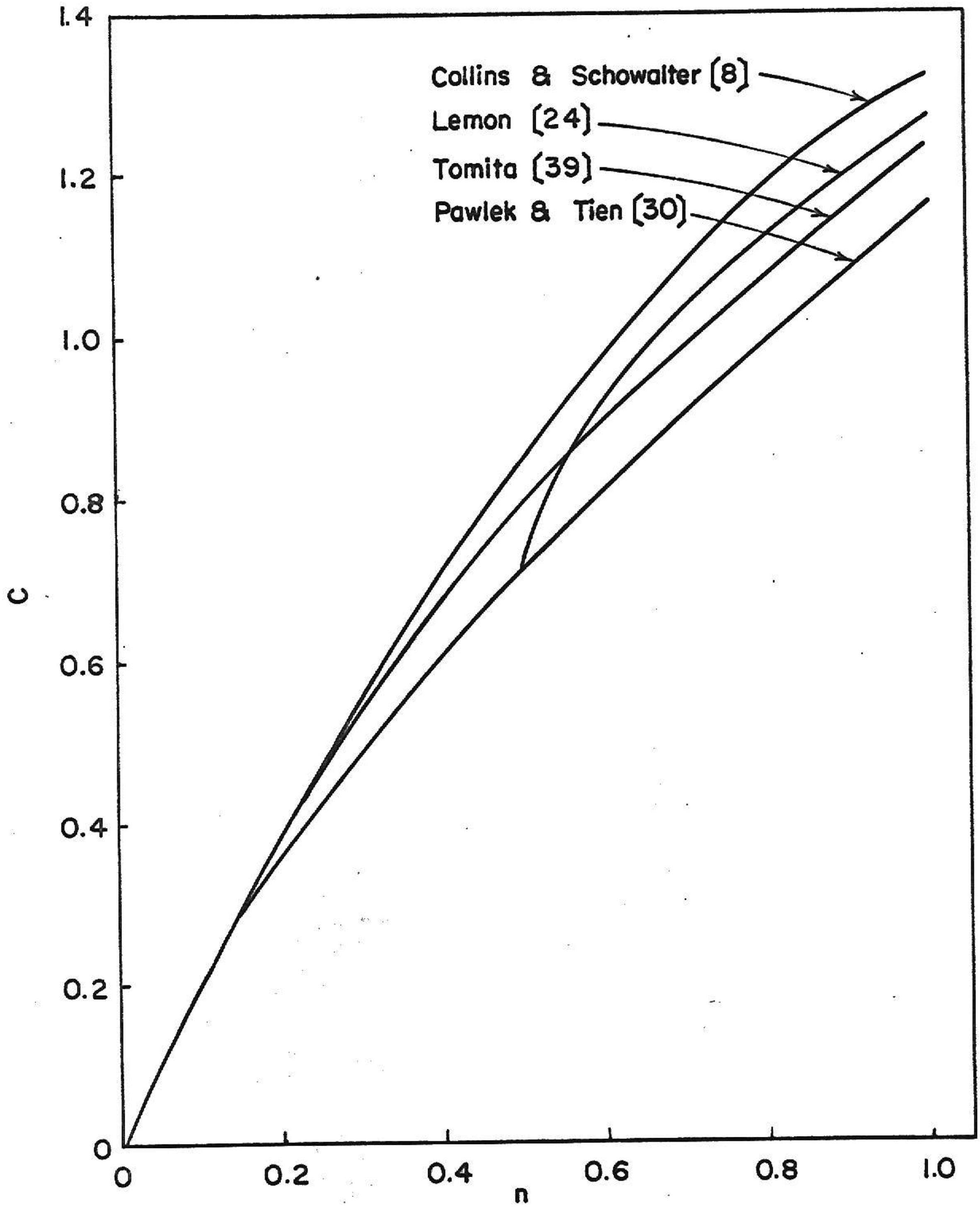


Fig. 16. Correction factor of pressure drop in the entrance region of a circular pipe for the power law fluid as a function of flow behavior parameter.

Table 5. Numerical results of momentum integral method for power law fluid in pipes

Parameter n	Entrance Length z_e^*	Fully developed center velocity U_e^*	Boundary layer thickness δ_e^*	Pressure drop p_e^*	Correction factor C
0.25	0.07104870	1.40	1.00000077	0.960000	0.407
0.50	0.08406192	1.667	1.00003453	1.778889	0.716
0.75	0.07411692	1.857	0.99999505	2.448440	0.952
1.0	0.05754545	2.0	1.0	2.0	1.159

from the present method are in good agreement with those from the matching and finite difference methods and also with the experimental data by Dodge (8) for n=0.725 and n=0.515. The pressure correction factor C plotted against n as shown in Fig. 16 is, in general, consistent with the results obtained from other three methods mentioned in the preceding paragraph.

4.3 VARIATIONAL METHOD

In this section the calculus of variation is used to analyze the entrance region flow problem. The so-called variational method was first introduced to solve entrance region flow problems for Newtonian fluid in circular pipes by Uematsu (40) and later it was extended to the power law fluid by Tomita (39).

The basic notion of the variational method is that instead of solving the equation of motion, the velocity field is determined so that a certain functional of velocity is an extremum.

Consider a power law fluid which flows from a large reservoir into a circular pipe with uniform velocity $\bar{u}$ as shown in Fig. 10. Under the flow conditions described in Section 3.1 the equations of continuity and motion are respectively

$$\frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0 \tag{38}$$

$$v_z\frac{\partial v_z}{\partial z} + v_r\frac{\partial v_r}{\partial r} = -\frac{1}{\rho}\frac{dp}{dz} + \frac{m'^{1/n'}}{\rho}\left[\frac{\partial}{\partial r}\left(\frac{\partial v_z}{\partial r}\right)^{1/n'} + \frac{1}{r}\left(\frac{\partial v_z}{\partial r}\right)^{1/n'}\right] \tag{39}$$

where v_z and v_r are velocity components along the coordinate axes z and r of the cylindrical coordinates respectively, p is the pressure, is the density, m' and n' are the rheological parameters for the power law model of the form

$$\tau_{rz} = -m'^{\frac{1}{n'}}\left|\frac{dv_z}{dr}\right|^{\frac{1-n'}{n'}}\frac{dv_z}{dr}$$

where τ_{rz} is shear stress, and n' and m' are related to n and m' by

$$n' = \frac{1}{n}$$

and

$$m' = m^{1/n}$$

To make the dimensionless form of Eqs. (38) and (39), we let

$$x = \frac{z}{RN_{Re}}, \quad y = r/R, \quad u = v_z/\bar{u}$$
$$v = \frac{v_r N_{Re}}{\bar{u}}, \quad p' = p/\rho\bar{u}^2 \tag{40}$$

where N_{Re} is the Reynolds number for the power law fluid defined by

$$N_{Re} = \frac{(2R)^{\frac{1}{n'}}\,\bar{u}^{2-\frac{1}{n'}}\,\rho}{m'^{\,1/n'}} \tag{41}$$

and R is the radius of the circular tube and $\bar{u}$ is the average velocity. Eqs. (38) and (39) respectively become

$$\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial}{\partial y}(yv) = 0 \tag{42}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp'}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{\frac{1}{n'}} + \frac{1}{y}\left(\frac{\partial u}{\partial y}\right)^{\frac{1}{n'}} \tag{43}$$

The dimensionless boundary conditions at the conduit entry and at the wall are expressed in the following form.

$$u = 1, \quad v = 0, \quad \text{at } x = 0,$$

$$u = 0, \quad v = 0, \quad \text{at } y = 1, \quad x > 0.$$

Combining Eqs. (42) and (43) yields

$$-\frac{u}{y}\frac{\partial}{\partial y}(yv) + v\frac{\partial u}{\partial y} = -\frac{dp'}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{\frac{1}{n'}} + \frac{1}{y}\left(\frac{\partial u}{\partial y}\right)^{\frac{1}{n'}} \tag{44}$$

Equation (44) can be rearranged to yield

$$-\frac{u}{y}\frac{\partial}{\partial y}(yv) + v\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{\frac{1}{n'}} - \frac{1}{y}\left(\frac{\partial u}{\partial y}\right)^{\frac{1}{n'}} = -\frac{dp'}{dx} \tag{45}$$

Note that the right-hand side of Eq. (45) is a function of x only. Therefore, if v is a function of y, Eq. (45) becomes a differential equation with respect to u over the cross section at given x. Letting

$$E[u(y)] = -\frac{u}{y}\frac{\partial}{\partial y}(yv) + v\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{1/n'} - \frac{1}{y}\left(\frac{\partial u}{\partial y}\right)^{1/n'} \tag{46}$$

Eq. (45) becomes

$$E[u(y)] = -\frac{dp'}{dx} \tag{47}$$

Now let us consider the functional $J(u, w)$ of $u(y)$ and $w(y)$ defined as

$$J(u(y), w(y)) = \frac{\int_0^1 wE(u)y\,dy}{\int_0^1 wy\,dy} \tag{48}$$

where $w(y)$ is an auxiliary function. In the following paragraphs it will be shown that the first necessary condition for the functional $J(u(y), w(y))$ to have an extremum corresponds to the differential equation of the form of Eq. (47). The second necessary condition will give the relationship between the auxiliary function $w(y)$ and the function $u(y)$. Hence to solve Eq. (47) corresponds to the variational problem of finding the extremal value of the functional defined by Eq. (48).

As shown in any standard text on the calculus of variations the necessary conditions for the functional $J(u, w)$ to have an extremum can be obtained by perturbing the functional by the amounts represented by the variations δu and δw of u and w. Let us first vary only $w(y)$ and keep $u(y)$ fixed. Choose a certain function δw that satisfies the conditions

$$\delta w(0) = 0, \quad \delta w(1) = 0 \tag{49}$$

If $|\varepsilon|$ is sufficiently small, the function

$$w(y) = \bar{w}(y) + \epsilon\,\delta w(y) \tag{50}$$

lies in the neighborhood of $\bar{w}(y)$ which gives the functional $J(w)$ an extremum.

Substituting Eq. (50) into Eq. (48) gives

$$J(u, \bar{w} + \varepsilon\,\delta w) = \frac{{}_0\int^1 (\bar{w} + \varepsilon\,\delta w)\, E(u) y dy}{{}_0\int^1 (\bar{w} + \varepsilon\,\delta w) y dy} \qquad (51)$$

which is now a function of ε only, that is,

$$J(u, \bar{w} + \varepsilon\,\delta w) = \phi(\varepsilon) \qquad (52)$$

This equation indicates that the necessary condition for the functional J to be at an extremum is equivalent to the derivative of the functional J (or the function ϕ) with respect to the parameter ε at $\varepsilon = 0$ be zero, that is

$$\left.\frac{\partial \phi(\varepsilon)}{\partial \epsilon}\right|_{\varepsilon=0} = 0 \qquad (53)$$

Thus differentiating Eq. (51) with respect to ε gives

$$\frac{\partial \phi(\varepsilon)}{\partial \varepsilon} = \frac{{}_0\int^1 (\bar{w} + \varepsilon\,\delta w) y dy \int_0^1 E(u) y\,\delta w dy - \int_0^1 (\bar{w} + \varepsilon\,\delta w) E(u) y dy \int_0^1 y\,\delta w dy}{\left(\int_0^1 (\bar{w} + \varepsilon\,\delta w) y dy\right)^2}$$

Setting $\varepsilon = 0$ in Eq. (54) yields

$$\frac{{}_0\int^1 \bar{w} y dy \int_0^1 E(u) y\,\delta w dy - \int_0^1 \bar{w}\, E(u) y dy \int_0^1 y\,\delta w dy}{\left(\int_0^1 \bar{w} y dy\right)^2} = 0$$

Knowing that $\int_0^1 \bar{w} ydy$ and $\int_0^1 \bar{w}E(u)ydy$ are constant we have

$$\int_0^1 \left\{ E(u) \int_0^1 \bar{w} ydy - \int_0^1 \bar{w}E(u)ydy \right\} y\delta w dy = 0 \qquad (56)$$

Thus, in order that Eq. (56) may hold for and arbitrary δw the following equation must be satisfied.

$$E(u) = \frac{\int_0^1 \bar{w}\, E(u) y dy}{\int_0^1 \bar{w}\, y dy} = \text{extremal value of } J \qquad (57)$$

On comparing Eq. (57) and Eq.(47) it can be seen that the necessary condition for the functional J to be at an extremum with respect to w gives rise to an equation which is of the same form as Eq. (47). Thus, we can conclude that the pressure gradient equal to the extremal value of the functional J, that is

$$-\frac{dp'}{dx} = \text{the extremal value of } J \tag{58}$$

Considering next the case of $\delta w = 0$, the condition of the variation of J to be zero is

$$\frac{\partial}{\partial \epsilon} J(u + \varepsilon\,\delta u,\ w)\Big|_{\varepsilon=0} = 0 \tag{59}$$

Substituting the expression

$$u = \bar{u} + \varepsilon\,\delta u \tag{60}$$

into Eq. (48) and carrying out the differentiation with respect to ε and setting $\varepsilon = 0$, we obtain

$$\int_0^1 \Big\{ 2w\,\frac{\partial(yv)}{\partial y} + yv\frac{\partial w}{\partial y} + \frac{1-n'}{n'^2}\,y\left(\frac{\partial u}{\partial y}\right)^{\frac{1-2n'}{n'}}\frac{\partial^2 u}{\partial y^2} + \frac{1}{n'}\left(\frac{\partial u}{\partial y}\right)^{\frac{1-n'}{n'}}\frac{\partial w}{\partial y}$$

$$+ \frac{1}{n'}\,y\left(\frac{\partial u}{\partial y}\right)^{\frac{1-n'}{n'}}\frac{\partial^2 w}{\partial y^2}\Big\}\,\delta u\,dy = 0 \tag{61}$$

Here the boundary conditions $\delta u = 0$, $w = 0$, at $y = 1$ are used. For arbitrary δu Eq. (61) becomes

$$2\frac{w}{y}\frac{\partial(yv)}{\partial y} + v\frac{\partial w}{\partial y} + \frac{1}{n'}\left(\frac{\partial u}{\partial y}\right)^{\frac{1-2n'}{n'}}\left\{\frac{1-n'}{n'}\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right\}\frac{\partial w}{\partial y} + \frac{1}{n'}\left(\frac{\partial u}{\partial y}\right)^{\frac{1-n'}{n'}}\frac{\partial^2 w}{\partial y^2}$$

$$= \text{constant} \tag{62}$$

Equation (62) determines the relationship between the auxiliary function w(y) and the function u(y).

Now we shall solve the variational problem by Ritz's method which is briefly outlined in AppendixIII for a general case. First we choose the function u(y) in the form

$$u(y) = \frac{\alpha+3}{\alpha+1}(1 - y^{\alpha+1}) \tag{63}$$

where α is the parameter and a function of x only. The assumed form of the velocity profile, Eq. (63), is similar to the form of the velocity profile in the fully developed region

$$u(y) = \frac{n'+3}{n'+1}(1 - y^{n'+1}) \tag{64}$$

and α has the following limiting values,

$$\alpha(0) \to \infty \qquad \text{and} \qquad \alpha(\infty) \to n' \tag{65}$$

The transverse velocity v is found by using the equation of continuity. The results is

$$v = \left[\frac{y}{(\alpha_o+1)^2}(1 - y^{\alpha_o+1}) + \frac{y^{\alpha_o+2}}{\alpha_o+1}\log_e y\right]\frac{d\alpha_o}{dx} \tag{66}$$

where α_o is the value of α in Eq. (63) which gives the functional J an extremal value. The auxiliary function w(y) in the entrance region can be chosen to be

$$w = (1 - y^{\alpha_o+1})(1 + cy^{n'+1}) \tag{67}$$

where c is function of x only. Equation (67) may be partially justified from the solution of Eq. (62) in the fully developed region. Note that both Eqs. (64) and (67) satisfy the boundary conditions

$$u(1) = 0, \tag{68}$$

$$w(1) = 0. \tag{69}$$

According to Ritz's method, the assumed forms of u(y), w(y) together with v(y) in Eqs. (64), (67) and (66) are substituted into the functional J in Eq. (48). The parameter $\alpha(x)$ and c(x) can be determined by the system of equations

$$\frac{\partial J(\alpha,c)}{\partial c} = 0 \tag{70}$$

$$\frac{\partial J(\alpha,c)}{\partial \alpha} = 0 \tag{71}$$

From Eq. (70) the relationship between α_o and x can be determined. This relationship is used in the assumed form of the velocity profile u(y), Eq.(63), to give the complete velocity profile of u(x, y) in the entrance region. For the details of derivation the reader is referred to the original paper by Tomita (39). The numerical results of the entrance length and pressure correction factor are shown in Figs, 14 and 16.

4.4 MATCHING METHOD

In this section the technique developed by Collins and Schowalter (7, 8) for solving the entrance region flow problem of the power law fluid is introduced. The technique is an extension of the approach employed by Schlichting (32) for the Newtonian flow.

Consider a horizontal parallel plate channel of height 2h as shown in Fig. 11. A rectangular coordinate system with its origin at the entry on the lower plate is used. Near the entry, the effect of viscosity of the fluid is important only in a relatively thin layer near the wall and, therefore, the boundary layer model is applicable. In a region approaching fully developed flow, the velocity profile can be described in terms of perturbations on the fully developed flow. These two solutions are matched at an appropriate value of x in the flow direction.

Solution Near the Entry

The boundary layer equation for the x-component velocity can be written as

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U \frac{dU}{dx} + \frac{m}{\rho} \frac{\partial}{\partial y}\left(\left| \frac{\partial v_x}{\partial y} \right|^{n-1} \frac{\partial v_x}{\partial y} \right) \qquad (78)$$

Since $\partial v_x / \partial y$ is always positive in the lower half region of the channel we have

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U \frac{dU}{dx} + \frac{mn}{\rho} \left(\frac{\partial v_x}{\partial y} \right)^{n-1} \frac{\partial^2 v_x}{\partial y^2} \qquad (79)$$

According to the concept of the boundary layer, Eq. (79) is valid in the boundary layer where the x-component velocity v_x is a function of y and x. Outside of the boundary layer v_x is a function of x only and denoted as U(x).

To reduce the partial differential equation, Eq. (79), to a set of ordinary differential equations, we define a stream function

$$\psi = \bar{u}h \sum_{i=1}^{\infty} \varepsilon^{i} f_{i-1}(\eta) \tag{80}$$

where

$$\eta = y\left[\frac{\rho\,\bar{u}^{2-n}}{m\,x}\right]^{\frac{1}{n+1}} = \frac{y}{\varepsilon h} \tag{81}$$

$$\varepsilon = \left[\frac{m\,x}{h^{n+1}\,\bar{u}^{2-n}\,\rho}\right]^{\frac{1}{n+1}} \tag{82}$$

The velocity components v_x and v_y are expressed in terms of stream functions by

$$v_x = \frac{\partial\psi}{\partial y} \tag{83}$$

$$v_y = -\frac{\partial\psi}{\partial x} \tag{84}$$

Therefore, combining Eqs. (80), (83) and (84) we obtain

$$v_x = \bar{u} \sum_{i=1}^{\infty} \varepsilon^{i-1} f'_{i-1}(\eta) \tag{85}$$

$$v_y = \frac{\bar{u}h}{(n+1)x} \sum_{i=1}^{\infty} \varepsilon^{i}\left[f'_{i-1}\eta - i f_{i-1}\right] \tag{86}$$

in which f'_i is the first derivative of f_i with respect to η. The velocity outside of the boundary layer is a function of x and can be expressed in terms of ε by

$$U(x) = \bar{u} \sum_{i=0}^{\infty} K_i\,\varepsilon^{i} \tag{87}$$

where $K_o = 1$ and K_i for i = 1, 2, ··· are to be determined later.

Combining Eqs. (79), (85), (86) and (87), we obtain a differential equation containing terms of ε raised to different powers $(0,1,2,\cdots)$ multiplied by coefficients which are functions of η alone. When coefficients of like powers of ε are equated, an infinite set of ordinary differential equations is obtained. For the case of zero power of ε we have

$$n(n+1)\, f_o''' + f_o(f_o'')^{2-n} = 0 \tag{88}$$

Boundary conditions are

$$\text{far from the boundary layer } (\eta\to\infty)\text{:} \quad f_o'(\infty) = 1 \tag{89}$$

$$\text{at the wall } (\eta=0)\text{:} \quad f_o'(0) = 0 \tag{90}$$

$$f_o(0) = 0 \tag{91}$$

For the case of the first power of ε we have

$$f_o'f_1' - 2f_1f_o'' - f_of_1'' - n(n+1)(n-1)(f_o'')^{n-2}f_o'''f_1''$$

$$-n(n+1)(f_o'')^{n-1}f_1''' - K_1 = 0 \tag{92}$$

Boundary conditions are

$$\text{far from the boundary layer } (\eta\to\infty)\text{,} \quad f_1'(\infty) = K_1 \tag{93}$$

$$\text{at the wall } (\eta=0)\text{:} \quad f_1(0) = 0 \tag{94}$$

$$f_1'(0) = 0 \tag{95}$$

Repeating the same process, we obtain equations for f_2 through f_5 from the coefficients of ε^2, ε^3, ε^4, and ε^5. These equations and boundary conditions are shown in the appendix of Collins and Schowalter's paper (7).

In order to solve Eqs. (88), (92), etc., we must have numerical values for coefficients K_1, K_2, ... The following method for obtaining K_1, K_2, ... is adopted from the paper by Collins and Schowalter (8) which is simpler than the original method used in the paper (7) mentioned in the preceding paragraph. Since v_x approaches U for large η, the expression in Eq. (85) for large becomes

$$U = \bar{u} \sum_{i=1}^{\infty} \varepsilon^{i-1} f'_{i-1}(\eta)$$

Comparing this equation with Eq. (87) and equating the coefficients of like power of ε we obtain

$$f'_{i-1} = K_{i-1}$$

Integrating this equation with respect to η yields

$$f_{i-1} = K_{i-1}\eta + A_{i-1}, \qquad i = 1, 2, 3, \ldots \tag{96}$$

where A_{i-1}'s are integration constants. From the continuity conditions we have

$$h\bar{u} = \int_0^h v_x dy = \int_0^h \frac{\partial \psi}{\partial y} dy = \psi \Big|_{y=0}^{y=h} \tag{97}$$

Since at $\eta = 0$ (or $y = 0$), $f_i(0) = 0$, it follows from Eq. (80) that

$$\psi \Big|_{y=0} = 0 \tag{98}$$

Therefore, from Eq. (97), we have

$$\psi \Big|_{y=h} = h\bar{u} \tag{99}$$

Combining Eq. (80) for $y = h$ and Eq. (99) and utilizing the expression in Eq. (96) we obtain

$$\sum_{i=1}^{\infty} \varepsilon^{i} \left(K_{i-1}\eta + A_{i-1}\right) = 1 \tag{100}$$

Since

$$\eta = \frac{1}{\varepsilon} \qquad \text{at} \quad y = h,$$

Eq. (100) can be written as

$$\begin{aligned}
&\sum_{i=1}^{\infty} \varepsilon^{i-1} \left(K_{i-1} + \varepsilon A_{i-1}\right) \\
&= K_o + \varepsilon(K_1 + A_o) + \varepsilon^2(K_2 + A_1) + \cdots \\
&= 1 + \sum_{i=1}^{\infty} \varepsilon^{i} \left(K_i + A_{i-1}\right) \\
&= 1
\end{aligned} \tag{101}$$

Equating the coefficients of like powers of ε on both sides of Eq. (101) we have

$$K_i + A_{i-1} = 0, \quad i = 1, 2, \ldots \tag{102}$$

Combining Eqs. (102) and (96) and letting η_1 be the large value of η we have

$$K_i = K_{i-1}\eta_1 - f_{i-1}(\eta_1) \tag{103}$$

Equation (103) shows that K_i can be obtained from the previous values of K_{i-1} and $f_{i-1}(\eta)$.

Solution Approaching Fully Developed Flow

The equation of motion in the region near the fully developed flow may be approximated by

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{mn}{\rho}\left(\frac{\partial v_x}{\partial y}\right)^{n-1}\left(\frac{\partial^2 v_x}{\partial y^2}\right) \tag{104}$$

with the assumptions that $\frac{\partial^2 v_x}{\partial y^2} \gg \frac{\partial^2 v_x}{\partial x^2}$ and $p = p(x)$. Substituting the velocity in the y direction

$$v_y = \int_y^0 \frac{\partial v_x}{\partial x} dy$$

and introducing the dimensionless quantity $Y = 1 - y/h$ into Eq. (104) gives

$$v_x\frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial Y}\int_Y^1 \frac{\partial v_x}{\partial x}dY = -\frac{1}{\rho}\frac{dp}{dx} + \frac{mn}{\rho}\frac{1}{h^{n+1}}\left(\frac{\partial v_x}{\partial Y}\right)^{n-1}\frac{\partial^2 v_x}{\partial Y^2} \tag{105}$$

Differentiating Eq. (105) partially with respect to Y yields

$$v_x\frac{\partial^2 v_x}{\partial x\partial Y} + \left[\int_Y^1 \frac{\partial v_x}{\partial x}\, dY\right]\frac{\partial^2 v_x}{\partial Y^2} = \frac{mn}{\rho h^{n+1}}\frac{\partial}{\partial Y}\left[\left(\frac{\partial v_x}{\partial Y}\right)^{n-1}\left(\frac{\partial^2 v_x}{\partial Y^2}\right)\right] \tag{106}$$

The velocity v_x can be expressed as a perturbation of the fully developed velocity profile as (32)

$$v_x = v_{xf} + u^* \tag{107}$$

where u^* is the perturbation velocity, and v_{xf} the fully developed velocity profile given by

$$v_{xf} = u\left(\frac{2n+1}{n+1}\right)\left(1 - Y^{\frac{n+1}{n}}\right) \tag{108}$$

Substitution of Eqs. (107), (108) into (106) we have

$$\left(\frac{2n+1}{n+1}\right)\left(1 - Y^{\frac{n+1}{n}}\right)\frac{\partial^2 u^*}{\partial X \partial Y} - \left(\frac{2n+1}{n^2}\right) Y^{\frac{1-n}{n}} \int_Y^1 \frac{\partial u^*}{\partial X}\, dY$$

$$= \frac{m(2n+1)^{n-1}}{\bar{u}^{2-n}\, n^n \rho h^{n+1}} \left\{ n^2 Y^{\frac{n-1}{n}} \frac{\partial^3 u^*}{\partial Y^3} + 2n(n-1) Y^{-\frac{1}{n}} \frac{\partial^2 u^*}{\partial Y^2} \right.$$

$$\left. - (n-1)\, Y^{-\left(\frac{n+1}{n}\right)} \frac{\partial u^*}{\partial Y} \right\} \tag{109}$$

where terms containing u^* or its derivative to power greater than one are small and can be neglected. The validity of this approximation was discussed in the original paper (7) and the error was reduced by using a special numerical scheme. Using the method of separation of variables, we set

$$u^* = \bar{u} \sum_{i=1} C_i \exp(-\lambda_i X)\, \varphi_i'(Y) \tag{110}$$

where

$$X = \frac{(1+2n)^{n-1}\, mx}{\bar{u}^{2-n} h^{n+1} n^n \rho} = \frac{(1+2n)^{n-1}}{n^n}\, \varepsilon^{\,n+1} \tag{111}$$

Substituting Eq. (110) into Eq. (109) gives

$$n^2 Y^{\frac{n-1}{n}} \varphi_i''' + 2n(n-1) Y^{-\frac{1}{n}} \varphi_i'''(Y) + (1-n)\, Y^{-\left(\frac{n+1}{n}\right)} \varphi_i''$$

$$+ \left(\frac{2n+1}{n+1}\right)\left(1 - Y^{\frac{n+1}{n}}\right)\lambda_i \varphi_i''(Y) + \left(\frac{2n+1}{n^2}\right) Y^{\frac{1-n}{n}} \lambda_i \varphi_i(Y) = 0 \tag{112}$$

The boundary conditions given below represent the symmetric conditions at the center of the channel and no slip conditions at the walls,

$$\varphi_i(0) = \varphi_i''(0) \tag{113}$$

$$\varphi_i(1) = \varphi_i'(1) = 0 \tag{114}$$

Since the values of C_i are still arbitrary $\varphi'(0)$ can have any non-zero value and can be conveniently set $\varphi_i'(0) = 1$. This additional boundary condition enables us to determine the eigenvalues λ_i, and solution of Eq. (112) results in a set of corresponding eigenfunctions φ_i satisfying the boundary conditions.

The values of C_i are those which will provide the best fit of the upstream solution to the downstream solution at the jointing point of the two solutions. C_i are determined by minimizing the integral

$$\int_0^1 \left[\bar{u} \sum_{i=1} C_i \exp(-\lambda_i X_j) \varphi_i'(Y) - v_x^* \right]^2 dY \tag{115}$$

by differentiating it partially with respect to C_i and setting the results equal to zero. v_x^* is the perturbation velocity of the solution near the entry at the point $X = X_j$ where two solutions are jointed. A set of simultaneous equations is obtained and C_i can be evaluated.

Pressure Loss

It follows from Eq. (38), Appendix I, that the volume rate of flow of the power law fluid in the fully developed region of the parallel plate channel is

$$Q = 2h\bar{u}W = \frac{n}{1+2n} \frac{2Wh^2}{m^{1/n}} \left(h \frac{|\Delta p|}{\Delta x}\right)^{1/n} \tag{116}$$

Hence the pressure drop $|\Delta p_f|$ in the fully developed region for a certain duct length Δx is

$$\frac{|\Delta p|}{\frac{1}{2}\rho \bar{u}^2} = \frac{2^{n+1} \left(\frac{2n+1}{n}\right)^n}{N_{Re}} \frac{\Delta x}{h} \tag{117}$$

where

$$N_{Re} = \frac{(2h)^n \bar{u}^{2-n} \rho}{m} \tag{118}$$

$|\Delta p_f|$ = pressure at the beginning of duct length Δx

- pressure at the end of duct length Δx.

The pressure loss $|\Delta p|$ between $x = 0$ and some point x in the fully developed region may be represented by

$$\frac{|\Delta p|}{\frac{1}{2}\ u^2} \quad \frac{2^{n+1}\left(\frac{2n+1}{n}\right)^n}{N_{Re}} \frac{x_e}{h} + C \tag{119}$$

C is the correction factor defined by

$$C = \frac{|\Delta p_e|}{\frac{1}{2}\ u^2} \quad \frac{2^{n+1}\left(\frac{2n+1}{n}\right)^n}{N_{Re}} \frac{x_e}{h} \tag{120}$$

where $|\Delta p_e|$ and x_e are the pressure drop and the length of the entrance region respectively. The pressure drop in the entrance region can be obtained by a momentum balance over the differential length dx as follows:

$$\rho \frac{d}{dx}\int_0^h v_x^2\, dy - h\frac{dP}{dx} - \tau_{yx}\Big|_{y=0} = 0 \tag{121}$$

Substituting the shear stress expression and dimensionless coordinate in Eq. (121) and integrating it from $x = 0$ to $x = x_e$ we have

$$\frac{|\Delta P_e|}{\frac{1}{2}\rho\bar{u}^2} = \frac{2}{\bar{u}^2}\left[\int_0^1 v_x^2\, dY\right]_{X=0}^{X=X_e} - \frac{2m}{\bar{u}^2\rho h^{n+1}}\int_0^{X_e}\left[\left|\frac{\partial v_x}{\partial Y}\right|^{n-1}\frac{\partial v_x}{\partial Y}\right]_{Y=1} dX \tag{122}$$

Substituting the velocity profile at $x = 0$ and $x = x_e$ into the first term on the right hand side of Eq. (122) and carrying out the integration, we obtain

$$\frac{|\Delta P_e|}{\frac{1}{2}\rho\bar{u}^2} \quad 2\left[\frac{2(2n+1)}{3n+2} - 1\right] - \frac{2m}{\bar{u}^2\rho h^{n+1}}\int_0^{X_e}\left[\left|\frac{\partial V_x}{\partial Y}\right|^{n-1}\frac{\partial V_x}{\partial Y}\right]_{Y=1} dX \tag{123}$$

Let

$$X = \frac{(2n+1)^{n-1}}{n^n}\,\varepsilon^{n+1} = \frac{(2n+1)^{n-1}}{n^n}\left(\frac{mx}{h^{n+1}\bar{u}^{2-n}}\right) \tag{124}$$

Then Eq. (123) can be rewritten in the form of

$$\frac{\Delta P_e}{\frac{1}{2}\rho \bar{u}^2} = 2\left[\frac{2(2n+1)}{3n+2} - 1\right] - \frac{2n^n}{\bar{u}^n(2n+1)^{n-1}} \int_0^{X_e} \left[\left|\frac{\partial V_x}{\partial Y}\right|^{n-1} \frac{\partial V_x}{\partial Y}\right]_{Y=1} dX \tag{125}$$

The correction factor C then becomes

$$C = 2\left[\frac{2(n+1)}{3n+2} - 1\right] - \frac{2n^n}{\bar{u}^n(2n+1)^{n-1}} \int_0^{X_e} \left[\left|\frac{\partial V_x}{\partial Y}\right|^{n-1} \frac{\partial V_x}{\partial Y}\right]_{Y=1} dX - \frac{2^{n+1}\left(\frac{2n+1}{n}\right)^n}{N_{Re}} \frac{X_e}{h} \tag{126}$$

It should be noted here that the expressions of pressure correction factor C both in the paper by Collins and Schowalter (7) and in the Ph. D. thesis by Collins have some errors. To correct the error, the first term on the right hand side of Eq. (28) in the paper should be $2\left[\frac{2(2n+1)}{3n+2} - 1\right]$ and the coefficient of the second term should be $2n^n/\bar{u}^n(2n+1)^{n-1}$.

Let us consider briefly the case of circular pipes. Here Collins and Schowalter (8) applied the same technique as in the case of the horizontal parallel plate channel but with a different stream function ψ, dimensionless similarity transform η and dimensionless coordinate σ difined for axial position.

Define a stream function ψ as follows:

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \tag{127}$$

and

$$\psi = -R^2 u \sum_{i=1}^{\infty} \sigma^i f_i(\eta) \tag{128}$$

where

$$\eta = \frac{\left(1 - \frac{r^2}{R^2}\right)}{2\sigma} \tag{129}$$

$$\sigma = \left(\frac{2z}{RN_{Re}}\right)^{\frac{1}{n+1}} \tag{130}$$

Note that ψ satisfies the continuity equation. From Eqs. (127) and (128), we have

$$v_z = \bar{u} \sum_{i=1} \sigma^{i-1} f_i'(\eta) \tag{131}$$

$$v_r = \frac{R^2\bar{u}}{r(n+1)z} \left(\sum_{i=1}^{\infty} i\sigma^i f_i - \eta \sum_{i=1}^{\infty} \sigma^i f_i' \right) \tag{132}$$

The velocity outside of the boundary layer U(z) can be expressed as a function of

$$U = \bar{u}(1 + \sum_{i=1}^{\infty} \sigma^{i} K_i) \tag{133}$$

where K_i is constant and can be determined in the same manner as shown in the case of flow through flat plates.

The expressions of Eqs. (131), (132), and (133) are then substituted into Eq. (68), Chapter 3, and the coefficients of like powers of σ are equated to obtain an infinite set of ordinary differential equations, for example, the first being

$$2^{n-1} n(n+1) f_1 + f_1 f_1 (f_1 f_1)^{\frac{1-n}{2}} = 0 \tag{134}$$

with the boundary conditions

$$f_1(0) = f_1(0) = 0 \tag{135}$$

$$f_1(\infty) = 1 \tag{136}$$

Equating coefficients of terms with powers of σ higher than one gives rise to equations for f_2, f_3, ... The procedure of the solution from here on is carried out in manner comparable to that described previously in the case of parallel plates and will not be repeated here (see reference (8)).

Note that the set of nonlinear ordinary differential equations of the two-point boundary value type, Eqs. (88), (92), and (134), which was solved by the modified Runge-Kutta technique by Collins and Schowalter, can also be solved by the technique of quasilinearization (23).

Results and Discussion

The values of the entrance length, pressure drop, and correction factor for the case of the tube entrance region flow of power law fluid are shown in Figs. (14), (15), and (16) respectively. It is seen from Fig. (14) that the dimensionless entrance length increases from 0 to the maximum value, 0.31, as the flow behavior parameter increases from 0 to 0.1. The pressure drops in the entrance region for various flow behavior parameters are in good agreement with Dodge's experimental data. The results for $n < 0.4$ are all interpolated from the values of $n=0$ and $n=0.4$ because of difficulties in the numerical calculation.

4.5 FINITE DIFFERENCE METHOD

The numerical method developed by Lemmom (24) for solving the entrance region flow of power law fluids in pipes is briefly presented in this section. The flow system is shown schematically in Fig. 10. The assumptions and boundary conditions are the same as those listed in Section 3.1. The governing equations are

$$v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r} = -\frac{1}{\rho}\frac{dP}{dz} + \frac{m}{\rho}\left|\frac{\partial v_z}{\partial r}\right|^{n-1}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{nr}\frac{\partial v_z}{\partial r}\right) \tag{137}$$

$$\frac{\partial v_z}{\partial z} + \frac{v_r}{r} + \frac{\partial v_r}{\partial r} = 0 \tag{138}$$

$$\int_0^R 2\pi v_z r dr = \pi R^2 \bar{u} \tag{139}$$

where Eq. (137) is the equation of motion in the z direction. Equations (138) and (139) are the equations of microscopic and macroscopic material balance respectively. In writing Eqs. (137) and (138), it is assumed by Lemmon that the pressure gradient across the diameter of the pipe is negligible and the only shear stress of importance is τ_{rz}. Actually, they are exactly the same as the boundary layer equations we have derived in Section 3.2.

Then the pipe is divided into N concentric annuli with radial distance h apart and the pipe is further cut by vertical planes with horizontal distance k apart as shown in Fig. 17. Equations (137) and (138) now can be written in appropriate finite difference form for each grid point. The pressure and (N+1) velocity are known at the entry and those of the next grid points in the flow direction can be determined by solving the N+1 equations with an assumed value for pressure such that the volume rate of flow is constant. The velocity profiles for n = 0.5, 1.0 obtained by finite differnce method are also shown in Fig. 13. The results corresponding to the Newtonian fluid are in good agreement with experimental data both for velocity profile and pressure drop. The entrance length and pressure correction factor obtained by this investigation for n=1 compared with experimental data and theoretical results are shown in Table 6. The entrance length and the pressure drop and correction factor are plotted aginst n in Figs. 14, 15, and 16, respectively. It should noted that for $n < 0.5$ the pressure correction factor from this investigation is not greater than the kinetic energy correction only. This is not true because the kinetic energy correction is based only on the change of kinetic energy, i.e., the difference of kinetic energy calculated from a flat profile at the entry and from the fully developed profile while the usual pressure correction is based on both the effects of change in kinetic energy and viscous friction in entrance region. As a result, the finite difference mehtod employed by Lemmon fails for $n < 0.5$.

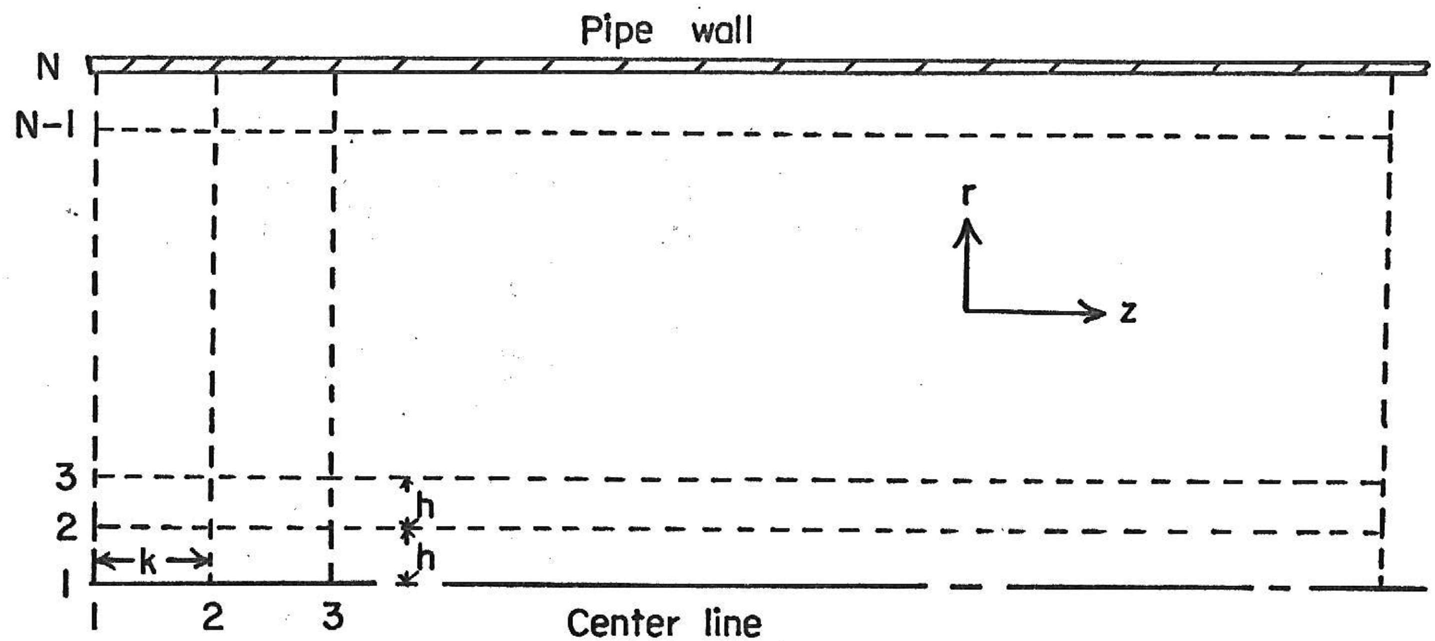


Fig. 17. The pipe is divided into a grid for the finite difference method.

Table 6.

Reported values of Newtonian entrance length z_e^* and pressure correction factors C in pipe flow

Theoretical results

Authors	Reference	C	$z_e^* = z_e/RN_{Re}$
Atkinson and Goldstein	(43)	1.41	0.130
Bogue	(2)	1.16	0.0576
Boussinesq	(3)	1.24	0.130
Campbell and Slattery	(4)	1.18	0.1350
Collins and Schowalter	(8)	1.33	0.122
Langhaar	(21)	1.28	0.1150
Lemmon	(24)	1.274	0.1108
Schiller	(31)	1.16	0.0576
Siegel (quartic profile)	(35)	1.106	0.0592
Siegel (cubic profile)	(35)	1.08	0.06
Sparrow, Lin, and Lundgren	(36)	1.24	
Tomita	(38)	1.22	0.133
Tomita	(39)	1.35	0.0517

Experimental results

Dorsey	(24)	1.08, 1.00	
Knibbs	(24)	1.27 ± 8%	
Kikuradse	(8)	1.32	0.125
Rieman	(24)	1.248 ± 1%	
Schiller	(31)	1.32 ± 10%	
Weltmann and Keller	(24)	1.20 ± 10%	

$(N_{Re} = 2R\bar{u}\rho/\mu)$

CHAPTER 5

ENTRANCE REGION FLOW OF BINGHAM FLUIDS

Consider that a Bingham fluid enters a horizontal circular pipe from a large chamber as shown in Fig. 10. Since the boundary layer model can be applied to the entrance region flow, near the entry of the pipe, we need only solve the approximate equation of motion, or boundary layer equation, of the Bingham fluid (see Chapter 3)

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \qquad (1)$$

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{dp}{dz} - \frac{1}{\rho r}\frac{\partial}{\partial r}(r\,\tau_{rz}) \qquad (2)$$

where

$$\tau_{rz} = -\left(\mu_o + \frac{\tau_o}{\left|\frac{\partial v_z}{\partial r}\right|}\right)\frac{\partial v_z}{\partial r} \qquad (3)$$

It is worth noting that in addition to the boundary layer (velocity) there exists a shear stress boundary layer for the flow of the Bingham fluid as shown in Fig. 18. At the edge of the velocity boundary layer, the velocity gradient is zero, while at the edge of the shear stress boundary layer, which is thicker than the velocity boundary layer, the shear stress becomes zero.

5.1 THE BASIC MOMENTUM INTEGRAL METHOD

The basic momentum integral method for the entrance region flow of Newtonian fluid was first developed by Schiller (31). He assumed a parabolic velocity profile in the boundary layer with a thickness $\delta(z)$ and a uniform velocity profile outside the boundary layer. From the form of the velocity profile in the fully developed region, it appears that the basic momentum integral method can also be used to analyze the entrance region flow of the

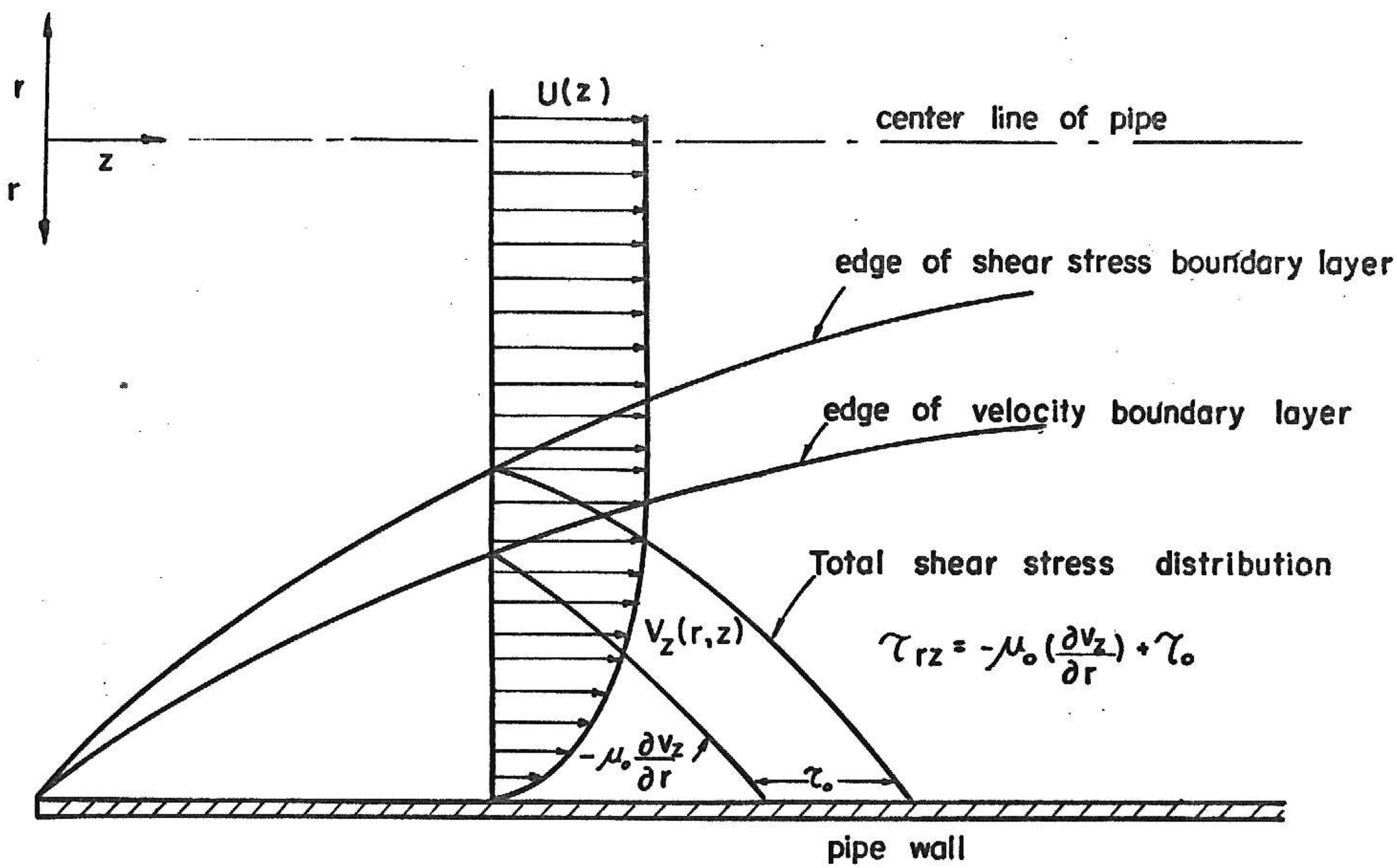


Fig. 18. Shear stress boundary layer and velocity boundary layer exist for the Bingham fluid in the entrance region of a circular pipe.

Bingham fluid because the uniform velocity distribution outside the boundary layer is analogous to the plug flow region of the Bingham flow. Hence it may be expected that this model can describe correctly the flow characteristics of the Bingham fluid in the entrance region.

It is assumed that the pressure gradient can be related to the plug flow velocity by

$$U\frac{dU}{dz} = -\frac{1}{\rho}\frac{dp}{dz} \tag{4}$$

where U is a function of z. Substitution of Eq. (4) into Eq. (2) gives

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = U\frac{dU}{dz} - \frac{1}{\rho r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{5}$$

Integrating Eq. (5) with respect to r from r=R at the solid wall to $r=R-\delta_s$ at the edge of the shear stress boundary layer where no shear stress exists, we have

$$\int_{r=R}^{r=R-\delta_s}\left(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} - U\frac{dU}{dz}\right) rdr = -\frac{1}{\rho}\Big[\,r\,\tau_{rz}\Big]_{r=R}^{r=R-\delta_s} \tag{6}$$

Combining Eq. (6) with the equation of continuity Eq. (1) and carrying out the integration we obtain (see Appendix A II.1)

$$\frac{dU}{dz}\int_R^{R-\delta_s}(U - v_z)rdr + \frac{d}{dz}\int_R^{R-\delta_s} v_z(U - v_z)rdr = -\frac{R}{\rho}\tau_w \tag{7}$$

where

$$\tau_w = (\tau_{rz}|_{r=R}) = -\mu_0\left(\frac{\partial v_z}{\partial r}\right)_{r=R} + \tau_0$$

Since $U - v_z$ becomes zero outside the velocity boundary layer, r=δ, Eq. (7) becomes

$$\frac{dU}{dz}\int_R^{R-\delta}(U-v_z)rdr + \frac{d}{dz}\int_R^{R-\delta} v_z(U-v_z)rdr = \frac{R\mu_0}{\rho}\left(\frac{\partial v_z}{\partial r}\right)_{r=R} - \frac{R\tau_0}{\rho} \tag{8}$$

As in the case of Newtonian entrance region flow, the following parabolic form of velocity profile inside the boundary layer is assumed.

$$\frac{v_z}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \tag{9}$$

where y is the new coordinate as shown in Fig. 10 and defined by

$$y = R - r \tag{10}$$

The assumed form of velocity profile satisfies the following boundary conditions.

$$\begin{aligned} v_z &= 0, \quad \text{at } y = 0 \\ v_z &= U, \quad \text{at } y = \delta \\ \frac{\partial v_z}{\partial y} &= 0, \quad \text{at } y = \delta \end{aligned} \tag{11}$$

Substituting the assumed velocity profile, Eq. (9), into Eq. (8) and carrying out the integration, we have (see Appendix AII.2)

$$\delta U\left(\frac{3}{5}R - \frac{11}{60}\delta\right)\frac{dU}{dz} + U^2\left(\frac{2}{15}R - \frac{1}{10}\delta\right)\frac{d\delta}{dz} = \frac{1}{\rho}\left(\frac{2R\mu_o U}{\delta} + R\tau_o\right) \tag{12}$$

If the dimensionless quantities

$$\begin{aligned} &\delta^* = \delta/R, \quad U^* = U/\bar{u}, \quad z^* = \mu_o z/R^2\bar{u} = z/RN_{Re}, \\ &\tau_o^* = R\tau_o/\mu_o\bar{u}, \quad N_{Re} = \rho\bar{u}R/\mu_o \end{aligned} \tag{13}$$

are introduced in Eq. (12), it becomes (see the second part of Appendix AII.2)

$$\delta^* U^*\left(\frac{3}{5} - \frac{11}{60}\delta^*\right)\frac{dU^*}{dz^*} + U^{*2}\left(\frac{2}{15} - \frac{1}{10}\delta^*\right)\frac{d\delta^*}{dz^*} = 2\frac{U^*}{\delta^*} + \tau_o^* \tag{14}$$

The relation between the boundary layer thickness, δ, and the center core velocity, U, can be obtained by using the macroscopic mass balance as follows:

$$\pi(R - \delta)^2 U + \int_{R-\delta}^{R} 2\pi v_z r\,dr = \pi R^2\bar{u} \tag{15}$$

When the assumed form of velocity profile given by Eq. (9) is used, Eq. (15) can be reduced to (see Appendix AII.3)

$$\delta^{*2} - 4\delta^{*} = 6(\frac{1}{U^{*}} - 1) \qquad (16)$$

Solving Eq. (16) for δ^{*} and noting that $\delta^{*} < 1$ and $U^{*} \geqslant 1$, we obtain

$$\delta^{*} = 2 - \sqrt{\frac{6}{U^{*}} - 2} \qquad (17)$$

Equation (14) now reduces to a first order differential equation if the relation of δ^{*} and U^{*} in Eq. (17) is used (see the second part of Appendix AII.3),

$$(\frac{5}{6} U^{*} + \frac{2}{15} U^{*}D - \frac{1}{5D} - \frac{4}{5}) \frac{dU^{*}}{dz^{*}} = \frac{2U^{*}}{2-D} + \tau_{o}^{*} \qquad (18)$$

where

$$D = \sqrt{\frac{6}{U^{*}} - 2} \qquad (19)$$

The initial condition of Eq. (18) is

$$U^{*} = 1 \quad \text{at} \quad z^{*} = 0 \qquad (20)$$

Equation (18) can be integrated numerically by using any standard mehtod of numerical integration. The solution of Eq. (18) will give the relation between the dimensionless center velocity, U^{*}, and the dimensionless tube length, z^{*}, for a given value of τ_{o}^{*}. Using the relationship of δ^{*} and U^{*} given by Eq. (17) and the velocity profile given by Eq. (9) we can also calculate δ^{*} and the velocity distribution for a given z^{*} and τ_{o}^{*}.

To obtain a clear visualization of the characteristics of the Bingham flow, it is preferable to use the dimensionless plug flow radius, $r_{o}^{*} = r_{o}/R$ in place of the dimensionless yield stress, τ_{o}^{*}, as a parameter. The relation between these two quantities can be obtained from the expression of the average velocity in the fully developed tube flow as follows:

$$\bar{u} = \frac{R\tau_w}{4}[1 - \frac{4}{3}(\frac{\tau_o}{\tau_w}) + \frac{1}{3}(\frac{\tau_o}{\tau_w})^4 \tag{21}$$

Here the shear stress at the wall, $_w$, and the yield stress, $_o$, are related by

$$\frac{\tau_o}{\tau_w} = \frac{r_o}{R} \tag{22}$$

Combining Eqs. (21) and (22) together with Eq. (13), we obtain

$$\tau_o^* = \frac{12r_o^*}{3 - 4r_o^* + r_o^{*4}} \tag{23}$$

For each r_o^* there exists a corresponding τ_o^* which is called the plasticity number. When r_o^* is zero the fluid is Newtonian. When r_o^* approaches unity, τ_o^* is infinite corresponding to the plug flow. Thus r_o^* may be used as a parameter together with the Reynolds number N_{Re} in the Bingham flow.

The entrance length, z_e, is the distance from the tube inlet to the point downstream where the boundary layer velocity profile asymtotically approaches that of the fully developed flow. In other words, at $z = z_e$, the boundary layer thickness δ_e is equal to $R - r_o$ and the center velocity U_e is equal to the velocity in the fully developed plug flow region, or

$$\text{at } z = z_e, \; \delta_e = R - r_o, \; U_e = R(\tau_w - \tau_o)^2/2\mu_o\tau_w$$

Using dimensionless quantities defined by Eq. (13) and the relations given by Eqs. (22) and (23), we obtain

$$\delta_e^* = 1 - r_o^* \tag{24}$$

$$U_e^* = \frac{6}{3 + 2r_o^* + r_o^{*2}} \tag{25}$$

For each given r_o^*, we can calculate U_e^* from Eq. (25). The corresponding dimentionless entrance length z_e^* is then obtained by utilizing Eq. (18).

The pressure drop, $|\Delta p|$, from the tube inlet to any section in the entrance region can be obtained by integrating Eq. (4) from z = 0 to z,

$$|\Delta p| = p_o - p = \frac{\rho}{2}(U^2 - \bar{u}^2) \tag{26}$$

where p_o is the pressure at the tube inlet without taking into account of contraction loss. If dimensionless pressure is defined as

$$p^* = p/\ \tfrac{1}{2}\rho\bar{u}^2,$$

Eq. (26) becomes

$$|\Delta p^*| = p_o^* - p^* = U^{*2} - 1 \tag{27}$$

Therefore, the pressure drop in the entrance region, $|\Delta p_e^*|$, can be expressed in terms of dimensionless center velocity at $z = z_e$,

$$|\Delta p_e^*| = p_o^* - p_e^* = U^{*2} - 1 \tag{28}$$

For the purpose of comparison, the dimensionless pressure drop based on the fully developed region for tube length Δz^* is (see Appendix AII.4)

$$|\Delta p_f^*| = \frac{48\Delta z^*}{3 - 4r_o^* + r_o^{*4}} = \frac{48(z^* - z_e^*)}{3 - 4r_o^* + r_o^{*4}} \tag{29}$$

Therefore, the pressure drop from the tube inlet to any cross section z^* in the fully developed region is customarily defined as

$$|\Delta p^*| = \frac{48z^*}{3 - 4r_o^* + r_o^{*4}} + C \tag{30}$$

where the correction factor C is

$$C = U_e^{*2} - 1 - \frac{48z_e^*}{3 - 4r_o^* + r_o^{*4}} \qquad (31)$$

Numerical Procedures and Results

The computational procedure is as follows;

(1) Given a value of plug flow radius r_o^*, the corresponding yield stress τ_o^* can be calculated from Eq. (23). The results are shown in Fig. 19. In the following numerical procedure, r_o^* in the range from 0 to 0.9, is used as the parameter.

(2) The relationship between center velocity U^* and tube distance z^* can be obtained by numerical integration of Eq. (18) using the trapezoidal rule. The results are shown in Fig. 20.

(3) For given values of U^*, the corresponding boundary layer thickness δ^* can be obtained from Eq. (17). Thus the relationship between δ^* and z^* is established using Step (2) and Eq. (17) and the results are shown in Fig. 21.

(4) The velocity profile at a given z^* is then calculated from Eq. (9). The results for the case of $r_o^* = 0$, 0.2 are shown in Figs. 22 and 23 respectively.

(5) The entrance length z_e^* is obtained in Step (2) when U^* approaches the fully developed center velocity U_e^*, which can be calculated from Eq. (45). The entrance length z_e^* is plotted against r_o^* in Fig. 24.

(6) The pressure drop from the tube entry to any cross section downstream in the pipe is obtained from Eqs. (30) and (31). The results are shown in Fig. 25.

(7) The correction factor defined in Eq. (31) can be calculated for given values of r_o^*. The results are plotted in Fig. 26.

Discussion

The characteristic behavior of the Bingham fluid is that it remains

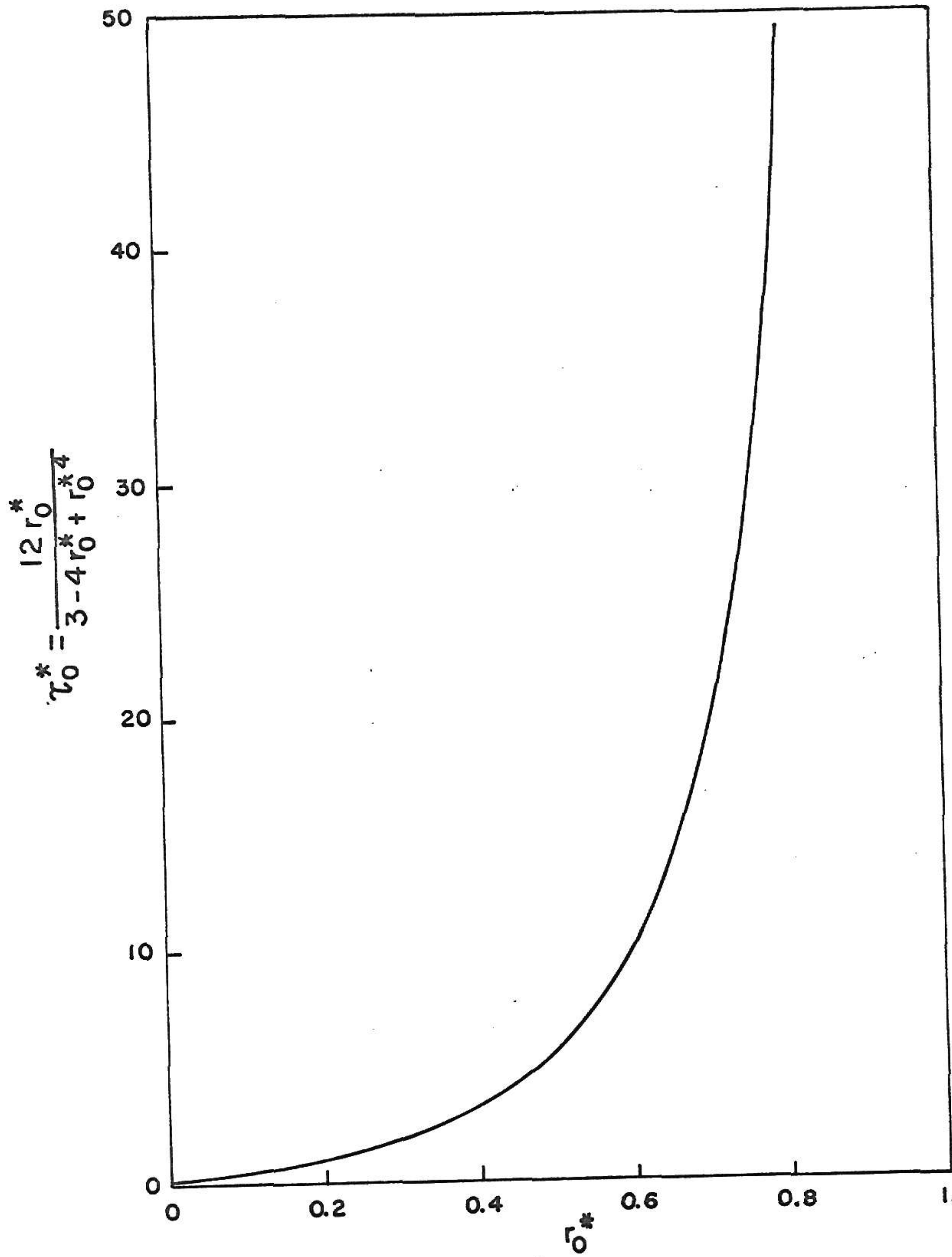


Fig. 19. Yield stresses of the Bingham fluid as a function of plug flow radius.

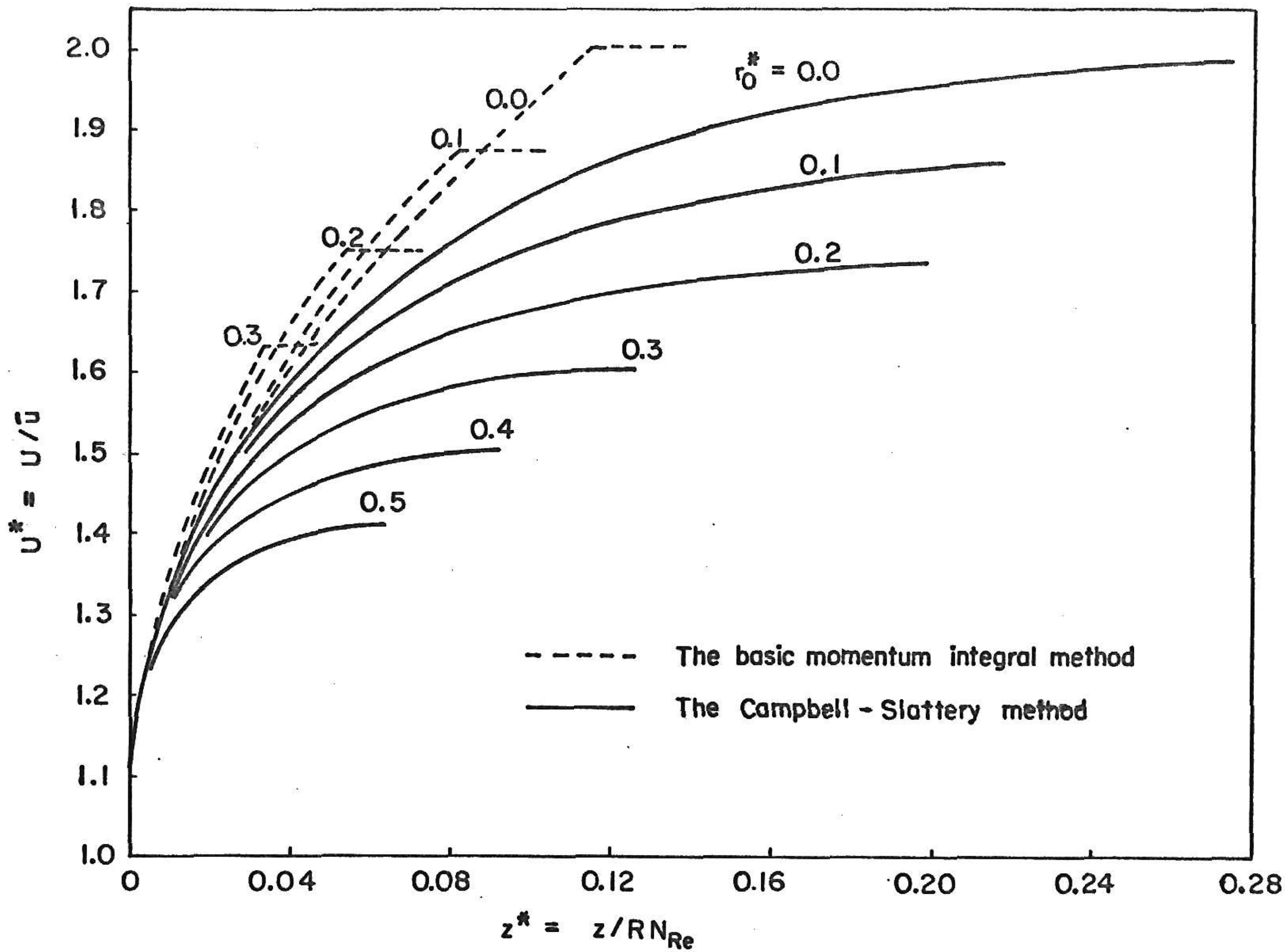


Fig. 20. Center-core velocity of the Bingham fluid as a function of tube distance by the basic momentum integral method and the Campbell-Slattery method.

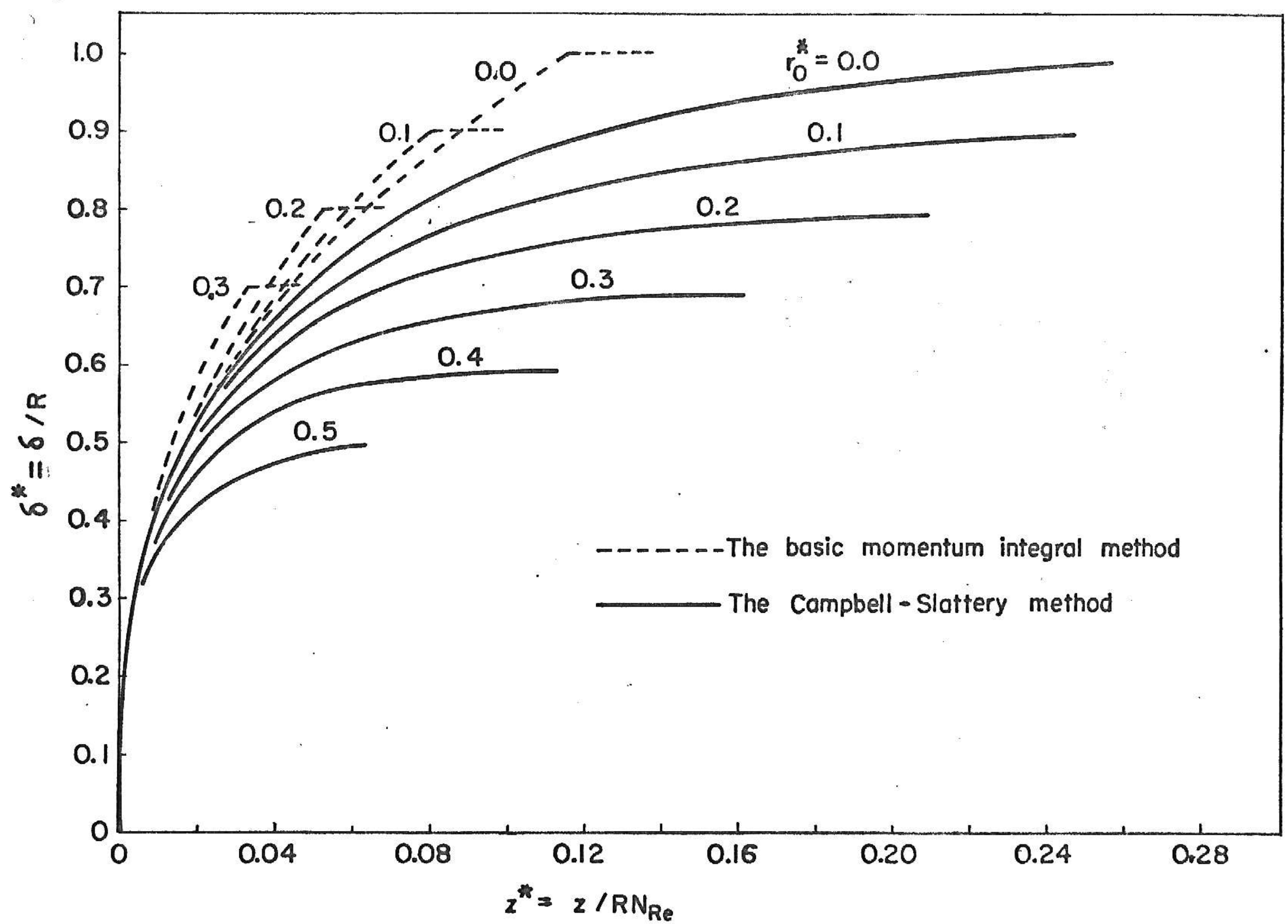


Fig. 21. Boundary layer thickness of the Bingham fluid as a function of tube length.

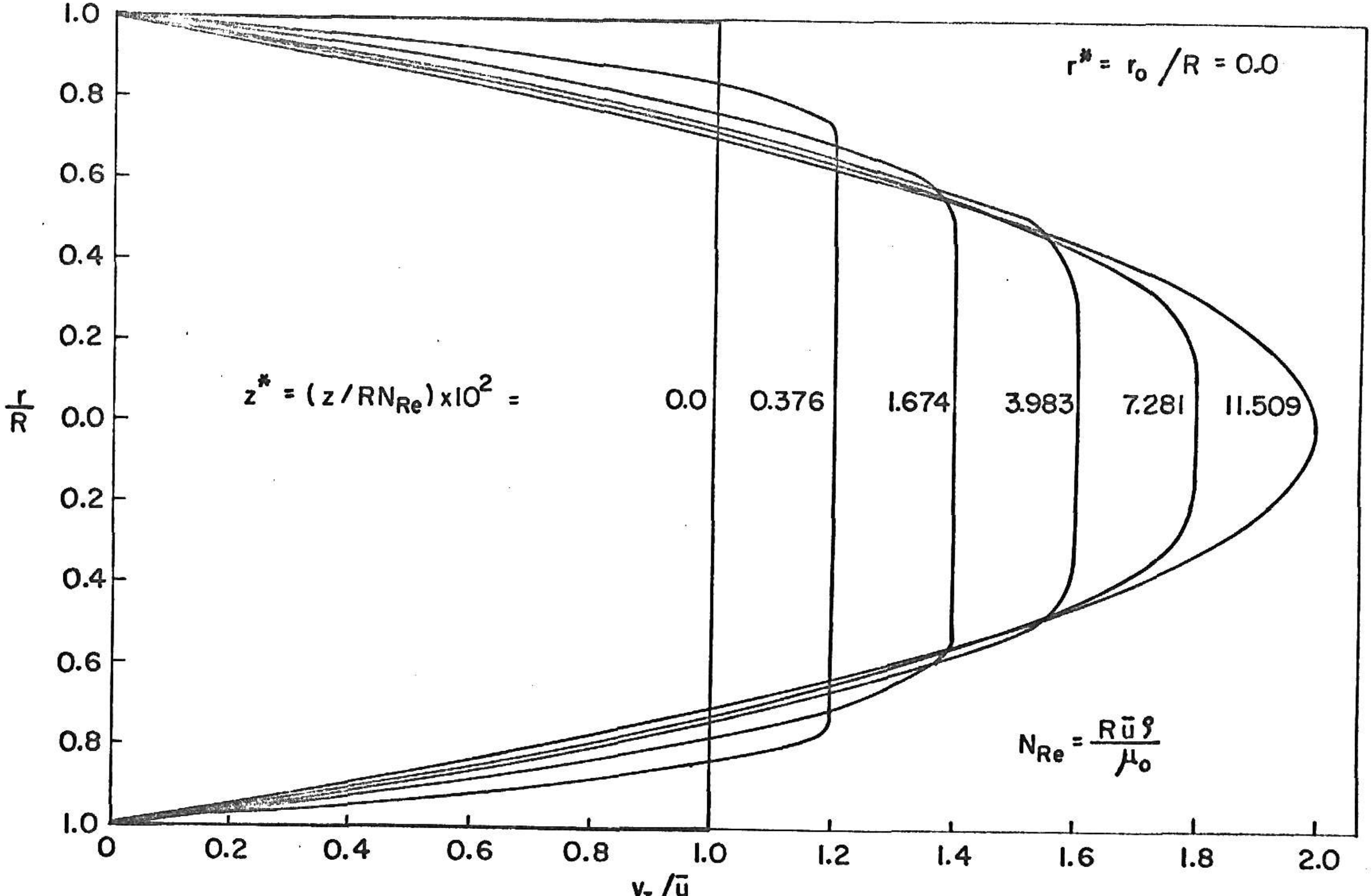


Fig. 22. Velocity profiles of Newtonian fluids in entrance region by the basic momentum integral method.

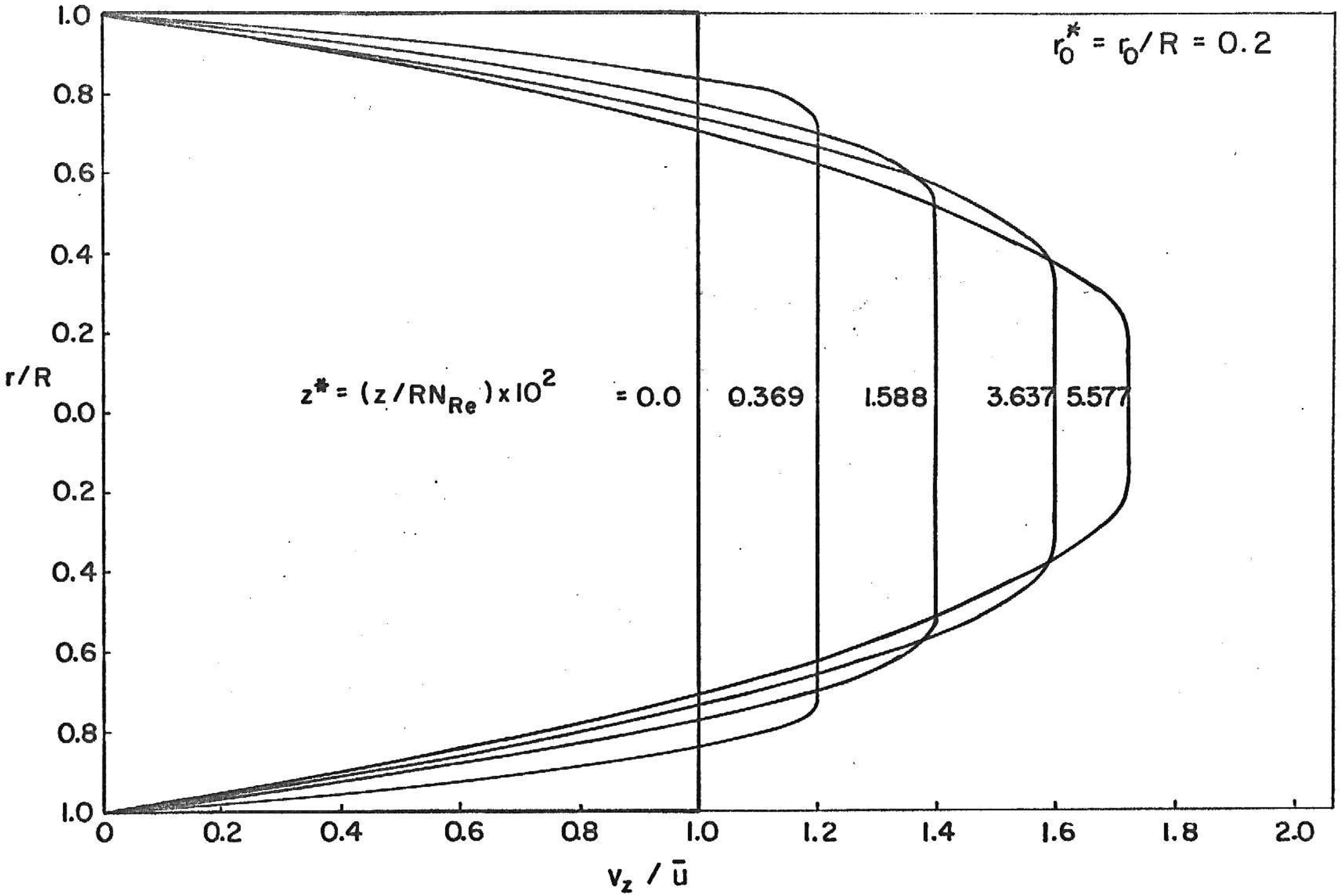


Fig. 23. Velocity profiles of the Bingham fluid in entrance region by the basic momentum integral method.

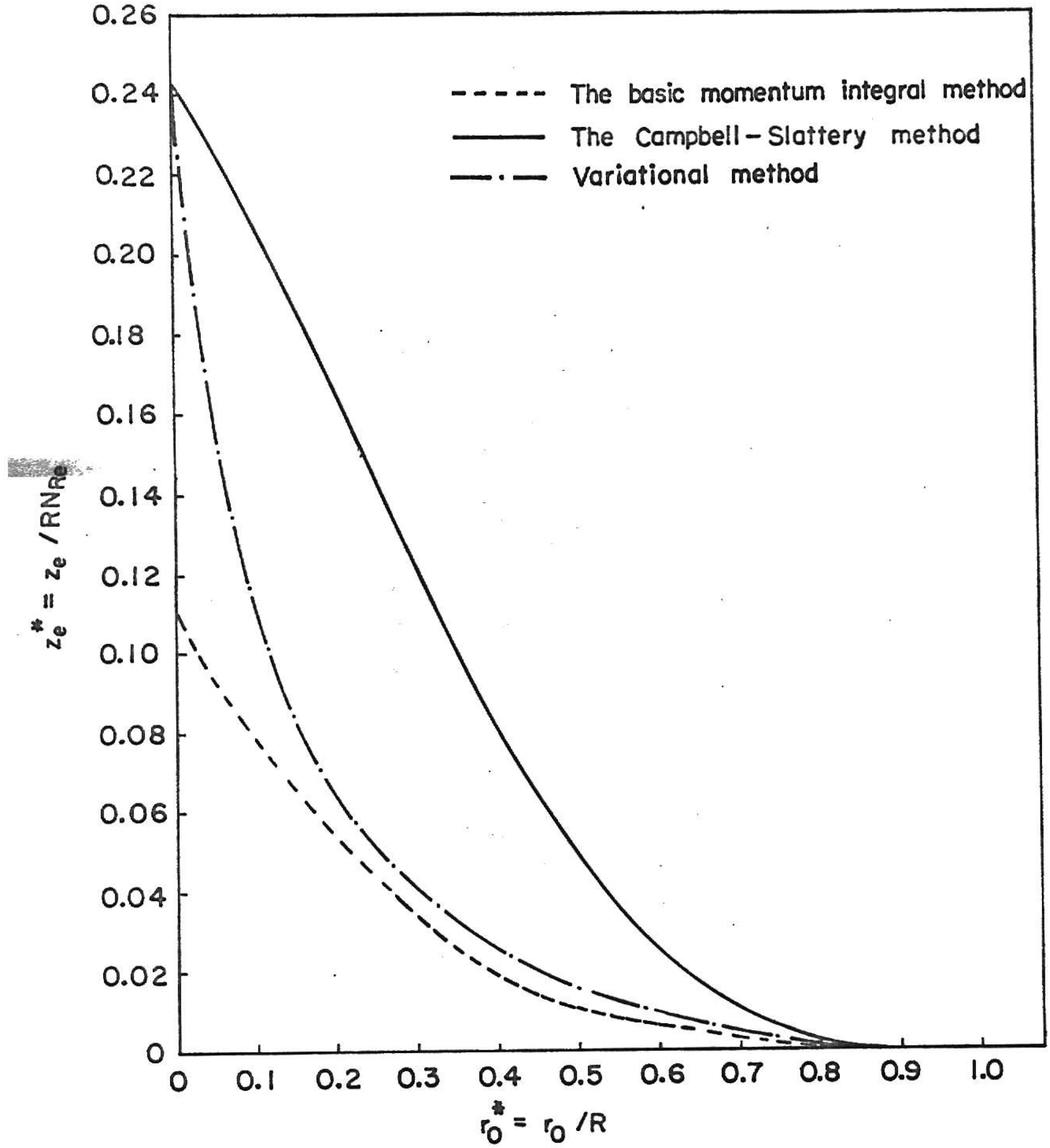


Fig. 24. Entrance length of the Bingham fluid as a function of plug flow radius by various methods.

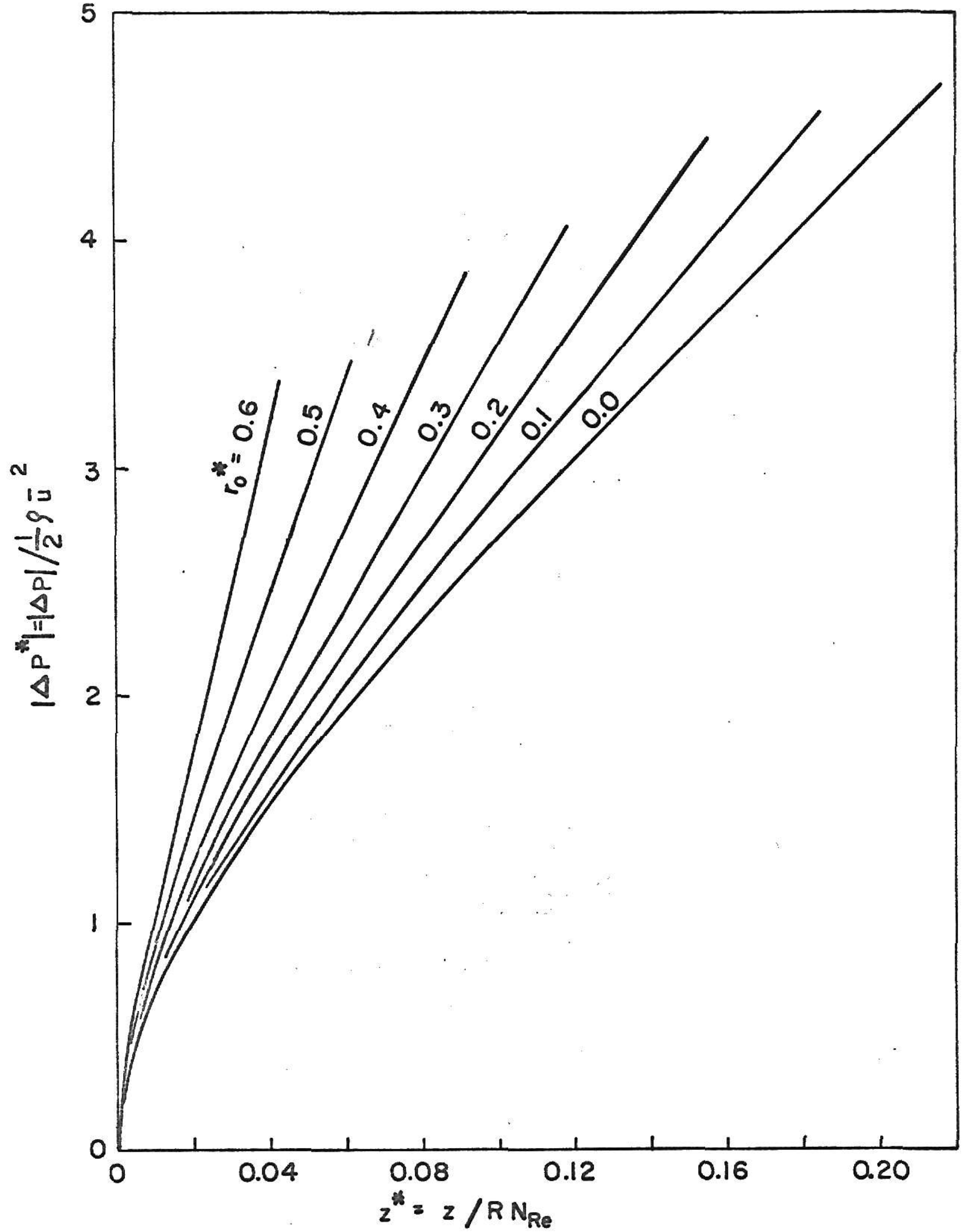


Fig. 25 . Pressure drop of Bingham fluids as a function of tube distance by the Campbell - Slattery method and the basic momentum integral method.

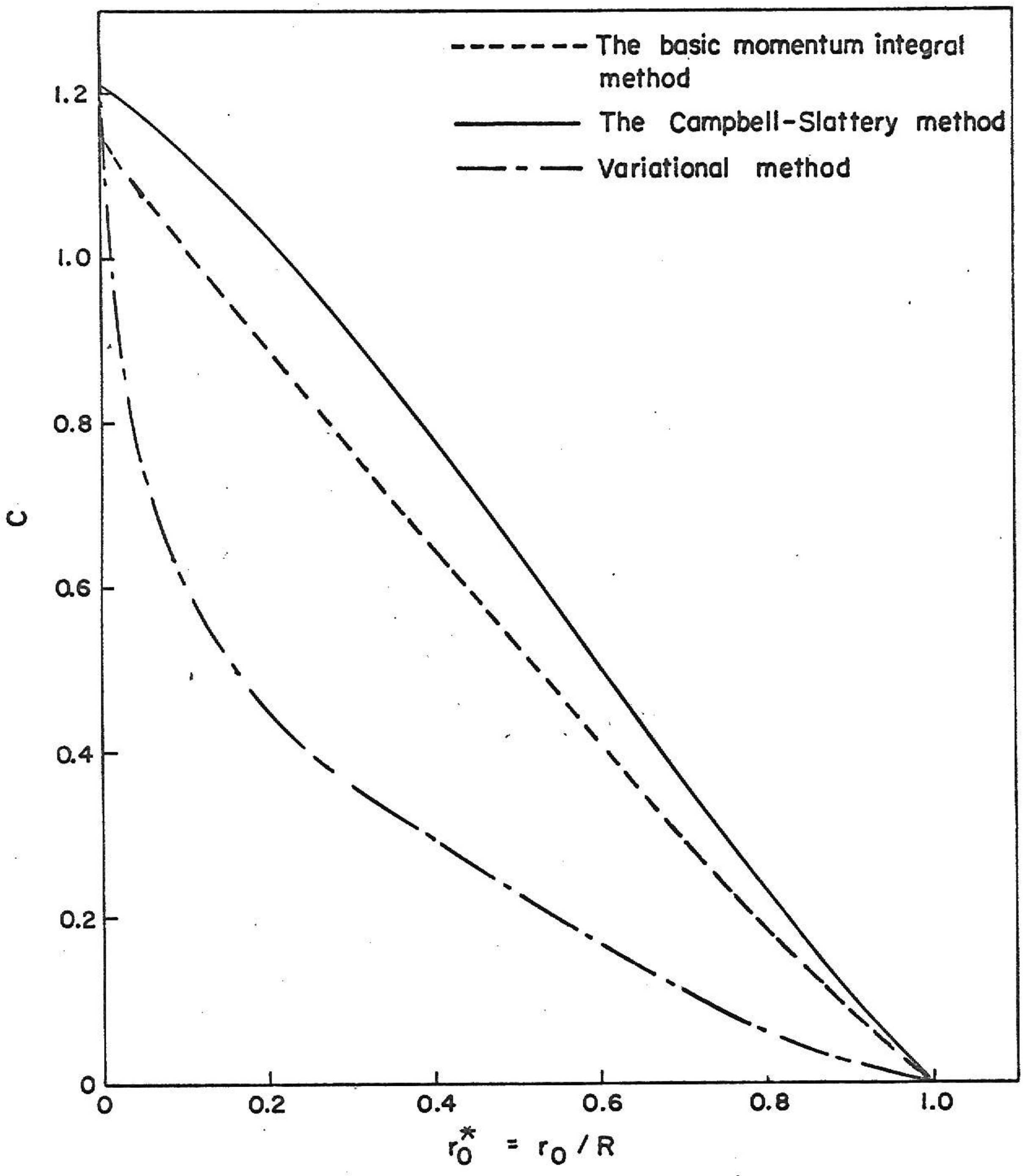


Fig. 26. Correction factor of entrance region pressure drop for the Bingham fluid as a function of plug flow radius computed by various methods.

rigid when the shear stress is less than the yield stress, but flows when the shear stress is larger than the yield stress. This flow behavior implies that there always exists a true plug flow region outside the boundary layer if the yield stress is not zero. At the edge of the boundary layer, the velocity gradient $\partial v_z/\partial r$ is zero, and the shear stress τ_{rz} is equal to the yield stress τ_o as can be visualized from Eq. (3). The fluid outside the boundary layer flows like a solid body with elastic properties because the fluid is under stress with τ_o at the edge of the boundary layer. This consideration prompted us to study the Bingham entrance region flow problem using the momentum integral method.

The center line velocity U^* plotted against z^* for each r_o^* (see Fig. 20) does not approach the fully developed value asymptotically. This results in shorter entrance length (see Fig. 24) compared with those obtained from the Campbell-Slattery method and the variational method, which will be described later. As shown in Figs. 20 and 21, the center line velocity and boundary layer thickness increase with increasing r_o^* at a given tube length z^*, but near the tube entry $z^* = z/RN_{Re} = 0.02$ and the variation of U^* and δ^* for different values of r_o^* is very small. Figure 24 indicates that the entrance length z_e^* decreases with increasing r_o^*.

5.2 SOLUTION BY THE CAMPBELL-SLATTERY METHOD

The unrealistic results in the region far from the entry, $z^* > 0.02$, obtained from the basic momentum integral method for Newtonian fluids were corrected by Campbell and Slattery (4). In addition to the basic assumptions employed in the basic momentum integral method, they used the macroscopic mechanical energy balance to account for viscous dissipation within the boundary layer. Their results are good in over- all description of the velocity field within the entrance region. The method of correction by Campbell and Slattery originally developed for Newtonian flow will be applied to Bingham fluids in this section.

The macroscopic momentum balance in the entrance region of a pipe may be written (42) as

$$\Delta(\rho < v_z^2 > S + pS) + F = 0 \tag{32}$$

where S is the cross-sectional area, F is the friction force of the fluid on the tube wall, and $< v_z^q >$ is defined as

$$< v_z^q > = \int_0^{2\pi}\int_0^R v_z^q r dr d\theta \Big/ \int_0^{2\pi}\int_0^R r dr d\theta = \int_0^{2\pi}\int_0^R v_z^q r dr d\theta / S \tag{33}$$

Here it is assumed that $v_z^2 >> v_r^2$. If Eq. (32) is applied to a differential length dz, we have (for the derivation of Eqs. (34), (37) and (38), see Appendix AII.5)

$$\rho \frac{d}{dz}\int_0^R v_z^2 r dr + \frac{R^2}{2}\frac{dp}{dz} + (\tau_{rz})_{r=R} R = 0 \tag{34}$$

The assumed form of the velocity distribution inside and outside of the boundary layer given in the preceding section,

$$\frac{v_z}{U} = \frac{2y}{\delta} - \frac{y^2}{\delta}, \qquad \text{for } y \leq \delta, \tag{35}$$

$$\frac{v_z}{U} = 1, \qquad \text{for } y \geq \delta \tag{36}$$

can be used in performing the integration of $\int_0^R v_z^2 \, r dr$ and calculating the shear stress at the wall $(\tau_{rz})_{r=R}$ in Eq. (34). The result is

$$\rho\frac{d}{dz}\left[U^2\left(\frac{R^2}{2} - \frac{7}{15}R\delta + \frac{2}{15}\delta^2\right)\right] + \frac{R^2}{2}\frac{dp}{dz} + \frac{2\mu_o RU}{\delta} + \tau_o R = 0 \tag{37}$$

The dimentionless form of this equation is

$$\frac{d}{dz^*}\left[U^{*2}\left(\frac{1}{2}-\frac{7}{15}\delta^*+\frac{2}{15}\delta^{*2}\right)\right]+\frac{1}{4}\frac{dp^*}{dz^*}+\frac{U^*}{2\delta^*}+\tau_o^*=0 \tag{38}$$

where dimentionless quantities are defined in Eq. (13).

The macroscopic mechanical energy balance (42) in the entrance region may be written as

$$\Delta\left(\frac{1}{2}\rho S\langle v_z^3\rangle\right)+w\int_{p_0}^{p}\frac{dp}{\rho}+E_v=0 \tag{39}$$

where w is mass flow rate, $w=\rho\langle v_z\rangle S$, and E_v is the total rate of irreversible conversion of mechanical to internal energy. Equation (39) can be applied to the fluid from the tube inlet to any cross section in the entrance region,

$$\frac{1}{2}\rho\int_0^R v_z^3 r dr-\frac{1}{4}\rho\bar{u}^3R^2+p\int_0^R v_z r dr-\frac{1}{2}p_o\bar{u}R^2+\frac{E_v}{2\pi}=0 \tag{40}$$

where

$$E_v=-\int_0^z\int_0^{2\pi}\int_0^R(\underline{\underline{\tau}}:\underline{\nabla}\,\underline{v})\,rdrd\theta dz \tag{41}$$

Using the scalar or expanded expression of $(\underline{\underline{\tau}}:\underline{\nabla}\,\underline{v})$ (42) and the usual boundary layer assumptions, i.e., neglecting the small term of velocity derivative except $\partial v_z/\partial r$, we have

$$-(\underline{\underline{\tau}}:\underline{\nabla}\,\underline{v})=\left(\mu_o-\frac{\tau_o}{\frac{\partial v_z}{\partial r}}\right)\left(\frac{\partial v_z}{\partial r}\right)^2 \tag{42}$$

Equation (41) can be integrated with respect to r and θ by using the expression of Eq. (42) and using the assumed form of the velocity profile, Eqs. (35) and (36). The result is (see Appendix AII.6)

$$E_v=2\pi\mu_o\int_0^z U^2\left(\frac{4}{3}\frac{R}{\delta}-\frac{1}{3}\right)dz-2\pi\tau_o\int_0^z U\left(\frac{\delta}{3}-R\right)dz \tag{43}$$

Combining Eqs. (40) and (43) and differentiating the resulting equation with respect to z we have (for the details of derivation of Eqs. (44), (45), and (66), see Appendix AII.7)

$$\frac{R^2\bar{u}}{z}\frac{dp}{dz} = -\frac{1}{2}\rho\frac{d}{dz}U^3\left(\frac{1}{2}R^2 - \frac{19}{35}R\delta + \frac{47}{280}\delta^2\right)$$

$$-\mu_o U^2\left(\frac{4}{3}\frac{R}{\delta} - \frac{1}{3}\right) + \tau_o U\left(\frac{1}{3}\delta - R\right) \tag{44}$$

The dimensionless form of this equation is

$$\frac{1}{4}\frac{dp^*}{dz^*} = -\frac{1}{2}\frac{d}{dz^*}\left[U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)\right]$$

$$-U^{*2}\left(\frac{4}{3}\frac{1}{\delta^*} - \frac{1}{3}\right) + U^*\tau_o^*\left(\frac{1}{3}\delta^* - 1\right) \tag{45}$$

Eliminating the pressure gradient dp^*/dz^* between Eqs. (38) and (45) gives

$$\frac{d}{dz^*}\left[U^{*2}\left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right)\right] - \frac{1}{2}\frac{d}{dz^*}\left[U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)\right]$$

$$-U^{*2}\left(\frac{4}{3}\frac{1}{\delta^*} - \frac{1}{3}\right) + \frac{2U^*}{\delta^*} + \tau_o^*\left[1 + U^*\left(\frac{1}{3}\delta^* - 1\right)\right] = 0 \tag{46}$$

The boundary layer thickness, δ^*, can be expressed in terms of center velocity U^* by

$$\delta^* = 2 - \sqrt{\frac{6}{U^*} - 2} \tag{47}$$

or

$$U^* = \frac{6}{\delta^{*2} - 4\delta^* + 6} \tag{48}$$

Using the relation Eq. (48) we can eliminate U^* from Eq. (46) and carry out the differentiation of the resulting equation to obtain (see Appendix AII.8)

$$\delta^*(112\delta^{*5} - 1036\delta^{*4} + 3130\delta^{*3} - 3574\delta^{*2} + 1014\delta^* + 432)\frac{d\delta^*}{dz^*}$$

$$= 140(\delta^{*6} - 118\delta^{*5} + 54\delta^{*4} - 148\delta^{*3} + 236\delta^{*2} - 204\delta^{*} + 72)$$
$$+ \frac{35}{3}\tau_o^* \delta^{*2}(\delta^{*7} - 14\delta^{*6} + 94\delta^{*5} - 340\delta^{*4} + 812\delta^{*3}$$
$$- 1224\delta^{*2} + 1080\delta^{*} - 432) \qquad (49)$$

Equation (49) is the differential equation which determines the relationship between δ^* and z^*. The boundary condition is $\delta^* = 0$ at $z^* = 0$. For Newtonian fluid, Eq. (49) can be integrated analytically (4). Equation (46) reduces to the differential equation which relates U^* to z^* if Eq. (47) is used (see the last part of Appendix AII. 8). The boundary condition is $U^* = 1$ at $z^* = 0$. Entrance length z_e^* is determined by the numerical integration of the resulting differential equation at $U^* = U_e^*$.

The pressure drop in the entrance region can be obtained by integrating either Eq. (38) or (45) from the tube inlet to any cross section in the entrance region

$$|\Delta p^*| = 4\left\{U^{*2}\left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right) - \frac{1}{2} + 2\int_0^{z^*} \frac{U^*}{\delta^*}\,dz^* + \tau_o^* z^*\right\} \qquad (50)$$

or

$$p^* = 2U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right) - 1$$
$$+ \frac{4}{3}\int_0^{z^*} U^{*2}\left(\frac{4}{\delta^*} - 1\right)dz^* + \frac{4}{3}\tau_o^* \int_0^{z^*} U^*(3 - \delta^*)\,dz^* \qquad (51)$$

If the expression of Eq. (50) is used to calculate the pressure drop, the correction factor becomes

$$C = 4\left\{U_e^{*2}\left(\frac{1}{2} - \frac{7}{15}\delta_e^* + \frac{2}{15}\delta_e^{*2}\right) - \frac{1}{2} + 2\int_0^{z_e^*} \frac{U^*}{\delta^*}\,dz^* + \tau_o^* z_e^*\right\}$$
$$- \frac{48 z_e^*}{3 - 4r_o^* + r_o^{*4}} \qquad (52)$$

The numerical procedure is , in general, similar to that in the basic momentum integral method. The determination of pressure drop as a function of z^* needs an additional integration. The center velocity and boundary layer thickness as shown in Figs. 20 and 21 as function of tube length are obtained by numerical integration of Eq. (49) combined with Eq. (47). The entrance lengths are plotted against r_o^* in Fig. 24. Pressure drop calculated from Eq. (50) is shown in Fig. 25 and correction factor C from Eq. (52) is shown in Fig. 26.

Results and Discussion

Since the present investigation is the first work applying the Campbell-Slattery method (4) originally developed for the Newtonian entrance region flow to the Bingham fluid, the numerical results obtained are compared with those obtained by the basic momentum integral method presented in the preceding section and with the variational method (29). Although the plug flow velocity and the boundary layer thickness plotted against pipe length for different values of plug flow radius indicate the opposite trend with respect to increasing plug flow radius as compared with those of the basic momentum integral method, both plug flow velocity and boundary layer thickness from this work approach the fully developed values asymptotically. Furthermore, the pressure drops in the entrance region are in good agreement with the results from the basic momentum integral method as shown in Fig 25. The results of pressure drop from the basic momentum integral method and the Campbell-Slattery method are also compared with those from the variational method in Fig. 27. The entrance lengths and correction factors for different values of r_o^* as shown in Figs. 24 and 26 are in general longer and higher than those obtained from the basic momentum integral method and the variational method.

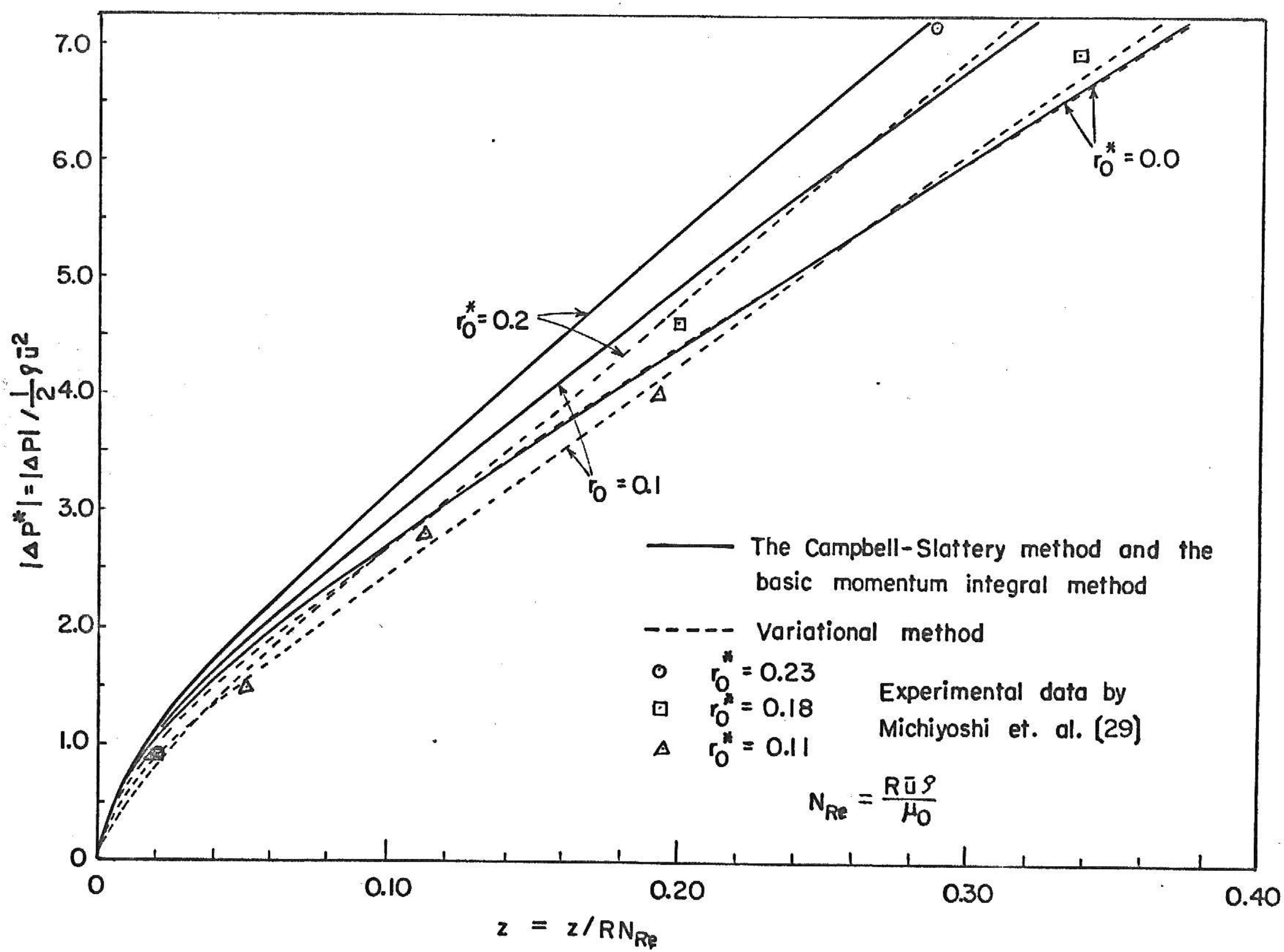


Fig.27. Comparison of pressure drop of the Bingham fluid calculated from various methods

5.3 THE VARIATIONAL METHOD

As indicated in the introduction of this report, the first work on the entrance region flow of the Bingham fluid was done by Michiyochi et al. (29) using the variational method. In reviewing their work it was found that their approximate form of the equation of motion based on the boundary layer model (Eq. (61) in this section) is different from the rigorous form derived in Section 3.2, Chapter 3. The validity of their approximate form of equation of motion merits further study. Nevertheless, it was assumed correct and the analysis part of their work will be briefly presented in this section.

Let us consider the flow system as shown in Fig. 10. The equation of continuity and the equation of motion in the z direction under the conditions that the flow is steady, laminar, and incompressible are (42)

$$\frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_\theta}{\partial \theta} = 0 \tag{53}$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] \tag{54}$$

The components of the stress tensor for Bingham fluids are (29),

$$\tau_{rr} = -2(\mu_o + \lambda)\frac{\partial v_r}{\partial r}$$

$$\tau_{\theta\theta} = -2(\mu_o + \lambda)\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\frac{1}{r}$$

$$\tau_{zz} = -2(\mu_o + \lambda)\frac{\partial v_z}{\partial z} \tag{55}$$

$$\tau_{r\theta} = \tau_{\theta r} = -(\mu_o + \lambda)\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\tau_{\theta z} = \tau_{z\theta} = -(\mu_o + \lambda)\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right)$$

$$\tau_{zr} = \tau_{rz} = -(\mu_o + \lambda)\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)$$

where

$$\lambda = \frac{\tau_o}{\phi} \tag{56}$$

Substituting the expressions τ_{rz}, $\tau_{\theta z}$, and τ_{zz} in Eq. (55) into Eq. (54) and using the axisymmetric condition, i.e., $\partial/\partial\theta = 0$ and $v_\theta = o$, we obtain

$$v_r\frac{\partial V_z}{\partial r} + V_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{(\mu_o + \lambda)}{\rho}\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

$$+ \frac{1}{\rho}\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)\frac{\partial \lambda}{\partial r} + \frac{2}{\rho}\frac{\partial V_z}{\partial z}\frac{\partial \lambda}{\partial z} \tag{57}$$

Here the following relation which is obtained by partial differentiation of Eq. (53) with respect to z has been used,

$$\frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial V_r}{\partial z} = 0$$

Equation (56) can be written as

$$\lambda = \frac{\tau_o}{\sqrt{2\left[\left(\frac{\partial V_r}{\partial r}\right)^2 + \left(\frac{\partial V_z}{\partial z}\right)^2 + \left(\frac{V_r}{r}\right)^2\right] + \left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)^2}} \tag{58}$$

It is assumed by the authors that by using the boundary layer approximation the following two expressions can be obtained.

$$\frac{\partial V_z}{\partial z} \approx -\frac{\partial V_r}{\partial r} \tag{59}$$

and

$$\phi \approx \frac{\partial V_z}{\partial r} + 2\,\frac{\partial V_r}{\partial r} \tag{60}$$

Under the assumption of Eqs. (59) and (60), Eq. (57) can be simplified as

$$v_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{dP}{dz} + \frac{\mu_o}{\rho}\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right)$$
$$+ \frac{\tau_o}{\rho}\frac{\partial}{\partial r}\left(\frac{\frac{\partial V_z}{\partial r}}{\frac{\partial V_z}{\partial r} + 2\,\frac{\partial V_r}{\partial r}}\right) \tag{61}$$

The equation of continuity is

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} = 0 \tag{62}$$

If the dimensionless variables

$$x = \frac{z}{R}, \quad y = \frac{r}{R} \tag{63}$$

$$u = \frac{v_z}{\bar{u}}, \quad v = \frac{v_r}{\bar{u}} \tag{65}$$

$$p' = \frac{p}{\rho \bar{u}^2} \tag{65}$$

$$N_{Re} = \frac{\rho R \bar{u}}{\mu_o} \tag{66}$$

$$\tau_o^* = \frac{\tau_o \; R}{\mu_o \; \bar{u}} \tag{67}$$

are intrduced, Eqs. (61) and (62) can be written as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{dp'}{dx} + \frac{1}{N_{Re}}\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right)$$

$$+ \frac{\tau_o^*}{N_{Re}} \frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}}\right) \tag{68}$$

$$\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial}{\partial y}(yv) = 0 \tag{69}$$

The boundary conditions are

$$\begin{aligned} u &= 1, \quad v = 0, \quad \text{at} \quad x = 0 \\ u &= 0, \quad v = 0, \quad \text{at} \quad y = 1, \quad x > 0 \end{aligned} \tag{70}$$

Combining Eqs. (69) and (68) yields

$$- \frac{u}{y} \frac{\partial}{\partial y}(yv) + v \frac{\partial u}{\partial y} = - \frac{dp'}{dx} + \frac{1}{N_{Re}}\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right)$$

$$+ \frac{\tau_o^*}{N_{Re}} \frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}}\right) \tag{71}$$

In the center core of uniform velocity region, it is assumed that

$$- \frac{dp'}{dx} = U^* \frac{dU^*}{dx} = \text{function of x only} \tag{72}$$

where $U^* = U/\bar{u}$ is the dimensionless center velocity.

If v is a function of y, Eq. (71) is a differential equation of u(y) for a cross section perpendicular to the pipe axis at given x. Let the functional E(u(y)) be

$$E(u(y)) = -\frac{u}{y}\frac{\partial}{\partial y}(yv) + v\frac{\partial u}{\partial y} - \frac{1}{N_{Re}}\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right)$$

$$+ \frac{\tau_0^*}{N_{Re}}\frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}}\right) \qquad (73)$$

Then, Eq. (71) becomes

$$E(u(y)) = -\frac{dp'}{dX} \qquad (74)$$

Let us consider the functional J(u, w) of u(y) and w(y),

$$J(u(y), w(y)) = \frac{\int_0^1 w\, E(u)\, y^2 dy}{\int_0^1 w\, y^2 dy} \qquad (75)$$

where w(y) is an auxiliary function. Here we want to show that solving the differential equation, Eq. (74), corresponds to the variational problem of finding the extremal of the functional J(u, w). To obtain the necessary conditions for the extremal of J(u, w), it is assumed that the variations δu and δw of u and w may be taken independently of each other. First, if we vary only w(y) and keep u(y) fixed, we obtain

$$\int_0^1 \left\{E(u)\int_0^1 \bar{w}\, y^2\, dy - \int_0^1 \bar{w}\, E(u)\, y^2\, dy\right\} y^2\, \delta w\, dy = 0 \qquad (76)$$

where $\bar{w}(y)$ is a piecewise-smooth arc which gives J(w) a weak relative extremum. For an arbitrary δw, Eq. (76) yields the relation

$$E(u) = \frac{\int_0^1 \bar{w}\, E(u)\, y^2 dy}{\int_0^1 \bar{w}\, y^2 dy} \tag{77}$$

The right-hand side of Eq. (77) is the extremal of J(u, w), and is a constant for given x

$$\begin{aligned} E(u) &= \text{constant} = \text{sationary value of } J(u, w) \\ &= -\frac{dp'}{dx} \end{aligned} \tag{78}$$

Equation (78) is a differential equation of the same form as that of Eq. (74). Thus the solution of Eq. (74) is equivalent to finding the stationary values of u and corresponding v which can be obtained from equation of continuity, and the pressure gradient $-dp'/dx$ is equal to the stationary value of functional J(u, w).

Next, let u(y) vary and δw be zero. The condition that J(u) has a stationary value is

$$\int_0^1 \Big[2y^2 \frac{\partial v}{\partial y} w + 3yvw + y^2 v \frac{\partial w}{\partial y} + \frac{1}{N_{Re}} \Big(y^2 \frac{\partial^2 w}{\partial y^2} + 3y \frac{\partial w}{\partial y} + w \Big)$$

$$+ 2 \frac{\tau_0^*}{N_{Re}} \frac{\partial^2}{\partial y^2} \left\{ \frac{w y^2 \frac{\partial v}{\partial y}}{\left(\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}\right)^2} \right\} - 2 \frac{\tau_0^*}{N_{Re}} \frac{\partial}{\partial y} \left\{ \frac{w y^2 \frac{\partial^2 v}{\partial y^2}}{\left(\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}\right)^2} \right\} \Big] \delta u\, dy$$

$$- \Big[\Big\{ w y^2 + \frac{1}{N_{Re}} \Big(y^2 \frac{\partial w}{\partial y} + y w \Big) + 2 \frac{\tau_0^*}{N_{Re}} \frac{w y^2 \frac{\partial^2 v}{\partial y^2}}{\left(\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}\right)^2}$$

$$+ 2\frac{\tau_o^*}{N_{Re}}\frac{\partial}{\partial y}\left[\frac{wy^2\frac{\partial v}{\partial y}}{\left(\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}\right)^2}\right]\Big\}\delta u\Bigg]_0^1$$

$$+\left[\left[\frac{1}{N_{Re}}wy^2+2\frac{\tau_o^*}{N_{Re}}\frac{wy^2\frac{\partial v}{\partial y}}{\left(\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}\right)^2}\right]\frac{\partial(\delta u)}{\partial y}\right]_0^1 \tag{79}$$

The boundary condition, $u = 0$ at $y = 1$ gives $\delta u = 0$ at $y = 1$, The boundary condition for $w(y)$ is assumed to be

$$w = 0 \quad \text{at} \quad y = 1$$

Hence, the second and third terms on the left-hand side of Eq. (79) become zero. From the macroscopic mass balance

$$\int_0^1 uydy = \text{constant} \tag{80}$$

we have

$$\int_0^1 y\delta udy = 0 \tag{81}$$

For arbitrary δu, Eq. (79) becomes

$$2y\frac{\partial v}{\partial y}w + 3vw + yv\frac{\partial w}{\partial y} + \frac{1}{N_{Re}}\left(y\frac{\partial^2 w}{\partial y^2}+3\frac{\partial w}{\partial y}+\frac{w}{y}\right)$$

$$+\frac{2}{y}\frac{\tau_o^*}{N_{Re}}\frac{\partial^2}{\partial y^2}\left[\frac{wy^2\frac{\partial v}{\partial y}}{\left(\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}\right)^2}\right]$$

$$+\frac{2}{y}\frac{\tau_o^*}{N_{Re}}\frac{\partial}{\partial y}\left[\frac{wy^2\frac{\partial^2 v}{\partial y^2}}{\left(\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}\right)^2}\right] = \text{constant} \tag{82}$$

Now, we shall solve the foregoing variational problem by Ritz's method (see Appendix III). The velocity distribution in the fully developed region is given in Appendix I. From the relationship $\tau_o/\tau_w = r_o/R = r_o^*$, the dimensionless form of the fully developed velocity profiles are

$$u = \frac{\tau_o^*}{2r_o^*}\left[(1 - r_o^*)^2 - (y - r_o^*)^2\right]$$

$$= \frac{6}{3 + 2r_o^* + r_o^{*2}}\left[1 - \left(\frac{y - r_o^*}{1 - r_o^*}\right)^2\right] \quad \text{for } r_o^* \leq y \leq 1 \qquad (83)$$

$$u = \frac{\tau_o^*(1 - r_o^*)^2}{2r_o^*} = \frac{6(1 - r_o^*)^2}{3 - 4r_o^* - r_o^{*4}} \quad \text{for } 0 \leq y \leq r_o^* \qquad (84)$$

The velocity distribution in the entrance region can be generalized in a form similar to Eqs. (83) and (84) as follows:

$$u = a\left[1 - \left(\frac{y - r_o^*}{1 - r_o^*}\right)^b\right] \qquad \text{for } r_o^* \leq y \leq 1 \qquad (85)$$

$$u = a, \qquad \text{for } 0 \leq y \leq r_o^* \qquad (86)$$

where a and b are functions of x only and a can be determined from the macroscopic mass balance

$$2\pi\int_0^{r_o} \bar{u}ar\,dr + 2\pi\int_{r_o}^{R} \bar{u}a\left[1 - \left(\frac{y - r_o^*}{1 - r_o^*}\right)^b\right] r\,dr = \pi R^2\bar{u} \qquad (87)$$

which gives

$$2\pi\bar{u}a\frac{r_o^2}{2} + 2\pi\bar{u}a\left(\frac{R^2 - r_o^2}{2}\right) - \frac{2\pi\bar{u}aR^2}{(1-r_o^*)^b}\int_{r_o^*}^{1}(y - r_o^*)^b y\,dy = \pi R^2\bar{u} \qquad (88)$$

The integral of the last term on the left-hand side of Eq. (88) can be integrated by parts,

$$\int_{r_o^*}^{1}(y - r_o^*)^b y\,dy = \frac{y}{b+1}(y - r_o^*)^{b+1}\Big|_{r_o^*}^{1} - \int_{r_o^*}^{1}\frac{(y - r_o^*)^{b+1}}{b+1}dy$$

$$= \frac{(1 - r_o^*)^{b+1}}{b+1} - \frac{(1 - r_o^*)^{b+2}}{(b+1)(b+2)} \qquad (89)$$

Hence, Eq. (88) becomes

$$a\left[1 - \frac{2(1 + b - br_o^* - r_o^{*2})}{(b+1)(b+2)}\right] = 1 \qquad (90)$$

or

$$a = \frac{(b+1)(b+2)}{b^2 + (2r_o^* + 1)b + 2r_o^{*2}} \qquad (91)$$

Thus Eqs. (85) and (86) become

$$u = \frac{(b+1)(b+2)}{b^2 + (2r_o^* + 1)b + 2r_o^{*2}}\left[1 - \left(\frac{y - r_o^*}{1 - r_o^*}\right)^b\right] \quad \text{for } r_o^* \leq y \leq 1 \qquad (92)$$

$$u = \frac{(b+1)(b+2)}{b^2 + (2r_o^* + 1)b + 2r_o^{*2}} \quad \text{for } 0 \leq y \leq r_o^* \qquad (93)$$

At the tube entry, $x = 0$, $u = 1$, i.e., $b \to \infty$, and in the fully developed region, $x \to \infty$, $b \to 2$.

Let b_o be the value of b which gives the functional J an extremum. The distribution of u is then given by Eqs. (92) and (93) with b_o in place of b. The corresponding v can be obtained from the equation of continuity, Eq. (69).

In determining the auxiliary function w, let us first obtain w in the fully developed region. In the fully developed region, i.e., in the region where $x \to \infty$ and $v = 0$, Eq. (82) becomes

$$\frac{1}{N_{Re}}\left(y\frac{\partial^2 w}{\partial y^2} + 3\frac{\partial w}{\partial y} + \frac{w}{y}\right) = \text{constant} = k \tag{94}$$

The solution of Eq. (94) with the boundary condition, w = 0 at y = 1, is

$$w = -\frac{k N_{Re}}{4}\,\frac{1-y^2}{y} \tag{95}$$

The generalized auxiliary function w in the entrance region for arbitrary x is

$$w = \left(\frac{1-y^{b_o}}{y}\right)\left[1 + c\left(\frac{y - r_o^*}{1 - r_o^*}\right)^2\right] \qquad \text{for } r_o^* \leq y \leq 1 \tag{96}$$

$$w = \frac{1 - r_o^{*\,b_o}}{y} \qquad \text{for } 0 \leq y \leq r_o^* \tag{97}$$

where c is determined by the condition that J has an extremum.

Substituting the quantities u, v and w into the functional J in Eq. (75), we obtain J as a function of b and c,

$$J = J(b, c) \tag{98}$$

Since the results are lengthy the reader is referred to the original paper (29). The conditions that J has an extremum are

$$\frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial c} = 0 \tag{99}$$

The latter condition gives c = 0 and we obtain

$$\frac{1}{N_{Re}} \frac{dx}{db_o} = F(b_o, r_o^*) \tag{100}$$

Equation (100) can be integrated from x = 0, $b_o = \infty$ to x and b_o

$$\frac{x}{N_{Re}} = \int_{\infty}^{b_o} F(b_o, r_o^*)\, db_o \tag{101}$$

From Eq. (101) the relation between b_o and x can be obtained by numerical integration. Thus, from Eqs. (92) and (93) the velocity distribution at each value of x in the entrance region can also be obtained. The entrance length $x_e = z_e/R$ is obtained corresponding to the value of b_{oe}, where $(b_{oe}-2)/2 = 0.01$, i.e., 99% of the fully developed value of b_o. The entrance length $z_e^* = z_e/RN_{Re}$ is plotted against r_o^* as shown in Fig. 24. For Newtonian fluid, $r_o^*=0$, $z_e^*=0.244$.

Since the pressure gradient $- dp'/dx$ is equal to the stationary value of J, we put c=0 in Eq. (98) and integrat the resulting equation from the tube entry x=0 to an arbitrary cross section to obtain the pressure drop in the entrance region as shown in Fig. 27. The correction factor C defined by

$$\frac{|\Delta p|}{\frac{1}{2}\rho \bar{u}^2} = \frac{|\Delta p'|}{\frac{1}{2}} = \frac{48\, x}{N_{Re}\,(3 - 4\lambda_o^* + \lambda_o^{*4})} + C \qquad \text{for } x > x_e \tag{102}$$

and

$$C = \frac{|\Delta p_e'|}{\frac{1}{2}} - \frac{48\, x_e}{N_{Re}\,(3 - 4\lambda_o^* + \lambda_o^{*4})} \tag{103}$$

can be obtained by using the results of pressure drop in the entrance region and entrance length as shown in Fig. 26.

Discussion

It is worth noting that the assumed velocity profile, Eq. (92) will not give true uniform velocity in the region where $r_o^* \leq y \leq 1$ and where shear stress is less than the yield stress τ_o. Nevertheless, in the region near the tube entry the large portion of velocity profile will remain almost flat except near the wall. This is because b approaches infinity at the tube entry and decreases gradually to the fully developed value 2 at $x \to \infty$.

The experiments performed by Michiyochi et al. (29) with alumina-water slurry indicate that the entrance length for the Bingham fluid is shorter than the Newtonian fluid. Accordingly, the measurements were fairly difficult and the results can be used qualitatively only.

It has been pointed out at the beginning of this section that the validity of the approximate form of equation of motion, Eq. (61), is doubtful. We shall derive the equation of motion equivalent to the form of Eq. (61) by following the authors' assumptions and derivation. Let us start with Eq. (57). Equation (57) can be written as

$$v_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu_o}{\rho}\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right)$$

$$+ \frac{\lambda}{\rho}\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{\rho}\frac{\partial V_z}{\partial r}\frac{\partial \lambda}{\partial r} + \frac{\lambda}{\rho r}\frac{\partial V_z}{\partial r} + \frac{\mu_o}{\rho}\frac{\partial^2 V_z}{\partial z^2} + \frac{\lambda}{\rho}\frac{\partial^2 V_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial V_r}{\partial z}\frac{\partial \lambda}{\partial r}$$

$$+ \frac{2}{\rho}\frac{\partial V_z}{\partial z}\frac{\partial \lambda}{\partial z} \tag{104}$$

If Eq. (60) is assumed to be correct the third and fourth terms on the right-hand side of Eq. (104) can be rewritten as follows:

$$\frac{\lambda}{\rho}\frac{\partial^2 V_z}{\partial r^2} - \frac{1}{\rho}\frac{\partial V_z}{\partial r}\frac{\partial \lambda}{\partial r} = \frac{1}{\rho}\frac{\partial}{\partial r}\left(\lambda\frac{\partial V_z}{\partial r}\right) \approx \frac{\tau_0}{\rho}\frac{\partial}{\partial r}\left(\frac{\frac{\partial V_z}{\partial r}}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}}\right) \tag{105}$$

Combining Eqs. (104) and (105) we have

$$\begin{aligned} v_r\frac{\partial V_z}{\partial r} + V_z\frac{\partial V_z}{\partial z} = {} & -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu_0}{\rho}\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right) + \frac{\tau_0}{\rho}\frac{\partial}{\partial r}\left(\frac{\frac{\partial V_z}{\partial r}}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}}\right) \\ & + \frac{\tau_0}{\rho r}\frac{\frac{\partial V_z}{\partial r}}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}} + \frac{\mu_0}{\rho}\frac{\partial^2 V_z}{\partial z^2} + \frac{\tau_0}{\rho}\left(\frac{\frac{\partial^2 V_z}{\partial z^2}}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}}\right) \\ & + \frac{\tau_0}{\rho}\left(\frac{\frac{\partial V_r}{\partial z}}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}}\right) + \frac{2\tau_0}{\rho}\frac{\partial V_z}{\partial z}\frac{\partial}{\partial z}\left(\frac{1}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}}\right) \end{aligned} \tag{106}$$

The reduction of Eq. (106) to Eq. (61) based on the boundary layer concept cannot be achieved because the fourth term on the right-hand side of Eq. (106) is not neglegible. Therefore, the term

$$\frac{\tau_0}{\rho r}\frac{\frac{\partial V_z}{\partial r}}{\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}}$$

should be added in Eq. (61). Later the authors made the assumption in the same paper that

$$\frac{\partial V_z}{\partial r} \approx \left(\frac{\partial V_z}{\partial r} + 2\frac{\partial V_r}{\partial r}\right) \tag{107}$$

If the assumption Eq. (107) is applied at this stage to Eq. (106), the following boundary layer equation derived in Section 3.2 and used in Section 5.1 and 5.2 is obtained.

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = - \frac{1}{\rho} \frac{dP}{dz} + \frac{\mu_0}{\rho} \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right) + \frac{\tau_0}{\rho r} \frac{\frac{\partial v_z}{\partial r}}{\left| \frac{\partial v_z}{\partial r} \right|} \tag{108}$$

5.4 CONCLUSION

The theoretical analysis of the entrance region flow of the Bingham fluid in a circular pipe was first presented in the paper by Michiyochi et al. (29), using the variational method. The method involved the boundary layer simplifications and variational arguments. The validity of these simplifications and arguments needs further verification.

The starting form of the governing equation used by the variational method should be modified. The proposed velocity profile used in this method would not provide the true flow behavior of the Bingham fluid. Furthermore, the method needs an excessive mathematic procedure as shown in the original paper. The results from this theoretical analysis can not be justified with one or two points of pressure drop in the entrance region from the experiments performed by the authors with an alumina-water slurry.

For the entrance length and the pressure correction factor, the results from the variational method show reasonable trend, i.e., both decrease with increasing plug flow radius. The pressure drop curves for various plug flow radius cross each other in the entrance and the fully developed region. This phenomenon may contradict the actual physical situation.

The basic momentum integral method and the Campbell-Slattery method have been used in this work to study the Bingham entrance region flow in a circular pipe. Solutions of the governing equations obtained by using both methods appear to give rise to the characteristics of the Bingham fluid. The momentum integral method is historically important in the field of fluid dynamics of the boundary layer type. The results obtained by the momentum integral method provide some general idea of the Bingham flow in the entrance region.

The Campbell-Slattery method for Newtonian entrance region flow problem has been accepted as a successful approach and is discussed in detail in their paper. The curves of entrance length and pressure drop correction factor for Bingham fluids show the same trend but are somewhat higher than those from the variational method.

The pressure drop curves in the entrance region have no crossing between each other and are consistent with the case of fully developed flow, i.e., for higher plug flow radius, the pressure drop is higher. Therefore, we may conclude that the Campbell-Slattery method which involves less mathematics provides a solution which is good in over-all description of actual physical behavior of a Bingham fluid in the entrance region of a circular pipe.

NOTATION

A = parameter in the Prandtl-Eyring model

A = area

A_i = constant defined by Eq. (96), Chapter 4

a = parameter used in Eq. (85), Chapter 5

B = parameter in the Prandtl-Eyring model

b = parameter used in Eq. (85), Chapter 5

b_0 = optimal value of b

C = pressure drop correction factor

C_1, C_2... = constant defined by Eqs. (15) through (18), Chapter 4

c = parameter used in Eq. (96), Chapter 5

E = function defined by Eq. (73), Chapter 5

E_v = total rate of viscous dissipation of mechanical energy

F = function defined in Eq. (100), Chapter 5

f = function used in the matching method, Chapter 4

h = half width of parallel-plate channel

J = functional defined by Eq. (75), Chapter 5

K_1, K_2, ... = constant defined by Eq. (20) through (23), Chapter 4

k = constant used in Eq. (95), Chapter 5

K_i = constant defined in Eq. (87), Chapter 4

L = axial length of a channel or a pipe or a characteristic length

M = minimum value defined in Appendix III

M_n = minimum value defined in Appendix III

m = parameter in the power law model

m' = parameter in the power law model, $m^{1/n}$

N = number of grid point used in the finite difference method, Chapter 4

n = parameter in the power law model, dimensionless

n' = parameter in the power law model, $1/n$

P = pressure in the fully developed flow

P = dimensionless pressure used in Chapter 3

p = fluid pressure

p' = dimensionless pressure, $p/\rho\bar{u}^2$, used in the variational method

p^* = dimensionless pressure, $p/(\frac{1}{2}\rho\bar{u}^2)$

p_o = pressure at the entry of a conduit

Q = volume rate of flow

q = quantity defined by Eq. (33), Chapter 5

R = radius of a circular pipe

R = dimensionless coordinate in the r direction, used in Chapter 3

r = radial distance in the cylindrical coordinates

r_o = radius of the plug flow region of the Bingham fluid in a pipe

r_o^* = dimensionless quantity of r_o, r_o/R

S = cross-sectional area

t = time

U = center-line or plug flow velocity

U = dimensionless velocity in the flow direction used in Chapter 3

U^* = dimensionless center-line or plug flow velocity, $U/\bar{u}$

U = characteristic velocity used in Chapter 3

u = velocity in the x direction

u = dimensionless velocity in the flow direction used in the variational method, $v_z/\bar{u}$

$\bar{u}$ = average velocity, $Q/\pi R^2$ or $Q/2hW$

u^* = perturbation velocity

$\underline{V}$ = velocity vector

V = dimensionless velocity in the y direction or r direction used in Chapter 3

$\underline{v}$ = velocity vector used in Chapter 5

v = dimensionless velocity in the r direction used in the variational method

v_i, v_j, v_r, v_x, v_y, v_z, v_θ = velocity components in the i, j, r, x, y, z, θ direction respectively.

W = width of a plat

w = auxiliary function used in the variational method

w = mass flow rate

$\bar{w}$ = optimal value of the function w

X = dimensionless x coordinate, x/L

X = dimensionless x coordinate defined by Eq. (111), Chapter 4

x = the rectangular coordinate

x = dimensionless coordinate in the z direction, z/R

x_i, x_j = the i and j coordinates

Y = dimensionless y coordinate, y/δ_L

Y = dimensionless y coordinate, $1 - y/h$

y = the rectangular coordinate

y = dimensionless coordinate in the r direction, r/R

Z = dimensionless z coordinate, z/L

z = the rectangular or cylindrical coordinate

z^* = dimensionless axial distance, z/RN_{Re}

Greek Letters

α = parameter in the Satterby or Ellis model

α = parameter defined in Eq. (63), Chapter 4

α_o = optimal value of α

β = parameter in the Satterby model

γ = shear strain

$\underline{\underline{\Delta}}$ = rate of deformation tensor

Δ_{ij} = component of rate of deformation tensor

$\Delta a = a_2 - a_1$, in which 1 and 2 refer to two control surfaces

δ, δ_L = boundary layer thickness

δ^* = dimensionless boundary layer thickness, δ/R

δ_s = shear stress boundary layer

δw = variation of the auxiliary function w

ϵ = parameter introduced in the variational method

ϵ = dimensionless coordinate of x, defined by Eq. (82), Chapter 4

η = non-Newtonian viscosity

η = dimensionless variable used in the matching method

η_o = parameter in the Satterby, Ellis or Meter-Bud model

η_∞ = parameter in the Meter-Bud model

θ =the cylindrincal coordinate

λ = quantity defined by Eq. (56), Chapter 5

λ_i = eigenvalue defined by Eq. (110), Chapter 4

μ = viscosity

μ_o = parameter in the Bingham or Reiner-Philippoff model

μ_∞ = parameter in the Reiner-Philippoff model

ρ = density

σ = dimensionless coordinate defined by Eq. (130), Chapter 4

$\underline{\underline{\tau}}$ = shear stress tensor

τ_{ij} = component of shear stress

τ_o = parameter in the Bingham model, or called yield stress

τ_o^* = dimensionless parameter in the Bingham model, $\tau_o R/\mu_o \bar{u}$

τ_w = shear stress at the solid wall

$\tau_{\frac{1}{2}}$ = parameter in the Ellis model

τ_m = parameter in the Meter-Bud model

τ_s = parameter in the Reiner-Philippoff model

ϕ = quantity defined by Eqs. (7), and (8), Chapter 3

ϕ = function defined by Eq. (52), Chapter 4

ψ = stream function used in the matching method

φ_i = eigenfunction defined by Eq. (110), Chapter 4

ψ_o, ψ_1 = parameters in the Ellis model

Subscripts

e = at entrance length

f = in the fully developed region

Dimensionless Groups

N_{Re} = Reynolds number

$\frac{R\bar{u}\rho}{\mu}$, $\frac{L\bar{u}\rho}{\mu}$ (Newtonian model)

$\frac{\rho(2R)^n\,\bar{u}^{2-n}}{m}$, $\frac{\rho(2h)^n\,\bar{u}^{2-n}}{m}$, $\frac{\rho\,L^n\,U^{2-n}}{m}$ (power law model)

$\frac{\rho R\bar{u}}{\mu_o}$, $\frac{\rho L U_\infty}{\mu_o}$ (Bingham model)

Mathematics Operations

$\frac{D}{Dt}$ = substantial derivative

$\underline{\nabla}$ = the " del" or " nabla" operator

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APPENDIX I

LAMINAR FULLY DEVELOPED FLOW OF SOME TIME-INDEPENDENT NON-NEWTONIAN FLUIDS IN PIPES AND CHANNELS

The laminar flow of time-independent non-Newtonian fluids in the fully developed region of pipes and channels is presented in Section 2.2-2. The general expressions of the shear stress distribution, pressure drop, velocity profile, and volume rate of flow are derived there without specifying the fluid model. In this Appendix we shall restrict these expressions to those of four fluid models, namely, Newtonian, power law, Bingham and Ellis fluids.

AI.1 Pipe Flow

Newtonian Fluids

From Newton's law of viscosity we have

$$\tau_{rz} = -\mu \frac{dv_z}{dr}$$

and

$$f(\tau_{rz}) = \frac{1}{\mu}\tau_{rz}$$

The volume rate of flow Q becomes

$$Q = \frac{\pi R^3}{\tau_w^3}\int_0^{\tau_w} \tau_{rz}^2 \left(\frac{1}{\mu}\tau_{rz}\right) d\tau_{rz}$$

$$= \frac{\pi R^3 \tau_w}{4\mu} = \frac{\pi R^4 |\Delta P|}{8L\mu}$$

The average velocity $\bar{u}$ is

$$\bar{u} = \frac{Q}{\pi R^2} = \frac{R\tau_w}{4\mu} = \frac{R^2|\Delta P|}{8L\mu}$$

The velocity distribution becomes

$$v_z = -\int \frac{1}{\mu}\left(\frac{\tau_w}{R} r\right) dr$$

$$= -\frac{\tau_w}{\mu R}\frac{r^2}{2} + \frac{\tau_w R}{2\mu}$$

$$= \frac{\tau_w R}{2\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

The maximum velocity U occurs at the center line of the tube $r = 0$,

$$U = \frac{\tau_w R}{2\mu} = \frac{R^2 |\Delta P|}{4\mu L}$$

In terms of U and $\bar{u}$, v_z can be expressed by

$$v_z = U\left[1 - \left(\frac{r}{R}\right)^2\right]$$

and

$$v_z = 2\bar{u}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Power Law Fluids

The flow equation of the power law fluid can be expressed as

$$\tau_{rz} = -m\left|\frac{dv_z}{dr}\right|^{n-1}\left(\frac{dv_z}{dr}\right)$$

Since du/dr is always negative in a circular tube, we have

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n$$

and

$$f(\tau_{rz}) = \left(\frac{\tau_{rz}}{m}\right)^{\frac{1}{n}}$$

The volume rate of flow Q becomes

$$Q = \frac{\pi R^3}{\tau_w^3}\int_0^{\tau_w} \tau_{rz}^2 \left(\frac{\tau_{rz}}{m}\right)^{\frac{1}{n}} d\tau_{rz}$$

$$= \pi R^3 \frac{n}{3n+1} \left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} = \frac{n\pi R^3}{3n+1} \left(\frac{R|\Delta P|}{2Lm}\right)^{\frac{1}{n}}$$

The average velocity , $\bar{u}$, is

$$\bar{u} = \frac{nR}{3n+1} \left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} = \frac{nR}{3n+1} \left(\frac{R|\Delta P|}{2Lm}\right)^{\frac{1}{n}}$$

The velocity distribution becomes

$$v_z = -\int \left(\frac{\tau_w}{mR}\right)^{\frac{1}{n}} r^{\frac{1}{n}} dr = -\left(\frac{\tau_w}{mR}\right)^{\frac{1}{n}} \frac{n}{n+1} r^{\frac{n+1}{n}} + \left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} \frac{nR}{n+1}$$

$$= \left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} \frac{nR}{n+1} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

and

$$U = \left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} \frac{nR}{n+1}$$

In terms of $\bar{u}$ and U, we have

$$v_z = \frac{3n+1}{n+1} \bar{u} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

and

$$v_z = U\left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

Bingham Fluids

The flow equation for a Bingham fluid is

$$\tau_{rz} = -\mu_0 \frac{dv_z}{dr} + \tau_0,$$

and

$$f(\tau_{rz}) = \frac{1}{\mu_0}(\tau_{rz} - \tau_0), \quad \text{if } |\tau_{rz}| > \tau_0$$

$$f(\tau_{rz}) = 0, \quad \text{if } |\tau_{rz}| < \tau_0$$

The volume rate of flow Q becomes

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau_{rz}^2 \left[\frac{1}{\mu_o}(\tau_{rz} - \tau_o)\right] d\tau_{rz}$$

$$= \frac{\pi R^3}{\mu_o \tau_w^3} \left(\frac{\tau_w^4}{4} - \frac{\tau_w^3 \tau_o}{3} - \frac{\tau_o^4}{4} + \frac{\tau_o^4}{3}\right)$$

$$= \frac{\pi R^3 \tau_w}{4\mu_o}\left[1 - \frac{4}{3}\frac{\tau_o}{\tau_w} + \frac{1}{3}\left(\frac{\tau_o}{\tau_w}\right)^4\right]$$

The average velocity is

$$\bar{u} = \frac{R\tau_w}{4\mu_o}\left[1 - \frac{4}{3}\frac{\tau_o}{\tau_w} + \frac{1}{3}\left(\frac{\tau_o}{\tau_w}\right)^4\right]$$

The velocity distribution becomes

$$v_z = -\int \frac{1}{\mu_o}\left(\frac{\tau_w}{R} r - \tau_o\right) dr$$

$$= \frac{\tau_o}{\mu_o} r - \frac{\tau_w}{2\mu_o R} r^2 + \frac{\tau_w R}{2\mu_o} - \frac{\tau_o R}{\mu_o}$$

$$= \frac{\tau_w R}{2\mu_o}\left[1 - \left(\frac{r}{R}\right)^2\right] - \frac{\tau_o R}{\mu_o}\left(1 - \frac{r}{R}\right) \qquad \text{(1-1)}$$

which is valid in the region $|\tau_{rz}| > \tau_o$. The location r_o at which $|\tau_{rz}| = \tau_o$ can be obtained from Eqs. (16) and (18) in Chapter 2.

$$r_o = \frac{2L}{|\Delta p|}\tau_o = \frac{\tau_o}{\tau_w} R$$

Therefore, the velocity distribution in the region $|\tau_{rz}| < \tau_o$ is constant and can be obtained by replacing r by r_o in Eq. (1-1)

$$U = \frac{\tau_w R}{2\mu_o}\left[1 - \left(\frac{\tau_o}{\tau_w}\right)^2\right] - \frac{\tau_o R}{\mu_o}\left(1 - \frac{\tau_o}{\tau_w}\right)$$

$$= \frac{\tau_w R}{2\mu_o}\left(1 - \frac{\tau_o}{\tau_w}\right)^2$$

The Ellis Model

The flow equation of the Ellis model is

$$\tau_{rz} = - \frac{\eta_0}{1 + \left(\frac{\tau_{rz}}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}} \left(\frac{dv_z}{dr}\right)$$

and

$$f(\tau_{rz}) = \frac{\tau_{rz}}{\eta_0}\left[1 + \left(\frac{\tau_{rz}}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$$

The volume rate of flow and the average velocity become

$$Q = \frac{\pi R^3}{\tau_w^3}\int_0^{\tau_w} \tau_{rz}^2 \, \frac{\tau_{rz}}{\eta_0}\left[1 + \left(\frac{\tau_{rz}}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right] d\tau_{rz}$$

$$= \frac{\pi R^3 \tau_w}{4\eta_0}\left[1 + \frac{4}{\alpha+3}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$$

and

$$\bar{u} = \frac{R\,\tau_w}{4\eta_0}\left[1 + \frac{4}{\alpha+3}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$$

The differential equation for velocity distribution can be integrated as follows:

$$v_z = -\int \frac{\tau_w r}{\eta_0 R}\left[1 + \left(\frac{\tau_w r}{\tau_{\frac{1}{2}} R}\right)^{\alpha-1}\right] dr$$

$$= -\frac{\tau_w}{2\eta_0 R}\left[1 + \frac{2}{\alpha+1}\left(\frac{\tau_w r}{\tau_{\frac{1}{2}} R}\right)^{\alpha-1}\right] r^2 + C$$

where

$$C = \frac{\tau_w R}{2\eta_0}\left[1 + \frac{2}{\alpha+1}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$$

Hence

$$v_z = \frac{\tau_w R}{2\eta_0}\left\{\left[1 - \left(\frac{r}{R}\right)^2\right] + \frac{2}{\alpha+1}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\left[1 - \left(\frac{r}{R}\right)^{\alpha+1}\right]\right\}$$

and

$$U = \frac{\tau_w R}{2\eta_o}\left[1 + \frac{2}{\alpha+1}\left(\frac{\tau_w}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$$

AI.2 Channel Flow

Newtonian Fluids

For Newtonian fluids, $f(\tau_{yx})$ is given by

$$f(\tau_{yx}) = \frac{1}{\mu}\tau_{yx}$$

The volume rate of flow Q and the average velocity $\bar{u}$ become

$$Q = \frac{2Wh^2\tau_w}{3\mu} = \frac{2Wh^3|\Delta P|}{3\mu L}$$

and

$$\bar{u} = \frac{h\tau_w}{3\mu} = \frac{h^2|\Delta P|}{3\mu L}$$

The velocity distribution and the center line velocity become

$$v_x = -\int_h^y \frac{\tau_w}{\mu h} y\,dy = -\frac{\tau_w}{\mu h}\left(\frac{y^2}{2} - \frac{h^2}{2}\right)$$

$$= \frac{\tau_w h}{2\mu}\left[1 - \left(\frac{y}{h}\right)^2\right] = \frac{h^2|\Delta P|}{2\mu L}\left[1 - \left(\frac{y}{h}\right)^2\right]$$

$$U = \frac{\tau_w h}{2\mu} = \frac{h^2|\Delta P|}{2\mu L}$$

Power Law Fluids

The function $f(\tau_{yx})$ for the power law fluid is

$$f(\tau_{yx}) = \left(\frac{\tau_{yx}}{m}\right)^{\frac{1}{n}}$$

The volume rate of flow Q and the average velocity become

$$Q = \frac{2Wh^2}{\tau_w^2}\int_0^{\tau_w} \tau_{yx}\left(\frac{\tau_{yx}}{m}\right)^{\frac{1}{n}} d\tau_{yx}$$

$$= \frac{2Wh^2 \tau_w^{\frac{1}{n}}}{m^{\frac{1}{n}}} \quad \frac{n}{2n+1}$$

$$= \frac{n}{2n+1} \quad \frac{2Wh^2}{m^{\frac{1}{n}}} \left(\frac{h}{L}\,|\Delta P|\right)^{\frac{1}{n}}$$

and

$$\bar{u} = \frac{n}{2n+1} \quad \frac{h\tau_w^{\frac{1}{n}}}{m^{\frac{1}{n}}} = \frac{nh}{2n+1}\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}$$

The velocity distribution and the maximum velocity become

$$v_x = -\int_h^y \left(\frac{1}{m}\,\frac{\tau_w}{h}\,y\right)^{\frac{1}{n}} dy$$

$$= -\frac{1}{m^{\frac{1}{n}}}\left(\frac{\tau_w}{h}\right)^{\frac{1}{n}} \frac{n}{n+1}\left(y^{\frac{n+1}{n}} - h^{\frac{n+1}{n}}\right)$$

$$= \frac{n}{n+1}\left(\frac{\tau_w}{h}\right)^{\frac{1}{n}} \frac{h^{\frac{n+1}{n}}}{m^{\frac{1}{n}}}\left[1 - \left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

and

$$U = \frac{n}{n+1}\left(\frac{\tau_w}{h}\right)^{\frac{1}{n}} \frac{h^{\frac{n+1}{n}}}{m^{\frac{1}{n}}} = \frac{nh}{n+1}\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}$$

<u>Bingham Fluids</u>

The function of $f(\tau_{yx})$ for the Bingham fluid is

$$f(\tau_{yx}) = \frac{1}{\mu_0}(\tau_{yx} - \tau_0)$$

The volume rate of flow Q and the average velocity become

$$Q = \frac{2Wh^2}{\tau_w^2}\int_0^{\tau_w} \frac{1}{\mu_0}(\tau_{yx}^2 - \tau_0\tau_{yx})\,d\tau_{yx}$$

$$= \frac{2Wh^2}{\mu_0\tau_w^2}\left\{\frac{1}{3}(\tau_w^3 - \tau_0^3) - \frac{1}{2}(\tau_0\tau_w^2 - \tau_0^3)\right\}$$

$$= \frac{2Wh^2\tau_w}{3\mu_o}\left[1 - \frac{3}{2}\left(\frac{\tau_o}{\tau_w}\right) + \frac{1}{2}\left(\frac{\tau_o}{\tau_w}\right)^3\right]$$

$$\bar{u} = \frac{h\tau_w}{3\mu_o}\left[1 - \frac{3}{2}\left(\frac{\tau_o}{\tau_w}\right) + \frac{1}{2}\left(\frac{\tau_o}{\tau_w}\right)^3\right]$$

The velocity distribution and the plug flow velocity which occurs at $y_o \geq |y|$, become

$$v_x = -\int_h^y \frac{1}{\mu_o}\left(\frac{\tau_w}{h}y - \tau_o\right)dy$$

$$= \frac{1}{\mu_o}\left[\tau_o y - h\tau_o - \frac{\tau_w}{2h}y^2 + \frac{\tau_w h}{2}\right]$$

$$= \frac{\tau_w h}{2\mu_o}\left[1 - \left(\frac{y}{h}\right)^2\right] - \frac{\tau_o h}{\mu_o}\left(1 - \frac{y}{h}\right), \quad y_o \leq y \leq h.$$

$$U = \frac{\tau_w h}{2\mu_o}\left[1 - \left(\frac{y_o}{h}\right)^2\right] - \frac{\tau_o h}{\mu_o}\left(1 - \frac{y}{h}\right)$$

$$= \frac{\tau_w h}{2\mu_o}\left[1 - \left(\frac{\tau_o}{\tau_w}\right)^2\right] - \frac{\tau_o h}{\mu_o}\left(1 - \frac{\tau_o}{\tau_w}\right)$$

$$= \frac{\tau_w h}{2\mu_o}\left(1 - \frac{\tau_o}{\tau_w}\right)^2$$

<u>The Ellis Model</u>

The function $f(\tau_{yx})$ for the Ellis model is

$$f(\tau_{yx}) = \frac{\tau_{yx}}{\eta_o}\left[1 + \left(\frac{\tau_{yx}}{\tau_{\frac{1}{2}}}\right)^{\alpha-1}\right]$$

Hence,

$$Q = \frac{2Wh^2}{\eta_o \tau_w^2} \int_0^{\tau_w} \tau_{yx}^2 \left[1 + \left(\frac{\tau_{yx}}{\tau_{\frac{1}{2}}} \right)^{\alpha - 1} \right] d\tau_{yx}$$

$$= \frac{2Wh^2}{\eta_o \tau_w^2} \left[\frac{1}{3} \tau_w^3 + \frac{1}{\alpha+2} \frac{1}{\tau_{\frac{1}{2}}^{\alpha-1}} \tau_w^{\alpha+2} \right]$$

$$= \frac{2Wh^2 \tau_w}{3\eta_o} \left[1 + \frac{3}{\alpha+2} \left(\frac{\tau_w}{\tau_{\frac{1}{2}}} \right)^{\alpha-1} \right]$$

$$\bar{u} = \frac{h\tau_w}{3\eta_o} \left[1 + \frac{3}{\alpha+2} \left(\frac{\tau_w}{\tau_{\frac{1}{2}}} \right)^{\alpha-1} \right]$$

$$v_x = -\int_h^y \frac{\tau_w y}{h\eta_o} \left[1 + \left(\frac{\tau_w y}{\tau_{\frac{1}{2}} h} \right)^{\alpha-1} \right] dy$$

$$= -\frac{\tau_w}{h\eta_o} \left\{ \frac{y^2}{2} - \frac{h^2}{2} + \left(\frac{\tau_w}{\tau_{\frac{1}{2}} h} \right)^{\alpha-1} \left[\frac{y^{\alpha+1}}{\alpha+1} - \frac{h^{\alpha+1}}{\alpha+1} \right] \right\}$$

$$= \frac{\tau_w h}{2\eta_o} \left\{ \left[1 - \left(\frac{y}{h} \right)^2 \right] + \left(\frac{\tau_w}{\tau_{\frac{1}{2}}} \right)^{\alpha-1} \frac{2}{\alpha+1} \left[1 - \left(\frac{y}{h} \right)^{\alpha+1} \right] \right\}$$

$$U = \frac{\tau_w h}{2\eta_o} \left[1 + \frac{2}{\alpha+1} \left(\frac{\tau_w}{\tau_{\frac{1}{2}}} \right)^{\alpha-1} \right]$$

APPENDIX II

DERIVATION OF EQUATIONS IN CHAPTER 5

AII.1 DERIVATION OF EQ. (7)

It follows from the equation of continuity, Eq. (1), that the transverse velocity component v_r can be obtained by multiplying each term by r and integrating it from the wall r=R to r, as follows

$$v_r = -\frac{1}{r}\int_R^r \frac{\partial v_z}{\partial z} r\,dr \tag{1-1}$$

in which no slip condition at the wall, $v_r = 0$ for $r = R$, has been used. This expression of v_r is then inserted to Eq. (6) to obtain

$$\int_R^{R-\delta_s} \left[-\frac{1}{r}\frac{\partial v_z}{\partial r}\int_R^r \frac{\partial v_z}{\partial z} r\,dr + v_z \frac{\partial v_z}{\partial z} - U\frac{dU}{dz}\right] r\,dr = \frac{R}{\rho}\tau_w \tag{1-2}$$

where

$$\tau_w = \tau_{rz}\big|_{r=R} \qquad = -\mu_o\left(\frac{\partial v_z}{\partial r}\right)_{r=R} + \tau_o \tag{1-3}$$

The first term can be integratrd by using the technique of integration by parts as follows:

$$-\int_R^{R-\delta_s} \frac{\partial v_z}{\partial r}\left[\int_R^r \frac{\partial v_z}{\partial z} r\,dr\right] dr$$

$$= -\left(\int_R^r \frac{\partial v_z}{\partial z} r\,dr\right)(v_z)\Big|_R^{R-\delta_s} + \int_R^{R-\delta_s} v_z \frac{\partial v_z}{\partial z} r\,dr$$

$$= -U\int_R^{R-\delta_s} \frac{\partial v_z}{\partial z} r\,dr + \frac{1}{2}\int_R^{R-\delta_s} \frac{\partial}{\partial z}(v_z^2)\, r\,dr$$

$$= -U \frac{d}{dz}\int_R^{R-\delta_s} v_z r\,dr + \frac{1}{2}\frac{d}{dz}\int_R^{R-\delta_s} v_z^2 r\,dr \qquad (1\text{-}4)$$

in which the boundary condition, $v_z = 0$ at $r = R$ and $v_z = U$ at $r = R - \delta_s$ are used. Substituting the expressions in Eqs. (1-3) and (1-4) into Eq. (1-2) gives

$$- U\frac{d}{dz}\int_R^{R-\delta_s} v_z r\,dr + \frac{1}{2}\frac{d}{dz}\int_R^{R-\delta_s} v_z^2 r\,dr + \frac{1}{2}\frac{d}{dz}\int_R^{R-\delta_s} v_z^2 r\,dr$$

$$-U\frac{dU}{dz}\int_R^{R-\delta_s} r\,dr = -\frac{\mu_o R}{\rho}\left(\frac{\partial v_z}{\partial r}\right)_{r=R} + \frac{R}{\rho}\tau_o \qquad (1\text{-}5)$$

Since

$$\frac{d}{dz}\int_R^{R-\delta_s} U v_z r\,dr = \frac{d}{dz}\left[U\int_R^{R-\delta_s} v_z r\,dr\right]$$

$$= U\frac{d}{dz}\int_R^{R-\delta_s} v_z r\,dr + \frac{dU}{dz}\int_R^{R-\delta_s} v_z r\,dr \qquad (1\text{-}6)$$

and

$$U\frac{dU}{dz}\int_R^{R-\delta_s} r\,dr = \frac{dU}{dz}\int_R^{R-\delta_s} U r\,dr \qquad (1\text{-}7)$$

we can add the terms on the left hand side of Eq. (1-5) and rearrange it in the form

$$\frac{dU}{dz}\int_R^{R-\delta_s}(U - v_z) r\,dr + \frac{d}{dz}\int_R^{R-\delta_s} v_z(U - v_z) r\,dr$$

$$= \frac{\mu_o R}{\rho}\left(\frac{\partial v_z}{\partial r}\right)_{r=R} - \frac{\tau_o R}{\rho} = -\frac{R}{\rho}\tau_w \qquad (7)$$

This is Eq. (7).

AII.2 DERIVATION OF EQ. (12) AND EQ. (14)

When the assumed form of velocity profile is substituted into the integral equation, Eq. (8), and the coordinate r is replaced by y by using the relation, y = R- r, we obtain

$$\frac{dU}{dz}\int_0^\delta U\left[1-2\frac{y}{\delta}+\left(\frac{y}{\delta}\right)^2\right](R-y)\,dy$$

$$+\frac{d}{dz}\left\{\int_0^\delta U^2\left[2\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2\right]\left[1-2\frac{y}{\delta}+\left(\frac{y}{\delta}\right)^2\right](R-y)\,dy\right.$$

$$=\frac{R\mu_0 U}{\rho}\left[\frac{\partial}{\partial y}\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\right]_{y=0}+\frac{R\tau_0}{\rho} \qquad (2\text{-}1)$$

Let the first term in the left-hand side of Eq. (2-1) be A, then

$$A = U\frac{dU}{dz}\int_0^\delta\left[R\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)-\left(y-2\frac{y^2}{\delta}+\frac{y^3}{\delta^2}\right)\right]dy$$

$$=U\frac{dU}{dz}\left[R\left(\delta-\delta-\frac{\delta}{3}\right)-\left(\frac{\delta^2}{2}-\frac{2}{3}\delta^2+\frac{\delta^2}{4}\right)\right]$$

$$=\frac{\delta U}{6}\left(2R-\frac{\delta}{2}\right)\frac{dU}{dz} \qquad (2\text{-}2)$$

Also let the second term in the left-hand side of Eq. (2-1) be B, then

$$B=\frac{d}{dz}\int_0^\delta U^2(R-y)\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}-4\frac{y^2}{\delta^2}+2\frac{y^3}{\delta^3}+2\frac{y^3}{\delta^3}-\frac{y^4}{\delta^4}\right)dy$$

$$=\frac{d}{dz}\left\{U^2\left[R\left(\delta-\frac{5}{3}\delta+\delta-\frac{1}{5}\delta\right)-\left(\frac{2}{3}\delta^2-\frac{5}{4}\delta^2+\frac{4}{5}\delta^2-\frac{\delta^2}{6}\right)\right]\right\}$$

$$=\frac{d}{dz}\left[U^2\left(\frac{2}{15}R\delta-\frac{1}{20}\delta^2\right)\right] \qquad (2\text{-}3)$$

Combining Eqs. (2-2) and (2-3) with Eq. (2-1), we have

$$\frac{\delta U}{6}\left(2R-\frac{\delta}{2}\right)\frac{dU}{dz}+\frac{d}{dz}[U^2\left(\frac{2}{15}R\delta-\frac{1}{20}\delta^2\right)=\frac{1}{\rho}\left(\frac{2R\mu_0 U}{\delta}+R\tau_0\right) \qquad (2\text{-}4)$$

Simplifying Eq. (2-4) gives

$$\frac{\delta U}{12}(4R-\delta)\frac{dU}{dz}+U\delta\left(\frac{4}{15}R-\frac{1}{10}\delta\right)\frac{dU}{dz}$$

$$+\,U^2\left(\frac{2}{15}R-\frac{1}{10}\delta\right)\frac{d\delta}{dz}=\frac{1}{\rho}\left(\frac{2R\mu_0 U}{\delta}+R\tau_0\right) \qquad (2\text{-}5)$$

Rearrangement of Eq. (2-5) gives

$$\delta U\left(\frac{3}{5}R-\frac{11}{60}\delta\right)\frac{dU}{dz}+U^2\left(\frac{2}{15}R-\frac{1}{10}\delta\right)\frac{d\delta}{dz}=\frac{1}{\rho}\left(\frac{2R\mu_0 U}{\delta}+R\tau_0\right) \quad (2\text{-}6)$$

This is Eq. (12).

Multiplying each term of Eq. (2-6) by $/\ {}_{o}\bar{u}$, we obtain

$$\frac{\delta}{R}\frac{U}{\bar{u}}\left(\frac{3}{5}-\frac{11}{60}\frac{\delta}{R}\right)\frac{d\left(\frac{U}{\bar{u}}\right)}{d\left(\frac{\mu_0 z}{\rho R^2\bar{u}}\right)}+\frac{U^2}{\bar{u}^2}\left(\frac{2}{15}-\frac{1}{10}\frac{\delta}{R}\right)\frac{d\left(\frac{\delta}{R}\right)}{d\left(\frac{\mu_0 z}{\rho R^2\bar{u}}\right)}$$

$$=2\,\frac{\left(\frac{U}{\bar{u}}\right)}{\left(\frac{\delta}{R}\right)}+\left(\frac{R\tau_0}{\mu_0\bar{u}}\right)\left(\frac{R}{R}\right)$$

which can be rewritten as

$$\delta^* U^*\left(\frac{3}{5}-\frac{11}{60}\delta^*\right)\frac{dU^*}{dz^*}+U^{*2}\left(\frac{2}{15}-\frac{1}{10}\delta^*\right)\frac{d\delta^*}{dz^*}=2\frac{U^*}{\delta^*}+\tau_0^* \qquad (2\text{-}7)$$

This is Eq. (14).

AII.3 DERIVATION OF EQS. (16) AND (18)

When the assumed form of the velocity profile is used and r is replaced by R - y, Eq. (15) can be written as

$$(R-\delta)^2 U + \int_0^{\delta} 2\pi U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)(R-y)\,dy = \pi R^2 \bar{u} \tag{3-1}$$

The second term may be integrated as follows

$$\int_0^{\delta} 2\pi U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)(R-y)\,dy$$

$$= 2\pi U \int_0^{\delta} \left[R\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) - \left(\frac{2y^2}{\delta} - \frac{y^3}{\delta^2}\right)\right] dy$$

$$= 2\pi U \left[R\left(\delta - \frac{\delta}{3}\right) - \left(\frac{2}{3}\delta^2 - \frac{1}{4}\delta^2\right)\right]$$

$$= 2\pi U \left[\frac{2}{3} R\delta - \frac{5}{12}\delta^2\right]$$

Thus Eq. (3-1) becomes

$$\pi(R - \delta)^2 U + 2\pi U\left(\frac{2}{3} R\delta - \frac{5}{12}\delta^2\right] = \pi R^2 \bar{u} \tag{3-2}$$

Dividing both sides of Eq. (3-2) by $R^2\bar{u}$ we have

$$(1 - \delta^*)^2 U^* + 2U^*\left(\frac{2}{3}\delta^* - \frac{5}{12}\delta^{*2}\right) = 1 \tag{3-3}$$

Rearranging Eq. (3-3) we have

$$\delta^{*2} - 4\delta^* = 6\left(\frac{1}{U^*} - 1\right) \tag{16}$$

This is Eq. (16)

Solving this for δ^* we obtain

$$\delta^* = 2 \pm \sqrt{\frac{6}{U^*} - 2} \tag{3-4}$$

Since the dimensionless center core velocity, U^*, is always larger than unity and the dimensionless boundary layer thickness, δ^*, is less than unity, the negative sign in Eq. (3-4) is used,

$$\delta^* = 2 - \sqrt{\frac{6}{U^*} - 2} \tag{17}$$

Differentiating δ^* with respect to z^* in Eq. (17) gives

$$\frac{d\delta^*}{dz^*} = \frac{3}{\sqrt{\frac{6}{U^*}\quad 2}} \frac{1}{U^{*2}} \frac{dU^*}{dz^*} \tag{3-5}$$

Substituting the expressions in Eqs. (17) and (3-5) into Eq. (14) yields

$$\left(2 - \sqrt{\frac{6}{U^*} - 2}\right) U^* \left[\frac{3}{5} - \frac{11}{60}\left(2 - \sqrt{\frac{6}{U^*} - 2}\right)\right] \frac{dU^*}{dz^*}$$

$$+ U^{*2} \left[\frac{2}{15} - \frac{1}{10}\left(2 - \sqrt{\frac{6}{U^*} - 2}\right)\right] \frac{3}{\sqrt{\frac{6}{U^*} - 2}} \frac{1}{U^{*2}} \frac{dU^*}{dz^*}$$

$$= 2 \frac{U^*}{2 - \sqrt{\frac{6}{U^*} - 2}} + \tau_0^*$$

or

$$\left\{U^* \left[\frac{5}{6} - \frac{2}{15}\sqrt{\frac{6}{U^*} - 2} - \frac{11}{10}\frac{1}{U^*}\right] - \frac{1}{5}\frac{1}{\sqrt{\frac{6}{U^*} - 2}} - \frac{3}{10}\right\} \frac{dU^*}{dz^*}$$

$$= 2 \frac{U^*}{2 - \sqrt{\frac{6}{U^*} - 2}} - \tau_0^*$$

or

$$\left(\frac{5}{6}U^* + \frac{2}{15}U^* D - \frac{1}{5D} - \frac{4}{5}\right)\frac{dU^*}{dz^*} - \frac{2U^*}{2 - D} + \tau_0^* \tag{18}$$

where

$$D = \sqrt{\frac{6}{U^*} - 2}$$

This is Eq. (18).

AII.4 DERIVATION OF EQ. (29)

The pressure drop in the fully developed region, Δp_f, can be obtained from the relation

$$\tau_w = \frac{|\Delta p_f|}{2L} R$$

or

$$|\Delta p_f| = \frac{2L\tau_w}{R} = \frac{2L\tau_0}{r_0} \tag{4-1}$$

where L is the tube length in the fully developed region. Dividing both sides of Eq. (4-1) by $\rho \bar{u}^2/2$ yields

$$|\Delta p_f^*| = \frac{|\Delta p_f|}{\frac{1}{2}\rho\bar{u}^2} = \frac{2L\tau_0}{r_0 \frac{1}{2}\rho\bar{u}^2} = \frac{4L\left(\frac{\tau_0 R}{\mu_0 \bar{u}}\right)}{R\left(\frac{\rho\bar{u}R}{\mu_0}\right)\left(\frac{r_0}{R}\right)} = 4\left(\frac{L}{R N_{Re}}\right)\frac{\tau_0^*}{r_0^*}$$

Since $L = z - z_e$, we have

$$|\Delta p_f^*| = 4\left(\frac{z - z_e}{R N_{Re}}\right)\frac{\tau_0^*}{r_0^*} = 4(z^* - z_e^*)\frac{\tau_0^*}{r_0^*} \tag{4-2}$$

It follows from Eq. (23) that

$$\frac{\tau_0^*}{r_0^*} = \frac{12}{3 - 4r_0^* + r_0^{*4}}$$

Hence, Eq. (4-2) becomes

$$|\Delta p_f^*| = \frac{48(z^* - z_e^*)}{3 - 4r_o^* + r_o^{*4}} = \frac{48\Delta z^*}{3 - 4r_o^* + r_o^{*4}} \tag{29}$$

This is Eq. (29).

AII.5 DERIVATION OF EQS. (34), (37), AND (38)

If the control volume of the fluid is taken from the tube inlet to any cross section z of the tube in the entrance region, not including the solid wall, Eq. (32) becomes

$$\pi R^2 \rho \left(\frac{2}{R^2}\int_0^R v_z^2 r\,dr - \bar{u}^2\right) + \pi R^2 (p - p_o) + 2\pi R \int_0^z \tau_{rz}\Big|_{r=R} dz = 0$$

or

$$\rho \int_0^R v_z^2 r\,dr - \frac{1}{2}R^2 \rho \bar{u}^2 + \frac{1}{2}R^2 (p - p_o) + R\int_0^z \tau_{rz}\Big|_{r=R} dz = 0 \tag{5-1}$$

Differentiating Eq. (5-1) with respect to z we obtain

$$\rho \frac{d}{dz}\int_0^R v_z^2 r\,dr + \frac{R^2}{2}\frac{dp}{dz} + R\,\tau_{rz}\Big|_{r=R} = 0 \tag{34}$$

This is Eq. (34).

Since the shear stress τ_{rz} for the Bingham fluid in the boundary layer is

$$\tau_{rz} = -\mu_o \frac{\partial v_z}{\partial r} + \tau_o \tag{5-2}$$

we have

$$\tau_{rz}\Big|_{r=R} = \left[-\mu_o \frac{\partial v_z}{\partial r} + \tau_o\right]_{r=R} = -\mu_o \left(\frac{\partial v_z}{\partial r}\right)_{r=R} + \tau_o$$

$$= \frac{2U\mu_o}{\delta} + \tau_o \tag{5-3}$$

Here we use the fact that (see velocity profile, Eq. (35)

$$\frac{\partial V_z}{\partial r} = \frac{\partial}{\partial r}\left[U\left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right)\right] = -\frac{\partial}{\partial y}\left[U\left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right)\right]$$

$$= U\left(\frac{2y}{\delta^2} - \frac{2}{\delta}\right),$$

and

$$\left(\frac{\partial V_z}{\partial r}\right)_{r=R} = \left(\frac{\partial V_z}{\partial r}\right)_{y=0} = \frac{2U}{\delta} \tag{5-4}$$

The integration of $\int_0^R v_z^2\, r dr$ can be carried out as follows:

$$\int_0^R v_z^2\, r dr = \int_0^{R-\delta} V_z^2\, r dr + \int_{R-\delta}^{R} V_z^2\, r dr$$

$$= \int_R^{\delta} U^2 (R-y)(-dy) + \int_{\delta}^{0} U^2\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)^2 (R-y)(-dy)$$

$$= U^2\int_{\delta}^{R}(R-y)dy + U^2\int_0^{\delta}\left(\frac{4y^2R}{\delta^2} - \frac{4y^3}{\delta^2} - \frac{4y^3R}{\delta^3} + \frac{4y^4}{\delta^3} + \frac{Ry^4}{\delta^4} - \frac{y^5}{\delta^4}\right)dy$$

$$= U^2\left[R(R-\delta) - \frac{1}{2}(R^2-\delta^2)\right] + U^2\left(\frac{4}{3}R\delta - \delta^2 - R\delta + \frac{4}{5}\delta^2 + \frac{1}{5}R\delta - \frac{1}{6}\delta^2\right)$$

$$= U^2\left[\frac{R^2}{2} + \frac{-30+20+3}{15}R\delta + \frac{15-30+24-5}{30}\delta^2\right]$$

$$= U^2\left[\frac{R^2}{2} - \frac{7}{15}R\delta + \frac{2}{15}\delta^2\right] \tag{5-5}$$

Substitution of Eqs. (5-3) and (5-5) into Eq. (34) gives

$$\rho\frac{d}{dz}\left[U^2\left(\frac{R^2}{2} - \frac{7}{15}R\delta + \frac{2}{15}\delta^2\right)\right] + \frac{R^2}{2}\frac{dP}{dz} + \frac{2\mu_0 RU}{\delta} + \tau_0 R = 0 \tag{37}$$

This is Eq. (37).

Introducing the dimentionless quantities given in Eq. (13) into Eq. (37) we obtain

$$R^2\bar{u}^2 \frac{1}{\frac{R^2\bar{u}}{\mu_o}} \frac{d}{dz^*}\left[U^{*2}\left(\frac{1}{2}-\frac{7}{15}\delta^*+\frac{2}{15}\delta^{*2}\right)\right]+\frac{R^2}{2}\frac{\frac{1}{2}\rho\bar{u}^2}{\frac{R^2\bar{u}\rho}{\mu_o}}\frac{dP^*}{dz^*}+\frac{2\mu_o R U^*\bar{u}}{\delta^* R}+\tau_o R$$

$$= 0 \qquad (5\text{-}7)$$

or

$$\frac{d}{dz^*}\left[U^{*2}\left(\frac{1}{2}-\frac{7}{15}\delta^*+\frac{2}{15}\delta^{*2}\right)\right]+\frac{1}{4}\frac{dP^*}{dz^*}+2\frac{U^*}{\delta^*}+\tau_o^*=0 \qquad (38)$$

This is Eq. (38).

AII.6 DERIVATION OF EQ. (43)

The quantity $-\underline{\underline{\tau}} : \underline{\nabla}\,\underline{V}$ is always positive and can be written as (42)

$$-\underline{\underline{\tau}} : \underline{\nabla}\,\underline{V} = \left(\mu_o + \frac{\tau_o}{|\phi|}\right)\phi^2 \qquad (6\text{-}1)$$

where ϕ is the square root of the quantity shown in Eq. (16), Chapter 3. If the boundary layer assumption is applied to Eq. (6-10, i.e., if the velocity derivatives are neglected except $\partial v_z/\partial r$, we obtain

$$-\underline{\underline{\tau}} : \underline{\nabla}\,\underline{V} = \left(\mu_o - \frac{\tau_o}{\frac{\partial v_z}{\partial r}}\right)\left(\frac{\partial v_z}{\partial r}\right)^2 \qquad (6\text{-}2)$$

Note that $\partial v_z/\partial r$ is negative. Therefore, Eq. (41) becomes

$$E_v = 2\pi\int_0^z\int_0^R -(\underline{\underline{\tau}}:\underline{\nabla}\,\underline{V})drdz = 2\pi\int_0^z\int_0^{R-\delta} -(\underline{\underline{\tau}}:\underline{\nabla}\,\underline{V})rdrdz$$

$$+\,2\pi\int_0^z\int_{R-\delta}^{R} -(\underline{\underline{\tau}}:\underline{\nabla}\,\underline{V})rdrdz$$

$$= 2\pi\mu_o \int_0^z \int_{R-\delta}^{R} \left(\frac{\partial v_z}{\partial r}\right)^2 r\,dr\,dz - 2\pi\tau_o \int_0^z \int_{R-\delta}^{R} \frac{\partial v_z}{\partial r} r\,dr\,dz \qquad (6\text{-}3)$$

The integration of Eq. (6-3) can be completed as follows:

$$\int_{R-\delta}^{R} \left(\frac{\partial v_z}{\partial r}\right)^2 r\,dr = \int_0^{\delta} U^2 \left(-\frac{2}{\delta} + \frac{2y}{\delta^2}\right)^2 (R-y)\,dy$$

$$= 4U^2 \int_0^{\delta} \left(\frac{R}{\delta^2} - \frac{y}{\delta^2} - \frac{2yR}{\delta^3} + \frac{2y^2}{\delta^3} + \frac{y^2 R}{\delta^4} - \frac{y^3}{\delta^4}\right) dy$$

$$= 4U^2 \left(\frac{R}{\delta} - \frac{1}{2} - \frac{R}{\delta} + \frac{2}{3} + \frac{1}{3}\frac{R}{\delta} - \frac{1}{4}\right)$$

$$= U^2 \left(\frac{4}{3}\frac{R}{\delta} - \frac{1}{3}\right) \qquad (6\text{-}4)$$

and

$$\int_{R-\delta}^{R} \frac{\partial v_z}{\partial r} r\,dr = -\int_{\delta}^{0} \left(-\frac{2}{\delta} + \frac{2y}{\delta^2}\right) U (R-y)\,dy$$

$$= U \int_0^{\delta} \left(-\frac{2R}{\delta} + \frac{2y}{\delta} + \frac{2yR}{\delta^2} - \frac{2y^2}{\delta^2}\right) dy$$

$$= U\left(-2R + \delta + R - \frac{2}{3}\delta\right)$$

$$= U\left(-R + \frac{1}{3}\delta\right) \qquad (6\text{-}5)$$

Combining Eqs. (6-3), (6-4), and (6-5) gives

$$E_v = 2\pi\mu_o \int_0^z U^2 \left(\frac{4}{3}\frac{R}{\delta} - \frac{1}{3}\right) dz - 2\pi\tau_o \int_0^z U \left(-R + \frac{1}{3}\delta\right) dz \qquad (43)$$

This is Eq. (43).

AII.7 DERIVATION OF EQS. (44) AND (45)

The first integral in Eq. (40) is

$$\int_0^R v_z^3\, rdr = \int_0^{R-\delta} v_z^3\, r dr + \int_{R-\delta}^R v_z^3\, r dr$$

$$= \int_R^{\delta} v_z^3 (R-y)(-dy) + \int_{\delta}^{0} v_z^3 (R-y)(-dy)$$

$$= U^3 \int_{\delta}^{R} (R-y)\,dy + \int_0^{\delta} U^3 \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)^3 (R-y)\,dy$$

$$= U^3\left[R(R-\delta) - \frac{1}{2}(R^2-\delta^2)\right] + U^3 \int_0^{\delta}\left(\frac{8y^3}{\delta^3} - \frac{12y^4}{\delta^4} + \frac{6y^5}{\delta^5} - \frac{y^6}{\delta^6}\right)(R-y)\,dy$$

$$= U^3\left[\frac{R^2}{2} - R\delta + \frac{\delta^2}{2}\right] + U^3\left(2\delta R - \frac{8\delta^2}{5} - \frac{12}{5}\delta R + 2\delta^2 + R\delta - \frac{6}{7}\delta^2 - \frac{1}{7}R\delta + \frac{1}{8}\delta^2\right)$$

$$= U^3\left[\frac{1}{2}R^2 - \frac{-35+70-84+35-5}{35}R\delta + \frac{140-448+560-240+35}{280}\delta^2\right]$$

$$= U^3\left[\frac{1}{2}R^2 - \frac{19}{35}R\delta + \frac{47}{280}\delta^2\right] \qquad (7\text{-}1)$$

Also note that

$$\int_0^R v_z\, rdr = \frac{Q}{2\pi} = \frac{1}{2}R^2\bar{u} \qquad (7\text{-}2)$$

If Eqs. (7-1), (7-2), and (43) are substituted into Eq. (40) and the resulting equation is differentiated with respect to z, we have

$$\frac{R^2\bar{u}}{2}\frac{dp}{dz} = -\frac{1}{2}\rho\frac{d}{dz}\left[U^3\left(\frac{1}{2}R^2 - \frac{19}{35}R\delta + \frac{47}{280}\delta^2\right)\right]$$

$$-\mu_o U^2\left(\frac{4}{3}\frac{R}{\delta} - \frac{1}{3}\right) + \tau_o U\left(-R + \frac{1}{3}\delta\right) \qquad (44)$$

This is Eq. (44)

Intrducing dimensionless quantities in Eq. (44) gives

$$\frac{\frac{1}{2}\rho\bar{u}^2}{\frac{R^2\bar{u}\rho}{\mu_o}}\frac{dp^*}{dz^*}\frac{\pi R^2\bar{u}}{2\pi} = -\frac{1}{2}\frac{\rho\bar{u}^3R^2}{\frac{R^2\bar{u}\rho}{\mu_o}}\frac{d}{dz^*}\left[U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)\right]$$

$$-\mu_o\bar{u}^2U^{*2}\left(\frac{4}{3}\frac{1}{\delta^*} - \frac{1}{3}\right) + \tau_o\bar{u}U^*R\left(-1 + \frac{1}{3}\delta^*\right) \qquad (7\text{-}3)$$

or

$$\frac{1}{4}\frac{dp^*}{dz^*} = -\frac{1}{2}\frac{d}{dz^*}\left[U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)\right]$$

$$-U^{*2}\left(\frac{4}{3}\frac{1}{\delta^*} - \frac{1}{3}\right) + U^*\tau_o^*\left(-1 + \frac{1}{3}\delta^*\right) \qquad (45)$$

This is Eq. (45).

Eliminating the dimensionless pressure gradient dp^*/dz^* between Eqs. (38) and (45) yields

$$\frac{d}{dz^*}\left[U^{*2}\left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right)\right] - \frac{1}{2}\frac{d}{dz^*}\left[U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)\right]$$

$$-U^{*2}\left(\frac{4}{3}\frac{1}{\delta^*} - \frac{1}{3}\right) - \frac{2U^*}{\delta^*} + \tau_o^*\left[1 + U^*\left(\frac{1}{3}\delta^* - 1\right)\right] = 0 \qquad (46)$$

This is Eq. (46).

AII.8 DERIVATION OF EQ. (49)

The relationship between δ^* and U^* from Eq. (48) will be used in Eq. (46) to obtain a differential equation which determines the relation between δ^* and z^*. From Eq. (16),

$$\delta^{*2} - 4\delta^* = \frac{6}{U^*} - 6 \tag{8-1}$$

we have

$$U^* = \frac{6}{B} \tag{8-2}$$

where

$$B = \delta^{*2} - 4\delta^* + 6 \tag{8-3}$$

Hence

$$dU^* = -6\,\frac{2\delta^* - 4}{(\delta^{*2} - 4\delta^* + 6)^2}\,d\delta^*$$

$$= \frac{12(2-\delta^*)}{B^2}\,d\delta^* \tag{8-4}$$

$$U^{*2} = \frac{36}{(\delta^{*2} - 4\delta^* + 6)^2} = \frac{36}{B^2} \tag{8-5}$$

$$U^{*3} = \frac{216}{(\delta^{*2} - 4\delta^* + 6)^3} = \frac{216}{B^3} \tag{8-6}$$

$$dU^{*2} = \frac{-72(2\delta^* - 4)}{(\delta^{*2} - 4\delta^* + 6)^3}\,d\delta^* = \frac{-144(\delta^* - 2)}{B^3}\,d\delta^* \tag{8-7}$$

$$dU^{*3} = \frac{-648(2\delta^* - 4)}{(\delta^{*2} - 4\delta^* + 6)^4}\,d\delta^* = \frac{-1296(\delta^* - 2)}{B^4}\,d\delta^* \tag{8-8}$$

$$B^2 = \delta^{*4} - 8\delta^{*3} + 28\delta^{*2} - 48\delta^{*} + 36 \tag{8-9}$$

$$B^3 = \delta^{*6} - 12\delta^{*5} + 66\delta^{*4} - 208\delta^{*3} + 396\delta^{*2} - 432\delta^{*} + 216 \tag{8-10}$$

$$B^4 = \delta^{*8} - 16\delta^{*7} + 120\delta^{*6} - 544\delta^{*5} + 1624\delta^{*4} - 326\delta^{*3} + 4320\delta^{*2} - 3456\delta^{*} + 1296 \tag{8-11}$$

$$\frac{d}{dz^*}\left[U^{*2}\left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right)\right]$$

$$= \left[U^{*2}\left(-\frac{7}{15} + \frac{4}{15}\delta^*\right) + \left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right)\frac{144(2-\delta^*)}{(\delta^{*2} - 4\delta + 6)^3}\right]\frac{d\delta^*}{dz^*}$$

$$= \frac{1}{B^2}\left[36\left(-\frac{7}{15} + \frac{4}{15}\delta^*\right) - \frac{144(2-\delta^*)\left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right)}{B}\right]\frac{d\delta^*}{dz^*} \tag{8-12}$$

$$\frac{d}{dz^*}\left[U^{*3}\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)\right]$$

$$= \left[U^{*3}\left(-\frac{19}{35} - \frac{47}{140}\delta^*\right) + \left(\frac{1}{2} - \frac{19}{35}\delta^* - \frac{47}{280}\delta^{*2}\right)\frac{1296(2-\delta^*)}{B^4}\right]\frac{d\delta^*}{dz^*}$$

$$= \frac{1}{B^3}\left[216\left(-\frac{19}{35} + \frac{47}{140}\delta^*\right) + \frac{1296}{B}(2-\delta^*)\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{480}\delta^{*2}\right)\right]\frac{d\delta^*}{dz^*} \tag{8-13}$$

Substituting Eqs. (8-12) and (8-13) into Eq. (46) gives

$$\left\{\frac{1}{B^2}\left[36\left(-\frac{7}{15} + \frac{4}{15}\delta^*\right) + \frac{144(2-\delta^*)}{B}\left(\frac{1}{2} - \frac{7}{15}\delta^* + \frac{2}{15}\delta^{*2}\right)\right]\right.$$

$$- \frac{1}{2B^3}\left[216\left(-\frac{19}{35}+\frac{47}{140}\delta^*\right)+\frac{1296}{B}(2-\delta^*)\left(\frac{1}{2}-\frac{19}{35}\delta^*+\frac{47}{280}\delta^*\right)\right]\Bigg\}\frac{d\delta^*}{dz^*}$$

$$-\frac{36}{B^2}\left(\frac{4}{3}\frac{1}{\delta^*}-\frac{1}{3}\right)+U^*\tau_o^*\left(-1+\frac{1}{3}\delta^*\right)+2\left(\frac{6}{B\delta^*}\right)+\tau_o^*=0 \qquad (8\text{-}14)$$

Multiplying each term in Eq. (8-14) by B^4 yields

$$\Big\{36B^2\left(-\frac{7}{15}+\frac{4}{14}\delta^*\right)+144(2-\delta^*)B\left(\frac{1}{2}-\frac{7}{15}\delta^*+\frac{2}{15}\delta^{*2}\right)$$

$$-\frac{1}{2}\left[216B\left(-\frac{19}{35}+\frac{47}{140}\delta^*\right)+1296(2-\delta^*)\left(\frac{1}{2}-\frac{19}{35}\delta^*+\frac{47}{480}\delta^{*2}\right)\right]\Bigg\}\frac{d\delta^*}{dz^*}$$

$$=36B^2\left(\frac{4}{3}\frac{1}{\delta^*}-\frac{1}{3}\right)-\frac{12B^3}{\delta^*}-\tau_o^*B^4\left[1+\frac{6}{B}\left(\frac{1}{3}\delta^*-1\right)\right] \qquad (8\text{-}15)$$

$$36B^2\left(-\frac{7}{15}+\frac{4}{15}\delta^*\right)=\frac{36}{15}B^2(\delta^*-7)$$

$$=\frac{12}{5}(4\delta^{*5}-39\delta^{*4}+168\delta^{*3}-388\delta^{*2}+480\delta^*-252) \qquad (8\text{-}16)$$

$$144(2-\delta^*)B\left(\frac{1}{2}-\frac{7}{15}\delta^*+\frac{2}{15}\delta^{*2}\right)=\frac{144}{30}B(2-\delta^*)(4\delta^{*2}-14\delta^*+15)$$

$$=\frac{24}{5}(-4\delta^{*5}+38\delta^{*4}-155\delta^{*3}+334\delta^{*2}-378\delta^*+180) \qquad (8\text{-}17)$$

$$-\frac{1}{2}(216B)\left(-\frac{19}{35}+\frac{47}{140}\delta^*\right)=-\frac{108}{140}B(-76+47\delta^*)$$

$$= -\frac{27}{35}(47\delta^{*3} - 264\delta^{*2} + 586\delta^* - 456) \tag{8-18}$$

$$-\frac{1}{2}(1296)(2-\delta^*)\left(\frac{1}{2} - \frac{19}{35}\delta^* + \frac{47}{280}\delta^{*2}\right)$$

$$= -\frac{648}{280}(2-\delta^*)(47\delta^{*2} - 152\delta^* + 140)$$

$$= -\frac{81}{35}(-47\delta^{*3} + 246\delta^{*2} - 444\delta^* + 280) \tag{8-19}$$

$$36B^2\left(\frac{4}{3}\frac{1}{\delta^*} - \frac{1}{3}\right) = \frac{12}{\delta^*}B^2(4-\delta^*)$$

$$= \frac{12}{\delta^*}(-\delta^{*5} + 12\delta^{*4} - 60\delta^{*3} + 160\delta^{*2} - 228\delta^* + 144) \tag{8-20}$$

$$-\frac{12B^3}{\delta^*} = \frac{12}{\delta^*}(-\delta^{*6} + 12\delta^{*5} - 66\delta^{*4} + 208\delta^{*3} - 396\delta^{*2} + 432\delta^* - 216) \tag{8-21}$$

$$-\tau_0^* B^4\left[1 + \frac{6}{B}\left(\frac{1}{3}\delta^* - 1\right)\right] = -\tau_0^* B^3[\delta^{*2} - 4\delta^* + 6 + 2\delta^* - 6]$$

$$= -\tau_0^*\delta^*(\delta^{*7} - 14\delta^{*6} + 90\delta^{*5} - 340\delta^{*4} + 812\delta^{*3} - 1224\delta^{*2} + 1080\delta^* - 432) \tag{8-22}$$

Substituting Eqs. (8-16) through (8-22) into Eq. (8-15), we have

$$\left\{\left(\frac{48}{5} - \frac{96}{5}\right)\delta^{*5} + \left[-\frac{(12)(39)}{5} + \frac{(24)(38)}{5}\right]\delta^{*4}\right.$$

$$+\left[\frac{(12)(168)}{5}-\frac{(24)(155)}{5}-\frac{(27)(47)}{35}+\frac{(81)(47)}{35}\right]\delta^{*3}$$

$$+\left[-\frac{(12)(388)}{5}+\frac{(24)(334)}{5}-\frac{(27)(586)}{35}-\frac{(81)(246)}{35}\right]\delta^{*2}$$

$$+\left[\frac{(12)(480)}{5}-\frac{(24)(378)}{5}-\frac{(27)(586)}{35}+\frac{(81)(444)}{35}\right]\delta^{*}$$

$$+\left[-\frac{(12)(252)}{5}+\frac{(24)(180)}{5}+\frac{(20)(456)}{35}-\frac{(81)(280)}{35}\right]\Big\}\frac{d\delta^*}{dz^*}$$

$$=\frac{12}{\delta^*}\left(-\delta^{*6}+11\delta^{*5}-54\delta^{*4}+148\delta^{*3}-236^{*2}+204\delta^*-72\right)$$

$$-\tau_o^*\delta^*\left(\delta^{*7}-14\delta^{*6}+90\delta^{*5}-340\delta^{*4}+812\delta^{*4}-122\delta^{*2}+1080\delta^*-432\right) \quad (8\text{-}23)$$

or

$$\delta^*\left\{-112\delta^{*5}+1036\delta^{*4}-3130\delta^{*3}+3574\delta^{*2}-1014\delta^*-432\right\}\frac{dU^*}{dz^*}$$

$$=140\left(-\delta^{*6}+11\delta^{*5}-54\delta^{*4}+148\delta^{*3}-236\delta^{*2}+204\delta^*-72\right)$$

$$-\frac{35}{3}\tau_o^*\delta^{*2}\left(\delta^{*7}-14\delta^{*6}+90\delta^{*5}-340\delta^{*4}+812\delta^{*3}-1224\delta^{*2}+1080\delta^*-432\right) \quad (49)$$

This is Eq. (49)

If the relationship between δ^* and U^* are used in Eq. (49) we obtain a differential equation which determines the relationship of U^* and z^*. Since

$$\delta^* = 2 - \sqrt{\frac{6}{U^*}-2} = 2 - D$$

$$\frac{d\delta^*}{dz^*} = \frac{3}{\sqrt{\frac{6}{U^*} - 2}} \frac{1}{U^{*2}} \frac{dU^*}{dz^*} = \frac{3}{D\,U^{*2}} \frac{dU^*}{dz^*} \tag{8-24}$$

where

$$D = \sqrt{\frac{6}{U^*} - 2} \tag{8-25}$$

Equation (49) then becomes

$$\frac{3(2-D)}{DU^{*2}} \Big\{ -112(2-D)^5 + 1036(2-D)^4 - 3130(2-D)^3$$

$$+ 3574(2-D)^2 - 1014(2-D) - 432 \Big\} \frac{dU^*}{dz^*}$$

$$= 140 \Big\{ -(2-D)^6 + 11(2-D)^5 - 54(2-D)^4 + 148(2-D)^3$$

$$- 236(2-D)^2 + 204(2-D) - 72 \Big\}$$

$$- \frac{35}{3} \tau_0^* (2-D)^2 \Big\{ (2-D)^7 - 14(2-D)^6 + 90(2-D)^5 - 340(2-D)^4$$

$$+ 812(2-D)^3 - 1224(2-D)^2 + 1080(2-D) - 432 \Big\} \tag{8-26}$$

This is the differential equation which relates U^* to z^*.

APPENDIX III

RITZ'S METHOD OF SOLVING VARIATIONAL PROBLEMS

In Sections 4.3 and 5.3 Ritz's method of solving the variational prblems has been mentioned. In this Appendix the basic notion of Ritz's method will be briefly described.

Consider the problem of finding a minimum (or maximum) of the functional $J(y(x))$ with respect to a certain class of admissible function $y(x)$. By admissible function we mean the function $y(x)$ satisfying the boundary conditions, and the conditions like continuity and smoothness in a specified region. Let the set of functions $\{\phi_1(x), \phi_2(x), \ldots\}$ be continuous and differentiable functions in the specified region and assume that $y(x)$ can be expressed as a linear conbination of the ϕ_i's in the same region and converges, that is,

$$y(x) = \sum_{i=1}^{\infty} \alpha_i \phi_i \tag{1}$$

where the α_i are constants. Along $y(x)$ which is a linear combination of the ϕ_i's, the functional $J(y(x))$ becomes a function of the coefficients α_1, α_2 ..., that is,

$$J(y(x)) = J\left(\sum_{i=1}^{\infty} \alpha_i \phi_i \right) = J(\alpha_1, \alpha_2, \cdots) \tag{2}$$

These coefficients are so chosen that the function $J(\alpha_1, \alpha_2, \ldots)$ has an extremum. Therefore, $\alpha_1, \alpha_2, \cdots$ shall be determined by the system of equations

$$\frac{\partial J}{\partial \alpha_i} = 0, \qquad i = 1, 2, \cdots \tag{3}$$

Assume that the functional has a minimum. Substituting the function $y(x)$ in Eq. (1) with the coefficients obtained from Eq. (3) into the functional J we have

$$J(\bar{y}(x)) = M$$

where $\bar{y}(x)$ gives J a minimum.

Suppose that the function $y(x)$ can be approximated by a finite number of the ϕ_i's, that is,

$$y_n(x) = \sum_{i=1}^{n} \alpha_i \phi_i$$

in which $y_n(x)$ must satisfy the boundary conditions. Then the functional becomes

$$J(y_n(x)) = J\left(\sum_{i=1}^{n} \alpha_i \phi_i \right) = J(\alpha_1, \alpha_2, \cdots \alpha_n)$$

Choose the α_i's such that

$$\frac{\partial J}{\partial \alpha_i} = 0, \qquad i = 1, 2, \cdots, n.$$

Then we have

$$J(y_n(x)) = M_n$$

where M_n is the approximation to the true minimum value M. Since $\bar{y}_n(x)$ is an approximation of $\bar{y}(x)$,

$$M_n \geq M$$

In general,

$$M_1 \geq M_2 \geq \cdots \geq M_n \geq \cdots \geq M.$$

The convergence of $y(x)$ in Eq. (1) depends on both characteristics of the problem and the choice of functions ϕ_i.

This method can also be extended to find the maximum (or minimum) of a functional J with two or more than two dependent variables, y_1, y_2, ..., y_k, and independent variables, x_1, x_2, ..., x_m.

ANALYTICAL AND NUMERICAL SOLUTIONS OF SOME NON-NEWTONIAN ENTRANCE REGION FLOW PROBLEMS

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ABSTRACT

When a viscous fluid enters a conduit (a circular pipe or a flat duct) from a very large reservoir through a well-rounded entrance, the velocity profile which is flat at the entry will gradually change as the fluid moves down the conduit because the walls of the conduit tend to retard the flow. Eventually, the velocity profile will develop to a form which remains unchanged with respect to the direction of flow. The flow in the region of changing velocity profile is the so-called hydrodynamic entrance region flow.

This report is concerned with laminar, isothermal non-Newtonian entrance region flow, especially that associated with power law and Bingham fluids. The report is divided into five chapters. Chapter 1 contains a general introduction to the problem. Chapter 2 provides the classification of non-Newtonian fluids, their shear stress expressions, (or the rheological equations of state) and flow behaviors in the fully developed region of pipes and channels.

Chapter 3 presents the governing equations. The general form of the governing equations is first derived in terms of shear stress based on the physical situation of the entrance region flow. The various assumptions, approximations, and shear stress expressions in terms of velocity gradients and rheological parameters can be used to obtain simplified governing equations for each non-Newtonian entrance region flow problem. The boundary layer equations of power law and Bingham fluids are derived in the latter part of this chapter.

Some published work on the entrance region flow of power law fluid is reviewed in detail in Chapter 4. The approach of individual methods and

their results are compared and discussed.

In Chapter 5 the basic momentum integral method and the Campbell-Slattery method are used to analyze the entrance region flow of Bingham fluids. The numerical results are compared with results which were obtained by the variational method. It is found that the Campbell-Slattery method which involves less mathematical manipulation appears to provide a solution which is capable of describing behavior of a Bingham fluid in the entrance region of a circular pipe.