FUNCTIONS ANNIHILABLE BY SAMPLING

by

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B. S., University of Wisconsin, 1957

A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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1961

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INTRODUCTION

It has been known for some time that the sampler in a sampled-data control system does not reveal fully the nature of the sampler's input function. A non-monotonic input function having zero crossings at the sampling intervals is completely annihilated by the sampler. This fact has led Jury (1, 2) to introduce a modified Z-transform to obtain a non-zero value of the function between the sampling intervals. The modified Z-transform delays the annihilable function so that its zeros do not lie at the sampling points.

Annihilable functions are not dodged so easily. One can imagine a function of time whose zero crossings coincide with the sampling points after a delay operation, and the problem of annihilation has not been avoided.

The only resort, then, is to become thoroughly familiar with annihilable functions and not waste time avoiding them. Various properties of annihilable functions of time are investigated in this paper, and a solution for the partial recovery of the annihilable input to the sampled-data control system is proposed.

THE Z-TRANSFORM

The Z-transform technique was originated by Demoivre and Laplace (8, 15) in developing a form of transform calculus which could be applied to the solution of linear difference equations.

Hurewicz (6) later adapted this approach to the solution of pulse filters and sampled-data systems. After subsequent efforts by Barker, Laning, Ragazzini, and Zadeh (3, 7, 9, 10, 12), the Z-transform of present type was formulated.

The Z-transform is, in fact, a Laplace transform of a discrete function. It is defined to be the Laplace transform of a sampled function $Z\left[f(t)\right]$ and

$$Z\left[f(t)\right] = f(t) \cdot \sum_{n=0}^{\infty} \delta(t - nT) \tag{1}$$

where $\sum_{n=0}^{\infty} \delta(t - nT)$ is a periodic impulse sequence with period T seconds. The Z-transform of f(t) is then defined to be

$$Z\left[f(t)\right] = L\left[f(t) \cdot \sum_{n=0}^{\infty} \delta(t - nT)\right] \tag{2}$$

Further calculation yields

$$Z\left[\bar{f}(s)\right] = \frac{1}{2\pi j} \oint_{\bar{f}} \bar{f}(\lambda) \cdot \frac{1}{1 - \bar{e}^{(s-\lambda)T}} \, d\lambda \tag{3}$$

where the region of integration contains only the poles of $\bar{f}(\lambda)$. Letting $\bar{e}^{sT} = z$, the Z-transform of f(t) becomes

$$Z\left[\bar{f}(s)\right] = \frac{1}{2\pi j} \oint_{\bar{f}} \bar{f}(\lambda) \cdot \frac{1}{1 - ze^{\lambda T}} \, d\lambda$$

or

$$Z\bar{f} = \sum_{\text{poles of } \bar{f}(\lambda)} \text{residue of } \left[\bar{f}(\lambda_k) \cdot \frac{1}{1 - ze^{\lambda_k T}}\right] \tag{4}$$

The Z-transform possesses the following properties:

A. The Z-transformed functions can be added, multiplied, or divided (except division by zero), subject to the basic rules of algebra.

Commutativity of addition and multiplication:

$$Z\overline{f} + Z\overline{g} = Z\overline{g} + Z\overline{f},\ (Z\overline{f})(Z\overline{g}) = (Z\overline{g})(Z\overline{f}) \qquad (5)$$

Associativity of addition and multiplication:

$$Z\overline{f} + (Z\overline{g} + Z\overline{h}) = (Z\overline{f} + Z\overline{g}) + Z\overline{h}$$

$$Z\overline{f}(Z\overline{g} \cdot Z\overline{h}) = (Z\overline{f} \cdot Z\overline{g})Z\overline{h} \qquad (6)$$

Distributivity of addition and multiplication:

$$Z\overline{f}(Z\overline{g} + Z\overline{h}) = (Z\overline{f})(Z\overline{g}) + (Z\overline{f})(Z\overline{h})$$

The existence of identity elements:

$$Z\overline{f} + 0 = Z\overline{f}; \quad Z\overline{f} \cdot 1 = Z\overline{f}$$

B. The Z-transform of the sum (or difference) of two functions is equal to the sum (or difference) of their respective transformed function. This property does not apply to the product of functions. By way of example,

$$Z(\overline{f} \pm \overline{g}) = Z\overline{f} \pm Z\overline{g} \qquad (8)$$

while $$Z(\overline{fg}) \neq (Z\overline{f})(Z\overline{g});\ Z(\overline{f}/\overline{g}) \neq (Z\overline{f})/(Z\overline{g}) \qquad (9)$$

C. The Z-transform of a Z-transformed function is again the original Z-transformed function, and in general $(Z)^k\overline{f} = Z\overline{f}$. This implies that the performance of a set cascaded sampler with the same sampling rate is equivalent to those of a single sampler. To show this, let

$$Z\overline{f} = \frac{1}{2\pi j}\oint \overline{f}(\lambda)\ \frac{1}{1 - ze^{\lambda T}}\, d\lambda$$

$$= \sum_{n=0}^{\infty} f(nT)\ z^n$$

$$Z^2\overline{f} \doteq Z\left[Z\overline{f}\right] = \frac{1}{2\pi j}\oint\sum_n f(nT)e^{-npT}\ \frac{1}{1 - e^{-(s-p)T}}\, dp$$

$$= \sum_n f(nT)e^{-nsT} \cdot \frac{1}{2\pi j}\oint \frac{1}{1 - ze^{pT}}\, dp$$

Therefore $\quad Z^2\overline{f} = \sum\limits_n f(nT)\, Z^n = Z\overline{f}$

By induction, one can show that $(Z)^k\overline{f} = Z\overline{f}$ (10)

D. The Z-transform of the discrete convolution of a continuous and a sampled function is equal to the product of two sampled functions.

1. $\quad Z\left[\overline{f}(Z\overline{g})\right] = (Z\overline{f})(Z\overline{g})$ (11)

Proof. Since $Z\overline{g} = \sum\limits_n g(nT)z^n$

Therefore $\quad Z\left[\overline{f}(Z\overline{g})\right] = Z\left[\overline{f} \sum\limits_n g(nT)z^n\right]$

$$= \frac{1}{2\pi j} \oint \overline{f}(\lambda) \cdot \sum_n g(nT)e^{-\lambda Tn} \frac{1}{1 - ze^{\lambda T}}\, d\lambda$$

$$= \sum_n g(nT)z^n \cdot \frac{1}{2\pi j} \oint \overline{f}(\lambda) \cdot \frac{1}{1 - ze^{\lambda T}}\, d\lambda$$

$$= (Z\overline{g})(Z\overline{f})$$

A new identity, along the same lines, is

2. $\quad Z\left[\overline{f}/(Z\overline{g})\right] = (Z\overline{f})/(Z\overline{g})$ (12)

The proof is

$$Z\left[\overline{f}/(Z\overline{g})\right] = Z\left[\overline{f}/(\sum_n g(nT)z^n)\right]$$

$$= \frac{1}{2\pi j} \oint \frac{\overline{f}(\lambda)}{\sum_n g(nT)e^{-\lambda Tn}} \cdot \frac{1}{1 - ze^{\lambda T}}\, d\lambda$$

$$= \frac{1}{2\pi j} \oint \frac{\overline{f}(s - p)}{\sum_n g(nT)e^{-(s-p)Tn}} \cdot \frac{1}{1 - e^{-pT}} dp$$

$$= \sum_k \frac{\overline{f}(s - j^{2\pi k/T})}{\sum_n g(nT)e^{-(s-j\,(2\pi k/T))nT}}$$

$$Z\left[\bar{f}/(Z\bar{g})\right] = \frac{\sum_k \bar{f}(s - j\frac{2\pi k}{T})}{\sum_n g(nT)e^{-sTn}}$$

$$= (Z\bar{f})/(Z\bar{g})$$

FUNCTIONS ANNIHILATED BY Z-TRANSFORM

It is readily seen from the properties of the Z-transform that some functions which are non-monotonic and have zeros at the sampling intervals will be completely annihilated by Z-transform. A family of such functions takes the form

$$f(t) = q(t)\ P(q(t)) \tag{13}$$

where $q(t) \equiv \prod_{n=0}^{\infty} (t - nT)$, and $P(q(t))$ is either a finite or infinite degree polynomial having only real zeros, only complex zeros, or a mixture of both. This representation yields only continuous annihilable functions.

The discontinuous annihilable functions are best represented in the Laplace transform domain. Such functions bear the general form:

$$\bar{f}(s) = \bar{a}(s)\ Z\left[\bar{b}(s)\right] - \bar{b}(s)\ Z\left[\bar{a}(s)\right] \tag{14}$$

where $\bar{a}(s)$, $\bar{b}(s)$ are the Laplace transform of any two arbitrary functions $a(t)$ and $b(t)$, and $Z\bar{a}$, $Z\bar{b}$ are their respective Z-transforms. The validity of such form can be seen rather readily, for as Z-transform is applied to $\bar{f}(s)$, it becomes

$$Z\bar{f} = Z\left[\bar{a}(Z\bar{b}) - \bar{b}(Z\bar{a})\right]$$

$$= Z\left[\bar{a}(Z\bar{b})\right] - Z\left[\bar{b}(Z\bar{a})\right]$$

$$= (Z\bar{a})(Z\bar{b}) - (Z\bar{b})(Z\bar{a}) = 0$$

When the functions a(t) and b(t) are free of annihilable elements, the functions f(t), a(t), and b(t) are uniquely related to one another; for any given two functions the third one is completely defined. To show this, one needs to distinguish between functions of s alone and of z alone by defining

$$Z\overline{a} \equiv A(e^{-sT}) = A(z)$$

$$Z\overline{b} \equiv B(e^{-sT}) = B(z)$$

The fundamental form can be written as

$$\overline{f}(z, s) = \overline{a}(s)\, B(z) - \overline{b}(s)\, A(z) \tag{15}$$

which can further be partitioned into functions of s and z only, as follows:

$$\frac{\overline{f}(z, s)}{\overline{b}(s)\, B(z)} = \frac{\overline{a}(s)}{\overline{b}(s)} - \frac{A(z)}{B(z)} \tag{16}$$

If the functions a(t) and b(t) are not free of annihilable functions, then the function is determinable up to either known or unknown annihilable function. The problem context always determines the two possibilities.

Some examples are now displayed.

A. It is desired to determine the annihilable function generated by $\overline{a}(s) = \frac{1}{s}$ and $\overline{b}(s) = \frac{1}{s + a}$.

The annihilable function

$$\begin{aligned}\overline{f}(z, s) &= \overline{a}(s)\, B(z) - \overline{b}(s)\, A(z)\\ &= \frac{1}{s} \cdot \frac{1}{1 - ze^{-aT}} - \frac{1}{s + a} \cdot \frac{1}{1 - z}\\ &= \frac{a(1 - z) - sz(1 - e^{-aT})}{s(s + a)(1 - z)(1 - ze^{-aT})}\end{aligned} \tag{17}$$

B. Given $\bar{f}(z, s) = \dfrac{\pi}{s^2 + \pi^2}$, and an impulsive function $\bar{b}(s) = 1$, it is desired to determine the generating function $\bar{a}(s)$.

$$\frac{\bar{f}(z, s)}{\bar{b}(s)\ B(z)} = \frac{\pi}{s^2 + \pi^2}$$

Therefore $\quad \bar{a}(s) = \dfrac{\pi}{s^2 + \pi^2} \cdot \bar{b}(s) = \dfrac{\pi}{s^2 + \pi^2} \qquad (18)$

C. Given $\bar{b}(s) = \dfrac{1}{s}$, $\bar{f}(z, s) = \dfrac{1 - z(1 + Ts)}{s^2(1 - z)^2}$, it is desired to find $\bar{a}(s)$.

$$\frac{\bar{f}(z, s)}{\bar{b}(s)\ B(z)} = \frac{1 - z(1 + Ts)}{s^2(1 - z)^2} \cdot \frac{1}{\dfrac{1}{s} \cdot \dfrac{1}{1 - z}}$$

$$= \frac{1 - z(1 + Ts)}{s(1 - z)}$$

Separating the variables z and s yields

$$\frac{\bar{f}(z, s)}{\bar{b}(s)\ B(z)} = \frac{1}{s} - \frac{T\ z}{1 - z}$$

Thus the desired $\bar{a}(s) = \dfrac{1}{s} \cdot \bar{b}(s) = \dfrac{1}{s^2} \qquad (19)$

A plot of the annihilable function $\bar{f}(z, s) = \dfrac{1 - z(1 + Ts)}{s^2(1 - z)^2}$, with period T being one second, is given in Fig. 1 (Appendix) to exhibit the behavior of this function.

SAMPLED-DATA CONTROL SYSTEM AND ANNIHILABLE FUNCTIONS

Two aspects of annihilable functions should be studied in the analysis of sampled-data control systems.

1. The partial or complete annihilation of the input functions containing annihilable elements by the sampler in the system.

2. The generation of non-monotonic response between the sampling intervals due to the non-ideal demodulation process in the system that may introduce instability to the system.

The input function to the sampled-data control system bears the general form of $f(t) = f_1(t) + f_o(t)$, where $f_o(t)$ is a function whose zero crossings coincide with the sampling intervals of the system's sampler, and the function $f_1(t)$, while it may have some of its zeros at the sampling points, is not annihilable by the sampler. $f_o(t)$ is annihilated by the sampler, and the Z-transform method of analyzing the sampled-data control system cannot reveal the existence of such a function at the input terminals.

Several methods for obtaining non-zero values of the function response between the sampling intervals have been introduced, among which the modified Z-transform developed by Jury (1, 2) is considered by some authors as showing promise. Jury (2) proposes that instead of applying Z-transform directly to the function response, it is applied to the delayed response, the delay being a fraction of the sampling interval so as to yield a non-zero transform. This method, however, does not evade the fact of annihilation of the input function. The annihilable elements in the input function are still annihilated by the sampler; it is possible to arrange the zero crossings so that the delayed function is annihilated.

In a system whose demodulator is of such nature that it generates sinusoidal response between the sampling intervals, the application of modified Z-transform does give a non-zero value, but again, in this case, an oscilloscope can be used to observe oscillatory response in place of the delay and sampler elements of the modified Z-transform.

The sampled-data feedback control system with elements as shown in Fig. 2, has an input-output relation

$$\overline{Y}(s) = \frac{\overline{g}(s)}{s} \cdot \frac{1 - z}{s} \cdot \frac{1}{1 + Z\left[\frac{\overline{g}(s)}{s} \cdot \frac{1 - z}{s}\right]} Z\,\overline{x}(s)$$

$$= \frac{\overline{g}(s)}{s} \cdot \frac{1 - z}{s} \cdot \frac{1}{1 + (1 - z)\,Z\left[\frac{\overline{g}(s)}{s^2}\right]} \cdot Z\,\overline{x}(s) \qquad (20)$$

The trapezoidal approximation (Halijak, 16) of $Z\left[\overline{g}(s) \cdot \frac{1}{s^2}\right]$ is given as

$$Z\left[\overline{g}(s) \cdot \frac{1}{s^2}\right] \doteq \frac{T^2 z}{(1 - z)^2}\left[Z\overline{g} - 0.5\ g_o\right] \qquad (21)$$

where g_o is the initial value of g(t).

Substituting the above approximate value, and rearranging terms, the equation yields

$$\overline{Y} \doteq \overline{g}(s) \cdot \frac{1}{T}\left(\frac{1 - z}{s}\right)^2 \cdot \frac{T}{1 - z + T^2 z\left[Z\overline{g} - 0.5\ g_o\right]} \cdot Z\overline{x}$$

$$\doteq \overline{g}(s) \cdot \frac{1}{T} \left(\frac{1 - z}{s}\right)^2 \cdot Z(s) \cdot Z\overline{x} \qquad (22)$$

$\frac{1}{T}\left(\frac{1 - z}{s}\right)^2$ is the delayed linear interpolator. No instability at or between the sampling intervals can occur due to the delayed linear interpolator. $\overline{g}(s)$ can introduce instability between the sampling intervals, and $Z(z)$ may cause the system to be unstable at and necessarily in between the sampling points.

The non-monotonic response termed as "hidden" oscillations in the literature (1, 2, 11, 13, 14) arises only when $\overline{g}(s)$ contains complex conjugate poles coinciding with the sampling intervals on the j-axis, and such system would have been at most limitedly stable even without the sampler on it. In practice, it is very unlikely that $\overline{g}(s)$ will be in such a form, and thus eliminating the possibilities of the existence of such "hidden" instability. Indeed, something stronger can be said about $\overline{g}(s)$ than the "hidden" instability concept. The transfer function $\overline{g}(s)$ should be able to follow a unit ramp input and must have a form such as

$$\overline{g}(s) = \frac{1 + mTs}{1 + mTs + \frac{1}{3}(mTs)^2} \qquad (23)$$

where $0 < m < 1$.

"Hidden" oscillation cannot exist in a stable system. The output of the stable system does have ripples between the sampling intervals due to the use of an approximation to the delayed linear interpolator, but on no occasion will it contribute any instability to the system. The generated ripples, in fact, can easily be detected by an arrangement shown in Fig. 3. Ripple is arbitrarily defined to be the variations of the system's response from straight line interpolation between the sampling intervals.

When a sampler and its non-ideal demodulator are added to a stable feedback system, the system in some instances will become unstable. The instability occurs both between and at the sampling intervals, and there is no occasion for the system to be unstable in between yet stable at the sampling intervals. This is exhibited by a series of examples of increasing complexity.

For a feedback system with open loop transfer function $\frac{\overline{g}(s)}{s} = \frac{k}{s}$, the system's characteristic equation

$$1 + \frac{\overline{g}(s)}{s} = \frac{k + s}{s} \tag{24}$$

has zero at s = k. The continuous feedback system is thus stable for all positive value of k. As the sampler and its demodulator are added to the forward loop, the characteristic equation of the sampled-data feedback system becomes

$$1 + Z\left[\frac{k}{s} \cdot \frac{1 - e^{-sT}}{s}\right] = 1 + k(1 - z) \cdot \frac{Tz}{(1 - z)^2}$$

$$= \frac{1 - z + kTz}{1 - z} \tag{25}$$

The zero of the equation as $z = \frac{1}{1 - kT}$, and the system is stable for $\left|Z\right| = \left|\frac{1}{1 - kT}\right| > 1$, which implies $0 < kT < 2$. For $kT > 2$, where the continuous counterpart is still stable, the sampled-data feedback system is unstable. At the border line $kT = 2$, the output response of the system

$$\bar{y} = \frac{k}{s} \cdot \frac{(1 - z)}{s} \cdot \frac{1}{1 + Z\left[\frac{k(1 - z)}{s^2}\right]} \cdot Z\,\bar{x}$$

$$= k \cdot \left(\frac{1 - z}{s}\right)^2 \frac{1}{1 + z} \cdot Z\,\bar{x} \qquad (26)$$

Consider now the case where $\frac{\bar{G}(s)}{s} = \frac{k}{s + z}$. The characteristic equation of the continuous feedback system $1 + \frac{G(s)}{s}$ $= \frac{s + a + k}{s + a}$ has zero at $s = -(k + a)$, and the continuous system is stable for all values $k + a > 0$, i.e., $k > -a$.

The corresponding sampled-data system with zero order hold circuit as its demodulator will have characteristic equation, letting $T = 1$ second.

$$1 + Z\left[\frac{1 - e^{-s}}{s} \cdot \frac{k}{s + a}\right] = 1 + k(1 - z)\left[\frac{1}{a} \cdot \frac{1}{1 - z} - \frac{1}{a} \cdot \frac{1}{1 - ze^{-a}}\right]$$

$$= \frac{a(1 - ze^{-a}) + k(1 - ze^{-a}) - k(1 - z)}{a(1 - ze^{-a})} \qquad (27)$$

The zero of the equation is given as $z(k - ke^{-a} - ae^{-a}) + a = 0$, and when bilinear transformation $z = \frac{s - 1}{s + 1}$ which maps the exterior of the unit circle in z-plane into the left half of the s-plane is made on the system characteristic equation in z, the zero in s-domain becomes

$$\left[(k - ke^{-a} - ae^{-a}) + a\right]s + \left[a - (k - ke^{-a} - ae^{-a})\right] = 0 \qquad (28)$$

Applying Routh's stability criterion, the required conditions for a stable system are given as

$$k - ke^{-a} - ae^{-a} + a > 0 \qquad (29)$$

and

$$a - k + ke^{-a} - ae^{-a} > 0 \qquad (30)$$

The intersections of the areas represented by equations (29) and (30) are found, and this region in which the values of k and a satisfy the stability criterion is plotted together with the region of $k + a > 0$ required for a stable continuous system in Figs. 4 and 5.

Again consider the case where $G(s) = \dfrac{k}{s(s + a)}$. The continuous feedback system with such open loop forward transfer function yields characteristic equation

$$1 + \frac{\overline{G}(s)}{s} = \frac{s^2 + as + k}{s(s + a)} \qquad (31)$$

The system is stable for all values $k > 0$ and $a > 0$; i.e., all values of k and a in the first quadrant of the k versus a plane.

In the case of its counterpart sampled-data system, the zeros of the system's characteristic equation can be shown to be

$$\left[k + e^{-a}(a^2 - ka - k)\right] z^2 + \left[e^{-a}(k - a^2) + k(a - 1) - a^2\right] z + a^2 + 0 \qquad (T = 1 \text{ sec}) \qquad (32)$$

If the bilinear transformation which maps the exterior of unit circle into the left half of s-plane, and the Routh's stability criterion is applied, the stability conditions are found to be:

$$ak(1 - e^{-a}) > 0 \qquad (33)$$

$$a^2 - e^{-a}(a^2 - ak - k) - k > 0 \qquad (34)$$

$$e^{-a}\left[2a^2 - k(a + 2)\right] + k(2 - a) + 2a^2 > 0 \qquad (35)$$

The intersections of these three areas are plotted in Fig. 6, and it is compared with the region of stability of the continuous counterpart in Fig. 7.

It is readily seen from Figs. 4 and 5 that the region of stability for the sampled-data feedback control system using a holding filter demodulator is smaller than the stability region for its continuous counterpart. It is possible for a stable continuous feedback system to become unstable when the sampler and its non-ideal demodulator is added to the system, but the insertion of sampler and holding filter combination to an unstable continuous system, on the other hand, do not alter the instability condition of the system. The stability region of the modulator/demodulator configuration is always contained in that of the original servo, and hence the modulator/demodulator servo cannot be expected to perform as well as the parent servo.

PARTIAL RECOVERY OF AN ANNIHILABLE INPUT

Since "hidden" oscillations cannot occur when the forward transfer function G(s) is stable, the problem of the annihilation property of Z-transform is therefore reduced to the investigation of the input functions, and the development of a method with which the annihilated portion of the input can be either partially or fully recovered at the output terminals. It is also well to note that the solution to such a problem does not lie in the use of adaptive feedback system configurations as shown in Fig. 8, for any feedback errors containing the annihilable elements are unable to get through the sampler in the main forward loop. The annihilable elements of the input function have zeros at the sampling intervals and they are the high order harmonic components of the input. A high-pass filter having cut-off frequency a bit lower than the sampling frequency of the sampler if used in supplement to the sampler, will provide the recovery of the annihilated components by the sampler. The arrangement of the sampler and the high-pass filter is shown in Fig. 9.

Consider the sampled-data control system with a high-pass filter of transfer function $\bar{h}(s)$ in the forward branch loop shown in Fig. 2, where the sampling period of the sampler is assumed to be one second, and the cut-off frequency of the filter is set at $\lambda\pi$. $0 \leq \lambda \leq 1$. Let the input function be

$$f(t) = f_1(t) + f_0(t)$$

$$= a \sin m\, t + \sin \pi\, t$$

and

$$\bar{f}(s) = \frac{am}{s^2 + m^2} + \frac{\pi}{s^2 + \pi^2} \qquad (36)$$

The output response at the x terminals is therefore being

$$x(t) = \mathcal{L}^{-1}\left[\frac{1 - e^{-s}}{s} Z\bar{f}(s) + \bar{f}(s)\,\bar{h}(s)\right]$$

$$= x_1(t) + x_0(t) \qquad (37)$$

and it is desired that $x_0(t)$ be in the form of $f_0(t)$ if the annihilable components are to be restored.

Using first order high-pass filter with $\bar{h}(s) = \dfrac{s}{s + \lambda\pi}$, the $\bar{x}_0(s)$ component of the output response is given as

$$\bar{x}_0(s) = \bar{h}(s) \cdot \bar{f}(s)$$

$$= \frac{s}{s+\lambda\pi}\left[\frac{am}{s^2+m^2}+\frac{\pi}{s^2+\pi^2}\right]$$

$$= \left(\frac{am\lambda\pi}{m^2+\lambda^2\pi^2}\cdot\frac{s}{s^2+m^2}\right)+\left(\frac{am^2}{m^2+\lambda^2\pi^2}\cdot\frac{m}{s^2+m^2}\right)$$

$$-\left(\frac{am\lambda\pi}{m^2+\lambda^2\pi^2}\cdot\frac{1}{s+\lambda\pi}\right)+\left(\frac{1}{1+\lambda^2}\cdot\frac{\lambda s+\pi}{s^2+\pi^2}\right)$$

$$-\frac{\lambda}{1+\lambda^2}\cdot\frac{1}{s+\lambda\pi}$$

$$x_o(t) = \frac{am}{m^2+\lambda^2\pi^2}\left(\lambda\pi\cos mt + m\sin mt - \lambda\pi e^{-\lambda\pi t}\right)$$

$$+\frac{1}{1+\lambda^2}\left(\lambda\cos\pi t+\sin\pi t-\lambda e^{-\lambda\pi t}\right)$$

$$= \frac{am}{\sqrt{m^2+\lambda^2\pi^2}}\sin\left(mt+\tan^{-1}\frac{\lambda\pi}{m}\right)$$

$$+\frac{1}{\sqrt{1+\lambda^2}}\sin\left(\pi t+\tan^{-1}\lambda\right)$$

$$-\lambda e^{-\lambda\pi t}\left(\frac{am\pi}{m^2+\lambda^2\pi^2}+\frac{1}{1+\lambda^2}\right)$$

$$= \frac{a}{\sqrt{1+\left(\frac{\lambda\pi}{m}\right)^2}}\sin\left(mt+\tan^{-1}\frac{\lambda\pi}{m}\right)$$

$$+\frac{1}{\sqrt{1+\lambda^2}}\sin\left(\pi t+\tan^{-1}\lambda\right)$$

$$-\lambda e^{-\lambda\pi t}\left(\frac{am\pi}{m^2+\lambda^2\pi^2}+\frac{1}{1+\lambda^2}\right)$$

The steady-state response of this system with $\lambda = 1$ and $m \ll \pi$ is

$$x_o(t) \overset{s.s.}{\approx} 0.707\sin(\pi t + 45^\circ) \qquad (38)$$

Replacing the first order high-pass filter with second order approximation (Butterworth) transfer function $\bar{h}(s) = \dfrac{s^2}{s + \lambda\pi\sqrt{2}\, s + (\lambda\pi)^2}$, the Laplace transform of $x_o(t)$ is, then

$$x_o(s) = \frac{s^2}{s^2 + \lambda\pi\sqrt{2}\, s + \lambda^2\pi^2}\left[\frac{am}{s^2 + m^2} + \frac{\pi}{s^2 + \pi^2}\right]$$

$$= \frac{As + B}{s^2 + \lambda\pi\sqrt{2}\, s + \lambda^2\pi^2} + \frac{Cs + D}{s^2 + m^2} + \frac{Es + F}{s^2 + \lambda\pi\sqrt{2}\, s + \lambda^2\pi^2}$$

$$+ \frac{Gs + H}{s^2 + \pi^2}$$

where

$$A = -\frac{a\lambda\pi\sqrt{2}\, m^3}{m^4 + (\lambda\pi)^4}$$

$$B = -\frac{am\lambda^2\pi^2\,(m^2 - \lambda^2\pi^2)}{m^4 + (\lambda\pi)^4}$$

$$C = \frac{a\lambda\pi\sqrt{2}\, m^3}{m^4 + (\lambda\pi)^4}$$

$$D = \frac{am^3(m^2 - \lambda^2\pi^2)}{m^4 + (\lambda\pi)^4}$$

$$E = -\frac{\lambda\sqrt{2}}{1 + \lambda^4}$$

$$F = -\frac{\pi\lambda^2(1 - \lambda^2)}{1 + \lambda^4}$$

$$G = \frac{\lambda\sqrt{2}}{1 + \lambda^4}$$

$$H = \frac{\pi(1 - \lambda^2)}{1 + \lambda^4}$$

For $\lambda = 1$, and $m \ll \pi$, the steady-state response of $x_o(s)$ is approximately to be

$$x_1(s) \overset{s.s.}{\approx} \frac{\sqrt{2}/2 \quad s}{s^2 + \pi^2}$$

$$x_o(t) \overset{s.s.}{\approx} 0.707 \sin(\pi t + 90°) \tag{39}$$

If the transfer function of third order approximation

$$\bar{h}(s) = \frac{s^3}{s^3 + 2\lambda\pi s^2 + 2\lambda^2\pi^2 s + (\lambda\pi)^3}$$

is used, the response is given as

$$x_o(s) = \bar{h}(s) \cdot \bar{f}(s)$$

$$= \frac{s^3}{s^3 + 2\lambda\pi s^2 + 2\lambda^2\pi^2 s + (\lambda\pi)^3}\left(\frac{am}{s^2 + m^2} + \frac{\pi}{s^2 + \pi^2}\right)$$

$$= \frac{As^2 + Bs + C}{s^3 + 2\lambda\pi s^2 + 2\lambda^2\pi^2 s + (\lambda\pi)^2} + \frac{Ds + E}{s^2 + m^2}$$

$$+ \frac{Fs^2 + Gs + H}{s^3 + 2\lambda\pi s^2 + 2\lambda^2\pi^2 s + (\lambda\pi)^3} + \frac{Js + K}{s^2 + \pi^2}$$

Evaluating the constants, they are found to be

$$A = -D = \frac{am^3(\lambda^3\pi^3 - 2m^2\lambda\pi)}{m^6 + (\lambda\pi)^6}$$

$$B = \frac{am(\lambda^6\pi^6 - 2m^4\lambda^2\pi^2 + 2m^2\lambda^4\pi^4)}{m^6 + (\lambda\pi)^6}$$

$$C = \frac{am\,\lambda^3\pi^3\,(m^4 - 2m^2\lambda^2\pi^2)}{m^6 + (\lambda\pi)^6}$$

$$E = \frac{am^3(m^4 - 2m^2\lambda^2\pi^2)}{m^6 + (\lambda\pi)^6}$$

$$F = -J = \frac{(\lambda^3 - 2\lambda^4 - 2\lambda + 4\lambda^2)}{(1 - 2\lambda^2)(1 + \lambda^6)}$$

$$G = \frac{\pi(\lambda^6 - 2\lambda^8 + 2\lambda + 2\lambda^4 - 4\lambda^5 - 4\lambda^2 + 4\lambda^3)}{(1+\lambda^6)(1 - 2\lambda^2)}$$

$$H = \frac{2\lambda^4\pi^2 - a\lambda^3\pi^2}{1 + \lambda^6}$$

$$K = \frac{\pi(1 - 2\lambda)}{1 + \lambda^6}$$

Again, let $\lambda = 1$ and $m << \pi$, the approximate steady-state response $x_o(s)$ becomes

$$x_o(s) \overset{s.s.}{\approx} \frac{1}{2} \cdot \frac{s - \pi}{s^2 + \pi^2}$$

$$\begin{aligned} x_o(t) &\overset{s.s.}{\approx} \tfrac{1}{2}(\cos \pi t - \sin \pi t) \\ &= \tfrac{1}{2}\sqrt{2}\cos(\pi t + 45^\circ) \\ &= 0.707 \sin(\pi t + 135^\circ) \qquad (40) \end{aligned}$$

The results of using first, second, and third order high-pass filter are very encouraging. It is seen that for m very much greater than π, if the response is made to delay for certain appropriate time, the annihilable elements can be recovered at the output terminals of the filter with a slightly reduced magnitude. High order filter introduces a rather large phase shift, which may on occasions cause the system to be unstable.

CONCLUSIONS

It is evident from the discussions presented that the problem of dealing with the annihilation property of Z-transform is not at all solved by employing the modified Z-transform technique. Oscillations between the sampling intervals due to the non-ideal demodulation that cause the sampled-data system to become unstable in between the sampling intervals, cannot exist in a system where its continuous forward transfer function G(s) is stable, and the detection of such instability can be accomplished by investigating the existence of complex conjugate poles of G(s) whose imaginary component is an integral multiple of the sampling frequency.

When an input function to a sampled-data control system contains elements whose zero crossings coincide with the sampling intervals, the elements are annihilated by the sampler and the informations carried by these elements are totally lost in the process. The recovery of the annihilated elements cannot be done by any adaptive feedback system configurations, as the elements cannot pass through the sampler in the main forward loop without being annihilated.

The solution, however, is found to be rather simple and it turns out that a high-pass filter of second order approximation can be made to recover the elements except for phase shift. When the input function is fed through the filter, the annihilable elements, which have relatively higher frequency, are reproduced at the output with phase shift and reduced magnitude.

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Dr. Charles A. Halijak, Department of Electrical Engineering, for his guidance and encouragement in the preparation of this thesis.

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APPENDIX

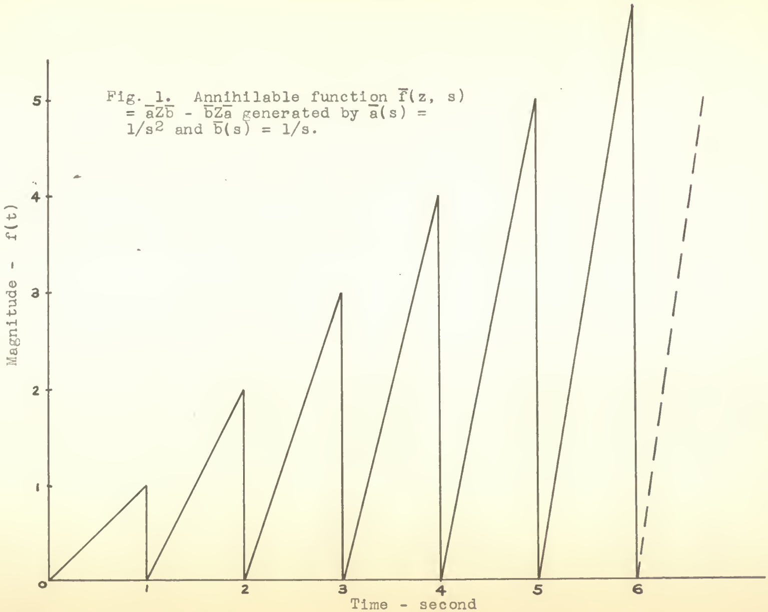

Fig. 1. Annihilable function $\overline{f}(z, s) = \overline{a}Z\overline{b} - \overline{b}Z\overline{a}$ generated by $\overline{a}(s) = 1/s^2$ and $\overline{b}(s) = 1/s$.

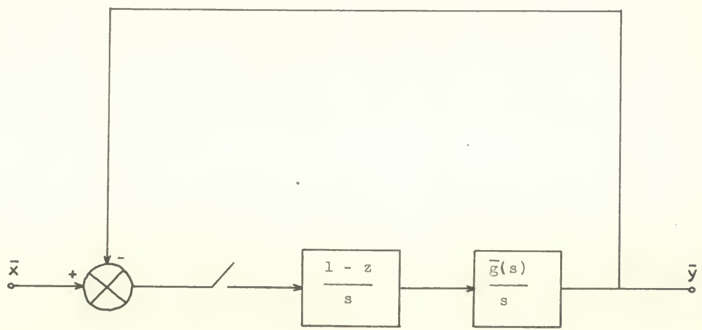

Fig. 2. Typical error-sampled feedback control system using zero-order data hold.

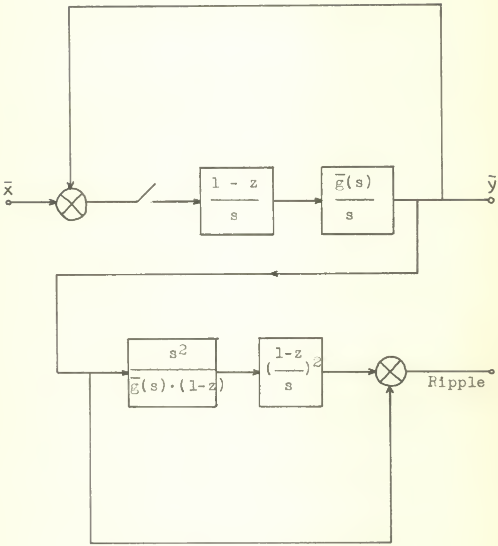

Fig. 3. Block diagram arrangement for detection of ripple.

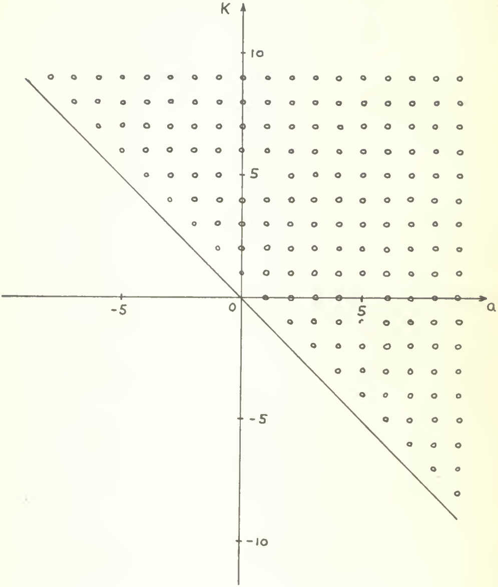

Fig. 4. Shaded area represents the region of stability for continuous feedback system with $\dfrac{\bar{g}(s)}{s} = \dfrac{k}{s + a}$.

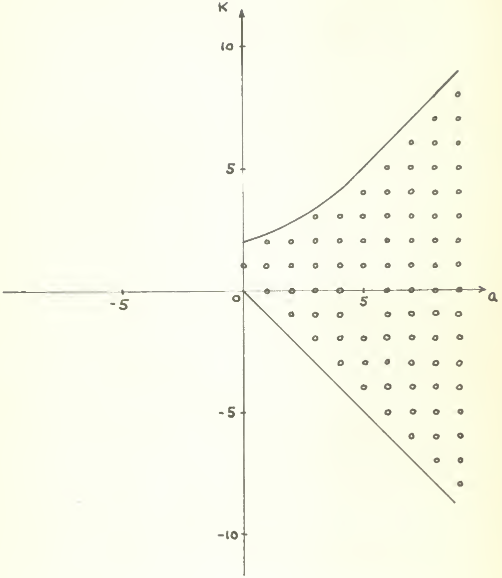

Fig. 5. Shaded area represents the region of stability of sampled-data feedback system with $\frac{\bar{g}(s)}{s} = \frac{k}{s + a}$.

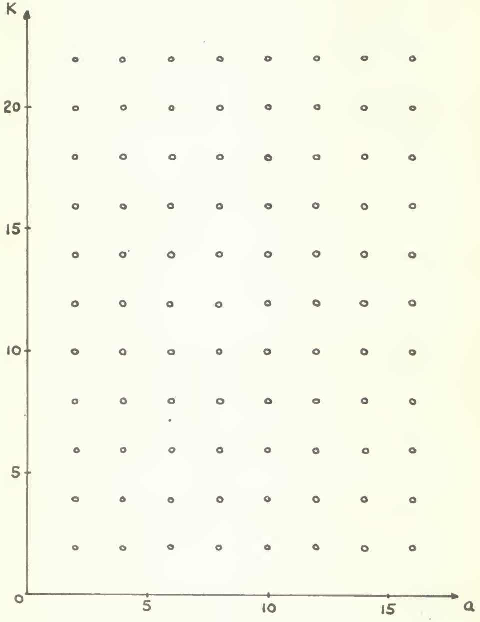

Fig. 6. Dotted area shows the region of stability of continuous feedback system with $\frac{\bar{g}(s)}{s} = \frac{k}{s(s + a)}$.

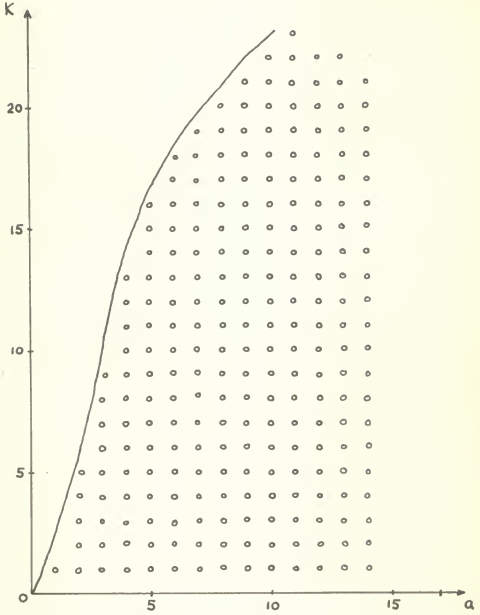

Fig. 7. Shaded area shows the region of stability of sampled-data feedback system with $\overline{F}(s)/s = \dfrac{k}{s(s + a)}$.

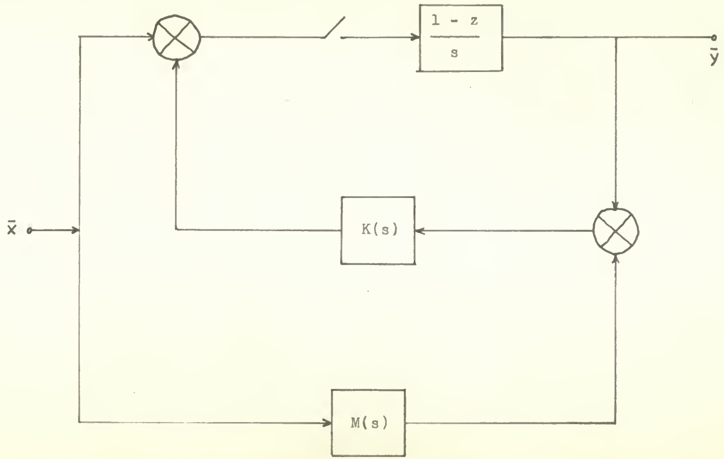

Fig. 8. Feedback system with adaptive configurations.

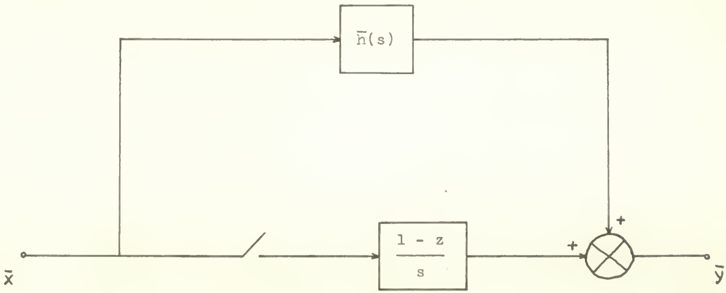

Fig. 9. Block diagram arrangement for partial recovery of annihilable functions.

FUNCTIONS ANNIHILABLE BY SAMPLING

by

JOSEPH PING-LIONG HO

B. S., University of Wisconsin, 1957

AN ABSTRACT OF
A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1961

The property of the sampler to annihilate non-monotonic input function having zero crossings at the sampling intervals has led Jury to propose a modified Z-transform to obtain a non-zero value of the function between the sampling intervals. The modified Z-transform delays the annihilable function so that its zero does not lie at the sampling points.

Annihilable functions are not dodged so easily. One can imagine a function whose zero crossing coincides with the sampling points after a delay operation and the problem of annihilation has not been circumvented.

Various properties of Z-transform are investigated, and it is shown that the Z-transform, aside from obeying commutative, associative, and distributive laws of algebra, has the following additional properties.

1. The Z-transform of a Z-transformed function is again the original Z-transformed function and in general $(Z)^k \, \overline{f} = Z\overline{f}$.

2. Z-transform of the discrete convolution of a continuous and a sampled function is equal to the product of two sampled functions.

3. $Z(\overline{f}/Z\overline{g}) = (Z\overline{f})/(Z\overline{g})$, provided $Z\overline{g} \neq 0$.

A family of functions which takes the form

$$f(t) = q(t) \; P\left[q(t)\right]$$

where $q(t) \equiv \prod_{m=0}^{\infty} (t - nT)$ and $P\left[q(t)\right]$ is either a finite or infinite degree polynomial, yields continuous annihilable functions having zero crossings at the sampling intervals. Their discontinuous counterparts are best represented in the Laplace transform domain. Such functions have the form

$$\bar{f}(s) = \bar{a}(s)\ Z\bar{b} - \bar{b}(s)\ Z\bar{a}$$

where $\bar{a}(s)$ and $\bar{b}(s)$ are the respective Laplace transform of any arbitrary functions (t) and b(t), and $Z\bar{a}$, $Z\bar{b}$ are their respective Z-transform.

Partial recovery of such annihilable input in a sampled-data control system is possible with a high-pass filter. The input functions are separately fed through a first or second order high-pass filter and then added to the output of the sampler. The annihilable elements are partially recovered at output of the filter with a reduced magnitude and a certain phase shift which increases with the order of the high-pass filter.