ELEMENTARY BOOLEAN ALGEBRA

by

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INTRODUCTION

In the last twenty-five years, Boolean algebra has developed from what was often regarded as just an interesting curiosity into an extensive and mature branch of mathematics (5, ix)[1]. Boolean algebra is named after the English mathematician, George Boole. He developed this algebra after the realization that an algebra is an abstract system. This gave Boole the opportunity he needed, and he separated the symbols of mathematical operations from the things upon which they operate and proceeded to investigate these operations in their abstract setting. He produced many notable mathematical works, but his main effort was in writing his book, "The Laws of Thought." As the name Boolean algebra suggests, it is part of that branch of mathematics known as modern or abstract algebra. It is an algebra usually studied in a basic course of modern algebra, and it has readily available applications to illustrate the theory. One phase of its development has been inspired by the applications of Boolean algebra to the design of switching circuits for telephone and control systems, and to design of logic circuits for computers. However, the subject has also developed into a significant branch of abstract algebra with important applications to topology (5, ix). Thus Boolean algebra is a proper sphere of interest for the pure mathematician as well as for those primarily interested in applications.

The purpose of this paper is to approach Boolean algebra from a basic set of postulates, then to develop this algebra from the postulates and theorems of lattices, and finally to present a class of simple

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[1]In this report the first number will be used to indicate the reference and the second number will indicate the page. The references are listed in the bibliography.

switching circuits as a model of Boolean algebra. A primary purpose in presenting this model is to illustrate how observation of a physical or logical system dictates the details of the mathematical system used to describe it. Since in the present instance the mathematical system used is different from the familiar ones of elementary algebra, geometry, and the calculus, it is well to consider some of the characteristics of a formal, mathematical description of a physical system.

In a formal, mathematical system one considers the following conditions. Because one cannot define every word in terms of simpler words, every mathematical system necessarily contains undefined terms. Similarly, because one cannot deduce every theorem as a logical consequence of simpler theorems, every mathematical system must also contain unproved theorems or postulates. From the undefined terms and the postulates one deduces theorems by means of the rules of logic. Then one introduces definitions of new terms and proves more theorems.

The choice of the undefined terms and the postulates of a mathematical system is by no means an easy task. Those of Euclidean geometry were the outgrowth of several thousand years' experience with experimental and intuitive geometry. In all other examples of postulational systems the undefined terms, postulates, and the definitions are likewise selected on the basis of physical or mathematical experience and in such a way as to yield useful results.

When mathematics is applied to a physical system, it is relatively rare that the system being studied is well enough understood so that even a reasonably complete set of undefined terms and postulates are suggested. Often, however, it is possible to give a set of postulates for a mathematical system which is a useful description of a physical system. An example of this is the use of Boolean algebra to represent switching circuits. One then has a mathematical system representing a particular physical system.

No mathematical system has ever provided all the answers to all of the problems concerning its corresponding physical system (5, x-xi). This is because it cannot take into account all of the conditions which affect the physical system in question. Normally one ignores all but what appears to be the most important factors. Taking these factors into account, one idealizes and symbolizes one's physical concepts and observations, thus utilizing a mathematical system which produces theorems which correlate closely with what is observed. When this is the case, the system in question is a useful one. Otherwise, the system is unsatisfactory and at least one conditional factor must be added to the list of vital ones.

This circumstance appears in the mathematical study of switching circuits. The simple mathematical system with which one begins is based on certain admittedly incomplete and inaccurate assumptions concerning the switching circuits which, however, make the mathematical system much more tractable. The resulting system is useful in solving a wide variety of problems because the factors invalidating the assumptions in question are not of major significance for the problems in question. When the invalidating assumptions do become significant, theory and observation will no longer correlate satisfactorily, and one must replace the system by a more general one which recognizes the importance of these factors.

In what follows an attempt will be made to point out where simplifying assumptions are made, and also to indicate the physical origins of the postulates being used.

DEFINITION OF A BOOLEAN ALGEBRA

For later ease of reference and in order to emphasize that Boolean algebra is, in fact, a mathematical structure independent of its applications, a Boolean algebra will be defined abstractly and the most useful rules will be derived. Some of the postulates may appear strange or artificial. The study of applications will help to make clear the necessity and naturalness of the postulates. The postulates are given in an order that is convenient in the remainder of the paper. They do not form an independent set.

Consider a set B of elements for which first of all equality is introduced and the familiar notation $x = y$ is used. If one attaches meanings to these statements, the meaning of $x = y$ would be that x and y are two names for identical objects. There are no restrictions placed on the nature of the objects, so that one has equality not only between numbers, as is common in mathematics, but between sets, or between functions, or indeed between the names of any objects.

Formulas involve operations, and it is assumed that in B there are two binary operations, that is, operations that may be applied to any ordered pair of elements of B to yield a unique third element of B. A Boolean algebra can then be defined in the following manner (3, 112).

<u>Definition</u>. A Boolean algebra B is a system consisting of a set B of elements, two operations $\cup$ and $\cap$ (usually read "cup" and "cap"), and the following postulates:

1 a. $\cup$ is commutative on B.

1 b. $\cap$ is commutative on B.

2 a. B contains an identity element 0 with respect to $\cup$.

2 b. B contains an identity element 1 with respect to $\cap$.

3 a. $\cup$ is distributive with respect to $\cap$.

3 b. $\cap$ is distributive with respect to $\cup$.

4. For each element b in B there is an element b' in B such that $b \cup b' = 1$ and $b \cap b' = 0$ (b' is called the complement of b).

Next some important theorems concerning Boolean algebra will be given.

<u>Theorem</u> 1-1. Every statement or algebraic identity deducible from the postulates of a Boolean algebra remains valid if the operations "$\cup$" and "$\cap$", and the identity elements 0 and 1 are interchanged throughout. (This is known as the principle of duality.)

The proof of this theorem follows from the symmetry of the postulates with respect to the two operations and the two identities.

By virtue of this principle, pairs of theorems will be stated. Furthermore, it will be necessary to prove only one of the theorems, for the steps in one proof are dual statements to those in the other, and the justification for each step is the dual postulate or theorem in one case of that in the other.

<u>Theorem</u> 1-2. For every b in B, $b \cup b = b$.

<u>Theorem</u> 1-2 a. For every b in B, $b \cap b = b$.

Proof: One has

$$b = b \cup 0 = b \cup (b \cap b') = (b \cup b) \cap (b \cup b') = (b \cup b) \cap 1 = b \cup b.$$

<u>Theorem</u> 1-3. For every b in B, $b \cup 1 = 1$.

Theorem 1-3 a. For every b in B, $b \cap 0 = 0$.

Proof: One has

$$b \cup 1 = 1 \cap (b \cup 1) = (b \cup b') \cap (b \cup 1) =$$
$$b \cup (b' \cap 1) = b \cup b' = 1.$$

Theorem 1-4. For every a and b in B, $a \cup (a \cap b) = a$.

Theorem 1-4 a. For every a and b in B, $a \cap (a \cup b) = a$.

Proof: One has

$$a \cup (a \cap b) = a \cup (b \cap a) = (b \cap a) \cup a =$$
$$(b \cap a) \cup (1 \cap a) = (b \cup 1) \cap a = 1 \cap a = a.$$

Theorem 1-5. $\cup$ is associative on B.

Theorem 1-5 a. $\cap$ is associative on B.

Proof: It is necessary to show that $a \cup (b \cup c) = (a \cup b) \cup c$ for any a, b, and c in B. Let $T = a \cup (b \cup c)$ and $S = (a \cup b) \cup c$. Then $a \cap T = a \cap [a \cup (b \cup c)] = a$ by theorem 4. Similarly

$$a \cap S = a \cap [(a \cup b) \cup c] = [a \cap (a \cup b)] \cup (a \cap c) =$$
$$a \cup (a \cap c) = a.$$

Thus $a \cap T = a \cap S$. Furthermore,

$$a' \cap T = a' \cap [a \cup (b \cup c)] = (a' \cap a) \cup [a' \cap (b \cup c)] =$$
$$0 \cup [a' \cap (b \cup c)] = a' \cap (b \cup c) \text{ and}$$
$$a' \cap S = a' \cap [(a \cup b) \cup c] = [a' \cap (a \cup b)] \cup (a' \cap c) =$$
$$[(a' \cap a) \cup (a' \cap b)] \cup (a' \cap c) =$$
$$(a' \cap b) \cup (a' \cap c) = a' \cap (b \cup c).$$

Thus $a' \cap T = a' \cap S$. Then

$$(a \cap T) \cup (a' \cap T) = (a \cap S) \cup (a' \cap S) \text{ or}$$
$$(a \cap a') \cup T = (a \cap a') \cup S$$

or $T = S$. Hence $a \cup (b \cup c) = (a \cup b) \cup c$.

Theorem 1-6. The element b' corresponding to each b in B is unique.

Proof: Assume there are two such elements, b_1' and b_2', satisfying postulate (4). Then

$$b_1' = 1 \cap b_1' = (b \cup b_2') \cap b_1' = (b \cap b_1') \cup (b_2' \cap b_1') =$$
$$0 \cup (b_2' \cap b_1') = 0 \cup (b_1' \cap b_2') = (b \cap b_2') \cup (b_1' \cap b_2') =$$
$$(b \cup b_1') \cap b_2' = 1 \cap b_2' = b_2'.$$

Theorem 1-7. For every b in B, $(b')' = b$.

Proof: $b' \cup b = 1$ and $b' \cap b = 0$, hence by theorem 6, $(b')' = b$.

The following pair of theorems is known as DeMorgan's laws.

Theorem 1-8. For every a and b in B, $(a \cup b)' = a' \cap b'$.

Theorem 1-8 a. For every a and b in B, $(a \cap b)' = a' \cup b'$.

Proof:

$$(a \cup b) \cup (a' \cap b') = \left[(a \cup b) \cup a'\right] \cap \left[(a \cup b) \cup b'\right] =$$
$$\left[(a \cup a') \cup b\right] \cap \left[a \cup (b \cup b')\right] =$$
$$(1 \cup b) \cap (a \cup 1) = 1 \cap 1 = 1$$

whereas

$$(a \cup b) \cap (a' \cap b') = \left[a \cap (a' \cap b')\right] \cup \left[b \cap (a' \cap b')\right] =$$
$$\left[(a \cap a') \cap b'\right] \cup \left[a' \cap (b \cap b')\right] =$$
$$(0 \cap b') \cup (a \cap 0) = 0 \cup 0 = 0.$$

Then, by Theorem 6 $(a \cup b)' = a' \cap b'$.

Definition. A binary relation $\leqq$ which, for simplicity's sake, is read "less than or equal to," is defined to be the following: $a \leqq b$ if $a \cap b = a$.

This relation differs from the linear order relation of the algebra of real numbers in that, given any two elements a and b of B, it may be that $a \leqq b$ or $b \leqq a$ or that neither of these holds. Some pairs of the elements then may be not comparable. From this definition the following theorems can be proved.

Theorem 1-9. For every a an element of B, $0 \leqq a \leqq 1$.

Proof: Since $0 \cap a = 0$, then $0 \leqq a$. Also, since $a \cap 1 = a$, $a \leqq 1$. Hence $0 \leqq a \leqq 1$.

Theorem 1-10. If for a, b elements of B, $a \leqq b$ and $b \leqq a$, then $a = b$.

Proof: If $a \leqq b$, then $a \cap b = a$, also if $b \leqq a$, then $b \cap a = b$. Since $a \cap b = b \cap a$, one has that $a = b$.

Theorem 1-11. If for a, b, c elements of B, $a \leqq b$ and $b \leqq c$, then $a \leqq c$.

Proof: If $a \cap b = a$, then $a \leqq b$. If $b \cap c = b$, then $b \leqq c$. Since $a \cap b = a$, and $b = b \cap c$, it follows that $a \cap b \cap c = a$, and $a \cap c = a$. So, $a \leqq c$.

Any collection B of elements, for which the definitions of equality and of " $\leqq$ " are satisfied and which satisfies the postulates and the theorems of the previous section is called a Boolean algebra.

ORDERED SETS AND LATTICES

Other methods could be used to define a Boolean algebra. One could choose a set of postulates different from those in the previous section. Also, as will be done in this section, one can define a Boolean algebra by use of other mathematical concepts.

In this section the concern will be with an ordering relation. First a relation is defined.

Definition. If, for any a and b in a set S of elements, either a is in relation R to b or a is not in relation R to b, then R is a binary relation (3, 2).

Frequently, a binary relation is referred to as a relation. The notation used in connection with relation is described in the following. The letter "R" will be used to denote a relation and " a R b " will denote that a is in relation "R" to b. Having defined the concept of relation, the concept of a special type of relation will be discussed in some detail.

Definition. An ordering relation is a relation R defined on a set S such that for elements a, b, c of S,

(1) a R a for every a in S,

(2) if (a R b) and (b R a), then a = b, and

(3) if (a R b) and (b R c), then a R c.

It is instructive to compare these three conditions with those which characterize an equivalence relation. Condition (1) and (3) are the familiar reflexive and transitive properties. Condition (2) differs from the symmetry property. If a R b is true and a and b are different, then b R a is false. A relation satisfying condition (2) is said to be antisymmetric.

Now one can define what is meant by an ordered set.

Definition. An ordered set is a set S together with an ordering relation R on S (1, 42). S is said to be ordered by the relation R.

The set S consisting of the set of all real numbers is ordered if for each pair a, b of real numbers, a R b means a is less than or equal to b in the usual order relation. This set is more restrictive than is necessary for a set to be ordered. This is indicated in the next example.

Let the set S be a collection of sets; for each pair A, B of sets in the collection S, "A R B" is defined as "A is a subset of B"; in symbols, $A \subset B$. This relation is called the relation of inclusion. This ordered set is important, especially in the case of the collection of all subsets of a given set. The relation of inclusion between subsets of a given set is a more general ordering relation than is the relation "less than or equal to" between real numbers. The reason for this is that every two real numbers a and b are comparable; that is, at least one of the relations "a is less than or equal to b" and "b is less than or equal to a" must be true. On the other hand, it is easy to find two sets A and B such that both of the relations "$A \subset B$" and "$B \subset A$" are false. Thus, for the inclusion relation, it is possible to have an incomparable pair of elements; that is, there may be two elements such that neither one is related to the other.

Next the idea of a greatest and least element of an ordered set is discussed. By definition, a greatest element of an ordered set S is an element g of S such that for each element a of S, $a \leqq g$, where $\leqq$ is an ordering relation, generally read "less than or equal to". A least element of an ordered set S is an element m of S, such that, for each element a of S, $m \leq a$.

There are ordered sets which have neither a least nor a greatest element, and also those which have both a greatest and a least element. The open unit interval is an example of a set which has neither a greatest nor a least element. Also the closed unit interval is an example of a set which has both a greatest and a least element.

In this section ordered sets will be studied as algebraic systems. First of all one begins by defining what is meant by an algebraic system. To define an algebraic system one must define a closed binary operation.

Definition. A closed binary operation on a set S is a mapping of S X S into S (3, 8).

For an operation * defined on a set S, it is required that, for each pair a, b of elements of S, a * b must be an element of S. It is not required that each element of S be expressible in the form a * b.

Now one can define an algebraic system.

Definition. An algebraic system is a set S together with certain relations or operations defined on S (1, 48).

Also, some new terminology is needed to state the definition of an algebraic system of an ordered set.

Let S be an ordered set and let A be a subset of S.

Definition. An upper bound of A is an element c of S such that, for each element a of A, $a \leq c$; a lower bound of A is an element d of S such that, for each element a of A, $d \leq a$.

Of course, there may be several upper bounds for a particular set A, or there may be none at all. This suggests that it might be convenient to choose the least of the upper bounds for A (if there is a least one) to represent all of the upper bounds. By definition let A be a subset of an ordered set S. A supremum of A, sup A, is an element of S such that sup A is an upper bound of A, and if c is any upper bound of A, then sup A $\leqq$ c. Similarly, the infimum of A, inf A, is an element of S such that, inf A is a lower bound of A, and if d is any lower bound of A, then d $\leqq$ inf A. The greatest lower bound's of sets are unique if they exist. If a and b are both greatest lower bound's of the same set A then a $\leqq$ b and b $\leqq$ a, whence a = b. The case for the least upper bound can be shown similarly. If the set A is finite, say $A = \{a_1, a_2, \cdots, a_n\}$, the elements sup A and inf A are sometimes denoted by $a_1 \vee a_2 \vee \cdots \vee a_n$ and $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ respectively. Also, the supremum of a set A can be referred to as the supremum of the elements of the set; for example, one can speak of the sup of a and b instead of sup$\{a, b\}$. One can speak of the infimum in a similar manner.

This leads one to the definition of a lattice.

Definition. A lattice is an ordered set such that each pair of elements has both a supremum and an infimum (1, 57).

As an example of a lattice, let S be a set and let $\mathcal{S}$ be the collection of all subsets of S, ordered by inclusion. Then $\mathcal{S}$ is a lattice. In fact, if A and B are any two elements of $\mathcal{S}$, the sup of A and B is just the union of the two sets where the union, A $\vee$ B, is the set of all objects which are elements of at least one of the two sets A and B. Similarly, the inf of A and B is their intersection, where the intersection,

$A \wedge B$, is the set of all objects which are elements of both of the sets A and B.

The operations sup and inf are not really binary operations on an ordered set S since it is not necessary to have exactly a pair of elements in order to perform these operations. It may be possible to perform these operations with any subset of S, finite or infinite, or with the entire set S. Of course, there may be subsets of S which do not have a sup or an inf; that is, subsets such that the operations sup or inf cannot be performed. The characteristic property of a lattice is that each of the operations sup and inf can be performed with any pair of elements. Thus, in a lattice it is frequently convenient to consider $\vee$ and $\wedge$ as binary operations. The following theorem shows that these operations are defined for all non-empty finite sets.

Theorem 2-1. In a lattice, any non-empty subset consisting of a finite number of elements has both a sup and an inf.

Proof: Let S be a lattice and consider a subset of S consisting of two elements $\{x_1, x_2\}$. Since this is a subset of a lattice, these two elements have a sup and an inf. $x_1 \vee x_2$ is the sup of the set $\{x_1, x_2\}$. Now assume the theorem is true for any set of n elements $\{x_1, x_2, \cdots, x_n\}$. The sup is denoted by $x_1 \vee x_2 \vee x_3 \cdots \vee x_n$. Now consider a subset which contains n + 1 elements $\{x_1, x_2, \cdots, x_n, x_{n+1}\}$. Now by assumption the set of elements $\{x_1, x_2, \cdots, x_n\}$ has a sup. Denote it by a. Since a is an element of S and x_{n+1} is an element of S, these two elements have a sup. Denote it by b. Now since the elements $x_1, x_2, \cdots, x_n$ are less than or equal to a and $a \leqq b$, then the elements $x_1, x_2, \cdots, x_n$ are less than or equal to b. Also $x_{n+1} \leqq b$, so the element b is an upper bound for the elements $x_1, x_2, \cdot \cdot \cdot, x_n, x_{n+1}$. Let c be any upper bound of $\{x_1, x_2, \cdot \cdot \cdot, x_{n+1}\}$, then the elements $x_1, x_2, \cdot \cdot \cdot, x_n$ are less than or equal to c. Therefore $a \leq c$ since a is the sup$\{x_1, x_2, \cdot \cdot \cdot, x_n\}$. Also $x_{n+1} \leq c$. Therefore $b \leq c$ since b is the sup of $\{a, x_{n+1}\}$. Hence $b = \sup\{x_1, x_2, \cdot \cdot \cdot, x_n, x_{n+1}\}$. Similarly one can prove this statement for the inf.

Theorem 2-2. Let a, b, and c be elements of a lattice S. The binary operations " $\vee$ " and " $\wedge$ " on S have the following properties:

(i) Commutative Laws.

(ii) Associative Laws.

(iii) Idempotent Laws.

(iv) Absorption Laws.

Proof: (i) Commutative Laws: $a \vee b = b \vee a$; $a \wedge b = b \wedge a$.

By definition $a \vee b = \sup\{a, b\}$ and $b \vee a = \sup\{b, a\}$, but the sets $\{a, b\}$ and $\{b, a\}$ are identical since they have the same elements. Thus $a \vee b = b \vee a$.

(ii) Associative Laws: $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

By definition $(a \vee b) \vee c \geq a \vee b$ and $(a \vee b) \vee c \geq c$. Since $a \vee b \geq a$ and $a \vee b \geq b$, then $(a \vee b) \vee c \geq a$ and $(a \vee b) \vee c \geq b$. Now suppose one has $x \geq a$, $x \geq b$, $x \geq c$, then $x \geq a \vee b$ and hence $x \geq (a \vee b) \vee c$. So $(a \vee b) \vee c$ is a least upper bound for $\{a, b, c\}$. Similarly, $a \vee (b \vee c)$ is a least upper bound for $\{a, b, c\}$. Now, since the least upper bound is unique $(a \vee b) \vee c = a \vee (b \vee c)$.

(iii) Idempotent Laws: $a \vee a = a$; $a \wedge a = a$. $a \vee a = \sup\{a\} = a$, hence $a \vee a = a$.

(iv) Absorption Laws: $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$. Since the ordering relation is reflexive, one has $a \leq a$. Also, since $a \wedge b$ is one of the lower bounds for $\{a, b\}$, one has $a \wedge b \leq a$. These two relations show that the element a is one of the upper bounds of the set $\{a, a \wedge b\}$. Evidently, if c is any upper bound of $\{a, a \wedge b\}$, then $a \leq c$. Thus, by definition, $\sup \{a, a \wedge b\} = a$.

The proofs of the remaining parts of the theorem are similar to those given.

The above theorem lists several properties of the binary operations " $\vee$ " and " $\wedge$ " in a lattice. Many other properties of these operations could have been mentioned, but the ones given are of particular importance; the following theorem shows that these properties actually characterize lattices.

<u>Theorem</u> 2-3. If an algebraic system is composed of a set S and two binary operations * and o on S such that, for all elements a, b, and c of S;

(i) $a * b = b * a$, $a \circ b = b \circ a$;

(ii) $a * a = a$, $a \circ a = a$;

(iii) $a * (b * c) = a * b) * c$, $a \circ (b \circ c) = (a \circ b) \circ c$;

(iv) $a * (a \circ b) = a$, $a \circ (a * b) = a$;

then there is a unique ordering relation in S which makes S a lattice and such that the given operations * and o are, respectively, $\vee$ and $\wedge$ in the lattice.

The proof of this theorem consists of the following lemmas.

<u>Lemma</u> 2-1. With S, *, and o as in the theorem, $a * b = b$ if and

only if $a \circ b = a$.

Proof: Suppose $a * b = b$. Then $a \circ b = a \circ (a * b) = a$ by the second of condition (iv) in the theorem. Suppose $a \circ b = a$, then $a * b = (a \circ b) * b = b * (a \circ b)$ by the commutative law. Similarly, $b * (a \circ b) = b * (b \circ a)$. But $b * (b \circ a) = b$ by the first part of (iv) in the theorem.

Lemma 2-2. With S, $*$, and $\circ$ as in the theorem define $\leqq$ by $a \leqq b$ if and only if $a * b = b$. Then $\leqq$ is an ordering relation on S; moreover, with this ordering relation, S is a lattice and the operations $\vee$ and $\wedge$ in the lattice are, respectively, $*$ and $\circ$.

Proof: It is evident that $\leqq$ is a relation on S; to prove it is an ordering relation, we must show that it is reflexive, antisymmetric, and transitive.

The relation $\leqq$ is reflexive since, by the first of conditions (ii) in the theorem, $a * a = a$ for each element a in S.

The relation $\leqq$ is antisymmetric since, if both of $a \leqq b$ and $b \leqq a$ are true, then $a * b = b$ and $b * a = a$. These equations, together with the first of conditions (i) in the theorem, imply $a = b$.

The relation $\leqq$ is transitive since, if $a * b = b$ and $b * c = c$, then $a * c = a * (b * c) = (a * b) * c = b * c = c$. This completes the proof that $\leqq$ is an ordering relation on S.

To complete the proof of the lemma it must be shown that each pair of elements a, b of S has a sup and an inf and that $a \vee b = a * b$ and $a \wedge b = a \circ b$. First, notice that $a \leqq a * b$ since $a * (a * b) = (a * a) * b = a * b$, and that $b \leqq a * b$ since $b * (a * b) = a * (b * b) = a * b$. Thus $a * b$ is an upper bound for $\{a, b\}$. Let c be any upper bound for $\{a, b\}$; then $a * c = c$ and $b * c = c$, so $(a * b) * c = a * (b * c) = a * c = c$.

Hence $a * b \leq c$; this proves that $a * b = a \vee b$.

Now, notice that $a \circ b \leq a$ because $(a \circ b) \circ a = a \circ (a \circ b) = (a \circ a) \circ b = a \circ b$ and $a \circ b \leq b$ because $(a \circ b) \circ b = a \circ (b \circ b) = a \circ b$.

Thus $a \circ b$ is a lower bound for $\{a, b\}$. Let c be any lower bound for $\{a, b\}$; then $c \circ a = c$ and $c \circ b = c$, so $c \circ (a \circ b) = (c \circ a) \circ b = c \circ b = c$. Hence, $c \leq a \circ b$; this proves that $a \circ b = a \wedge b$.

<u>Lemma</u> 2-3. With S, *, and o as in the theorem, the relation $\leq$ defined in the above lemma is the only ordering relation on S which makes S a lattice and such that the given operations * and o are, respectively, $\vee$ and $\wedge$ in the lattice.

Proof: Let " R " be any ordering relation on S which makes S a lattice, and such that "*" and "o" are, respectively, " $\vee$ " and " $\wedge$ ". If a R b, then $a \vee b = b$. Therefore, since " $\vee$ " is the same operation as "*", one has that $a * b = b$; thus by the definition in the previous lemma, $a \leq b$. The proof is completed by noting that each of the above steps is reversible. This also completes the proof of the theorem (1, 60-62).

So far only conditions which are satisfied by all lattices have been considered. There are several interesting conditions which are satisfied by some lattices, but not by others. The remainder of this section is devoted to such conditions.

A lattice S is distributive if and only if, for all elements a, b, c of S $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Thus, a lattice is distributive if and only if the operation " $\wedge$ " distributes over the operation " $\vee$ ". This seemingly unsymmetric treatment of the two lattice operations is misleading; the condition is actually symmetric in the two operations, as the following theorem shows.

Theorem 2-4. A lattice S is distributive if and only if, for all elements a, b, c of S, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Proof: Suppose S is distributive; then

$$(a \vee b) \wedge (a \vee c) = [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c].$$

By the absorption law and another use of the distributive property,

$$[(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] = a \vee [(a \wedge c) \vee (b \wedge c)].$$

The associative law, and another use of the absorption law, give

$$a \vee [(a \wedge c) \vee (b \wedge c)] = a \vee (b \wedge c).$$

Thus, if S is distributive, then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

To prove the converse assume

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

and use the above proof with the interchange of $\wedge$ and $\vee$. Then

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

An example of a distributive lattice is the lattice of all subsets of a given set S, ordered by inclusion. If A, B, and C are subsets of S, the set $A \wedge (B \vee C)$ is composed of all elements which are in A and also in at least one of B or C. The set $(A \wedge B) \vee (A \wedge C)$ is composed of all elements which are either in both of A and B or in both of A and C.

From this one can see that these two conditions on the elements of S are equivalent; thus $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.

It has been shown that an ordered set may or may not contain greatest and least elements. The same is true of a lattice.

The collection $\mathcal{S}$ of all subsets of a given set S forms a lattice when inclusion is the ordering relation. The greatest element g of this lattice is S itself, and the least element m is the empty set $\emptyset$. The familiar set operations of union and intersection are the operations sup and inf in the lattice. But there is another set operation, complementation, which has not been needed so far. The complement of a subset A of S is defined to be the collection of all elements of S which are not elements of A. Thus, complementation is an unary operation. If A' is the complement of A, one can see that $A \vee A' = S$, and $A \wedge A' = \emptyset$. This familiar set operation suggests the following definition.

Definition. In a lattice S with greatest element g and least element m, a complement of the element a of S is an element b of S such that $a \vee b = g$ and $a \wedge b = m$.

Definition. A complemented lattice is a lattice in which there is a greatest element and a least element and in which each element has at least one complement.

Ordered sets have been presented and a lattice has been defined as a special type of ordered set. A Boolean algebra is then a special type of lattice. It can be defined in the following manner. A Boolean algebra is a complemented, distributive lattice. Thus, a Boolean algebra is an ordered set in which each pair of elements has both a sup and an inf, each of the binary operations " $\vee$ " and " $\wedge$ " distributes over the other, there is a greatest element and a least element, and each element has at least one complement. Note that a Boolean algebra must be nonempty--in particular, it must contain a greatest element. From this beginning, one can derive the same properties which were presented in the beginning description of a Boolean algebra. This completes the development of a Boolean algebra by beginning with the concept of ordered sets.

APPLICATION

An important application of the Boolean algebra defined in the first section of this paper will be covered in this section. This application consists of using a Boolean algebra to represent a model of combinational relay circuitry.

In electronic digital computers, telephone switching systems, control systems for automatic factories, and other systems involving communication or processing of data, one finds many examples of electric circuits which employ what are known as two-state or bi-stable devices (5, 1). The simplest example of such a device is a switch or contact which may be in the open state or in the closed state. When a contact is operated with the aid of an electromagnet, the combination is called a relay. A switch or relay is called a bilateral circuit element since it permits the passage of current in either direction when it is closed. Devices which permit the passage of current in only one direction are called unilateral.

The methods and results of Boolean algebra and related subjects have been found useful in discussing circuits employing two-state devices. Initially, one uses contact networks to illustrate how this is done, for the mathematical system is particularly simple to develop in this case.

Two mathematical symbols are introduced as the first step in constructing the desired system. With an open contact or path in a circuit, one associates the symbol "0"; and with a closed contact or path, one associates the symbol "1". The postulates that follow will give 0 and 1 their mathematical meaning.

When the condition of a contact is variable in a problem, it is represented by a literal symbol such as a, b, x, y, etc. Such a symbol, called a circuit variable, takes on the value 0 when the contact is open, the value 1 when it is closed. With each symbol x, a symbol x' is associated called the complement of x, which is 1 when x is 0 and 0 when x is 1. The complement x' of x is the circuit variable associated with a contact which is open when the x- contact is closed and is closed when the x- contact is open. Note that the operation of complementation is thus defined in terms of the symbols 0 and 1. This definition implies that $0' = 1$ and $1' = 0$.

When two or more contacts always open and close simultaneously, they are denoted by the same circuit variable. If a "make contact" is denoted by x, then a "break contact" operated by the same electromagnet is denoted by x' (5, 3). Employing this notation, it may be verified that in the following diagram there is a path from t_1 to t_2 if and only if $x' = y' = 1$ or $x = y = 1$ and a path from t_3 to t_4 if and only if $x = 0$, $y = 1$ or $x = 1$, $y = 0$.

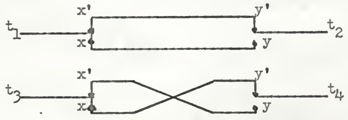


Figure 1.

In figure 1 above, one sees where the mathematical system does not represent all conditions of the switch. When the coil of such a transfer contact is energized, there is a brief time when neither contact is closed. Similarly in figure 2, there is a brief time when neither contact is open.

Figure 2.

Thus, the assumption that one may call one contact x and the other x' is not strictly justified. When this becomes a serious problem, it can be corrected by design technique, to eliminate undesirable effects (5, 4).

Next it will be shown how the physical situation can be interpreted in terms of two operations on the symbols 0, 1, x, y, etc.

Applied to two circuit variables, the operation of union, denoted by $\cup$, may be represented physically by the parallel connection of the contacts corresponding to these variables. Thus, in figure 3 A, contacts symbolized by a and b are shown connected in parallel, and the connection is represented in the mathematical system by the union a $\cup$ b. This is read as a or b since the circuit provides a closed path between its endpoints if and only if the a- contact, or the b- contact, or both, are closed.

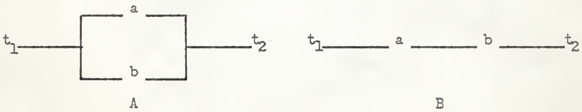


Figure 3.

Applied to two circuit variables, the operation of multiplication, denoted by $\cap$ or simply by juxtaposition, may be represented physically by the series connection of the contacts corresponding to these variables. In figure 3 B, contacts symbolized by a and b are shown connected in series and this connection is represented by the product a $\cap$ b. This is read as "a and b" when it is desired to emphasize the fact that the circuit provides a closed path between its endpoints if and only if the a- contact is closed and the b- contact is also closed.

The circuits drawn in figure 3 provide open or closed paths between the terminals t_1 and t_2 depending on the states of the contacts involved. Any circuit consisting of interconnected contacts whose purpose is to connect two fixed points with a conducting path under specified conditions of the contacts is called a two-terminal switching circuit, and it is such circuits that are included in this section.

There is a large class of two-terminal switching circuits and with each circuit, one can associate a function of the circuit variables. The characteristic property of each of these functions is that it takes on the value 1 for all combinations of values (0's or 1's) of the circuit variables which correspond to the circuit being closed and takes on the value 0 for all combinations which correspond to the circuit being open. This function is called the switching function of the two-terminal circuit.

Two switching functions of the same set of circuit variables will be regarded as equal and their circuits as equivalent if and only if both switching functions take on the value 1 for exactly the same combinations of values of the circuit variables and hence, also both take on the value 0 for the same combinations of values of the circuit variables.

For a permanently open circuit, the switching function is identically

0 and for a permanently closed circuit, the switching function is 1.

For a circuit containing a single contact represented by the circuit variable a, the switching function is just a.

In the case of a two-terminal circuit consisting of two contacts connected in parallel, $a \cup b$ is to be the corresponding switching function where a and b are the circuit variables corresponding to these contacts. Now, only when $a = b = 0$ does the circuit fail to provide a closed path between its terminals. Hence, only when $a = b = 0$ should the switching function $a \cup b$ take on the value zero. However, if $a = 1$ and $b = 0$, or $a = 0$ and $b = 1$, or $a = b = 1$, the circuit does provide a closed path between its terminals and hence, in all these cases, the switching function $a \cup b$ should take on the value 1. For all these conditions to be satisfied, define

$$0 \cup 0 = 0,\ 1 \cup 0 = 1,\ 0 \cup 1 = 1,\ 1 \cup 1 = 1.$$

In the case of a two-terminal circuit consisting of two contacts connected in series, $a \cap b$ is the corresponding switching function where a and b are circuit variables representing the contacts. If $a = b = 0$, or if $a = 0$ and $b = 1$, or if $a = 1$ and $b = 0$, the circuit fails to provide a closed path and hence, the switching function $a \cap b$ should take on the value 0 in all these cases. Only if $a = b = 1$ does the circuit provide a closed path between its terminals and hence $a \cap b$ should be 1 only in this case. For all these conditions to be satisfied, define

$$0 \cap 0 = 0 \cap 1 = 1 \cap 0 = 0,\ 1 \cap 1 = 1.$$

Now suppose one has given two circuits with switching functions f and g respectively, and suppose the circuits are connected in parallel.

Then the resulting circuit is closed if either one of the two given circuits is closed or if both are closed. Hence, for exactly those combinations of values of the circuit variables such that $f = 1$, $g = 0$ or $f = 0$, $g = 1$ or $f = g = 1$, the switching function of the parallel circuit must take on the value 1, and thus may be represented by $f \cup g$. To evaluate $f \cup g$ at a given combination of values of the circuit variables, first evaluate f and g individually, and then compute the union of these values as defined above.

If the two circuits are connected in series, the resulting circuit is closed only when the two given circuits are both closed. Hence, for exactly those combinations of values of the circuit variables such that $f = g = 1$, the switching function of the series circuit must take on the value 1, and this may be represented by $f \cap g$. To evaluate $f \cap g$ at any given combination of values of the circuit variables, multiply the individual values of f and g as defined above.

Also, the operation of complementation can be applied to arbitrary switching functions. Given a two-terminal circuit c with switching function f, a circuit denoted by c' is closed when c is open and open when c is closed. The switching function of c' will be denoted by f'. This is consistent with the earlier use of x and x' as circuit variables to denote normally open and normally closed contacts on the same relay.

These ideas can be illustrated by referring to figure 1. In this figure there is a path through the circuit joining t_1 to t_2 if the x-contact and the y- contact are closed or if the x'- contact and the y'-contact are closed, i.e., if $x = y = 1$ or $x' = y' = 1$. Each of these paths has two contacts in series and the two paths are in parallel. Hence, by the principles outlined above, the switching function is

$(x \cap y) \cup (x' \cap y')$.

In the case of the circuit joining t_3 to t_4, there is a closed path if the x'-contact and y-contact are colsed or if the x-contact and y'-contact are colsed, i.e., if $x' = y = 1$ or $x = y' = 1$. Again, each path has two contacts in series and the two paths are in paralled. Hence, the switching function must be $(x' \cap y) \cup (x \cup y')$.

These results are summarized in table 1.

Table 1.

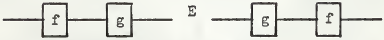
x	y	$(x \cap y) \cup (x' \cap y')$	$(x' \cap y) \cup (x \cap y')$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

From the table one can see that the two functions have complementary values. As a result, the two functions are complements of each other. The function $(x' \cap y) \cup (x \cap y')$ is said to be the exclusive or function since it is 1 if $x = 1$ or $y = 1$, but not when both are 1.

Now the postulates governing the application of the symbols $\cup$, $\cap$, and ' to arbitrary switching functions which come from the physical considerations can be given. In the diagrams to follow, equivalence of circuits is denoted by a capital "E". The symbols f, g, h denote arbitrary switching functions, which include 0, 1, and symbols for single contacts as special cases.

The Commutative Laws:

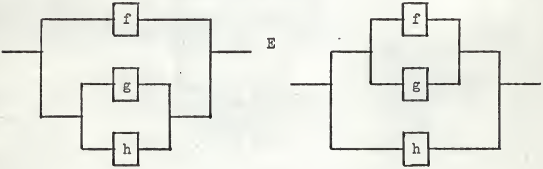
$f \cup g = g \cup f$



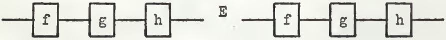
$f \cap g = g \cap f$

Figure 4.

The Associative Laws:



$f \cup (g \cup h) = (f \cup g) \cup h.$

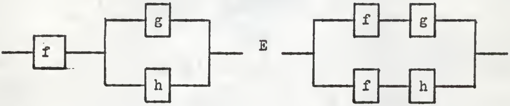


$f \cap (g \cap h) = (f \cap g) \cap h.$

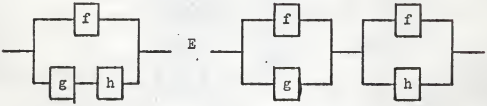
Figure 5.

Here connecting f in parallel with the parallel connection of g and h gives the same physical circuit as connecting the parallel connection of f and g in parallel with h. Similarly, in the series it is legitimate to write f $\cup$ g $\cup$ h and f $\cap$ g $\cap$ h as switching functions for these circuits.

The Distributive Laws:



$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h).$$

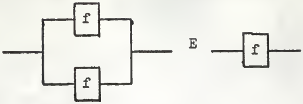


$$f \cup (g \cap h) = (f \cup g) \cap (f \cup h).$$

Figure 6.

The first of these laws indicates that multiplication distributes to each term of a union and the second indicates that union distributes to each factor in multiplication. In the first case, both circuits are closed if and only if the f-circuit is closed and at least one of the other two circuits is closed. In the second case, both circuits are closed if and only if the f-circuit is closed or both of the other two circuits are closed.

The Idempotent Laws:



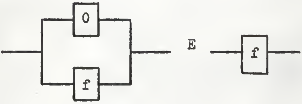
$f \cup f = f.$

$f \cap f = f.$

Figure 7.

These laws are dictated by the fact that each circuit here is closed if and only if the f circuit is closed. These laws account for the absence of conventional exponents and coefficients other than 0 and 1 in the algebra of switching circuits.

The Laws of Operation with 0 and 1:



$0 \cup f = f.$

$1 \cap f = f.$

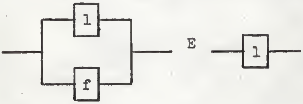
Figure 8.

Here a 0 designates a permanently open circuit and a 1 designates a permanently closed circuit.

Note that 0, with respect to union, and 1, with respect to multiplication, are identity elements; that is, they leave the function f unchanged.

Contrast this with the behavior of 0 in multiplication and of 1 in union:

$$0 \cap f = 0.$$

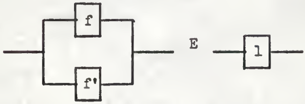


$$1 \cup f = 1.$$

Figure 9.

The Laws of Complementarity:

$$f \cap f' = 0.$$



$$f \cup f' = 1.$$

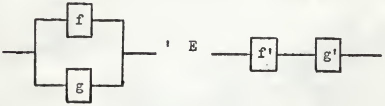
Figure 10.

In the first case, one of the circuits corresponding to f and f' is always open while in the second case one of these circuits is always closed.

A prime on a circuit denotes the complement of that circuit. One should notice that whenever a particular circuit is closed (open), the corresponding complementary circuit shown here is indeed open (closed). These laws enable one to compute the complement of an arbitrary switching function. The following is a statement of these laws. The complement of a product is the union of the separate complements and the complement of a union is the product of the separate complements.

The Laws of Dualization (DeMorgan's Laws):

$$(f \cap g)' = f' \cup g'.$$



$$(f \cup g)' = f' \cap g'.$$

Figure 11.

The Law of Involution:

$$\left(-\boxed{f'}-\right)' \quad E \quad -\boxed{f}-$$

$$(f')' = f.$$

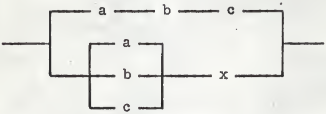
Figure 12.

If one forms the complement and then forms the complement again, the original function is recovered.

An important fact to note is that the above postulates, except for the last, appear in dual pairs, each member of a pair being obtainable from the other by replacing each of the operations " $\cup$ " and " $\cap$ " by the other and replacing each of the symbols 0 and 1 by the other, whenever they appear. In these simple cases, such interchanges may be interpreted as the interchange of series and parallel connections and of open and closed circuits. The last postulate may be regarded as its own dual since the required interchanges leave it unaltered.

The symbols $\cap$ and $\cup$ respectively have been used to represent the concepts "and" and "or" and these concepts have been interpreted with the aid of series and parallel connections, respectively. One can build up circuits of desired complexity by suitable successions of series and parallel connections. Such circuits are called series-parallel circuits. Any circuit obtainable by substituting a known series-parallel circuit for any contact of a known series-parallel circuit is also defined to be series-parallel. Any circuit not so obtainable is called a non-series-parallel or bridge circuit (5, 13). In every circuit, series-parallel or bridge, independent contacts are represented by distinct circuit variables, identically behaving contacts by the same circuit variable, and oppositely behaving contacts by a variable and its complement.

The switching function of any given series-parallel circuit can be written without difficulty, for the switching function is simply a mathematical statement of the various series and parallel connections. For example, with the following circuit one can associate the switching function,



$$f = (a \cap b \cap c) \cup \left[(a \cup b \cup c) \cap x\right].$$

Figure 13.

In the case of the bridge circuit, the "and" and "or" relationships of the various paths through the circuit are often less evident than they are in the series-parallel case. However, one can still write a switching function for the bridge circuit by first tracing all possible paths through it. For example, the bridge circuit of figure 14 provides four possible paths between the terminals t_1 and t_2. These are just the same paths as are provided by the series-paralled circuit of figure 13 so that the function f is also the function of this bridge.

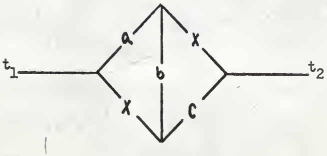


Figure 14.

It should be observed that whereas the given bridge realizes the same switching function as does the series-parallel circuit, it does so with two fewer contacts.

The general class of circuits which has n variables $x_1, x_2, \ldots, x_n$ and also their n complements is called a combinational circuit. One can always write a switching function $f(x_1, x_2, \ldots, x_n)$ for such a combinational circuit, using only the operations "$\cap$", "$\cup$", and " ' ".

To complete the mathematical system of combinational switching circuits, the relation of inclusion is introduced. Let $f_1(x_1, x_2, \ldots, x_n)$ and $f_2(x_1, x_2, \ldots, x_n)$ be switching functions associated with two-terminal combinational circuits s_1 and s_2. If s_1 is never closed unless s_2 is also closed, it shall be said that s_1 is included in s_2. Rather it means that the ability of s_1 to close a path between its terminals for certain combinations of relays operated is possessed also by s_2, so that s_2 includes the circuit closing ability of s_1.

If s_1 is included in this sense in s_2, then for every combination that makes $f_1 = 1$, f_2 must be 1 also. However, one may well have $f_2 = 1$ for some combinations for which $f_1 = 0$. When f_1 and f_2 are related in this way, one writes $f_1 \leqq f_2$ or $f_2 \geqq f_1$ reading the symbols "f_1 is equal to or less than f_2" and "f_2 is equal to or greater than f_1", respectively. This terminology is of course suggested by the resemblance of our notation to that of ordinary arithmetic. It implies in particular that $0 \leqq 0$, $0 \leqq 1$, and $1 \leqq 1$. These statements are all weaker than is necessary since in fact $0 = 0$, $1 = 1$, and $0 < 1$. Where $0 < 1$ means "0 is less than 1". Indeed, the weaker statements are often more useful than the stronger ones.

Now the basic properties of the relation " $\leqq$ " will be established.

Here f, g, h are arbitrary switching functions of the same circuit variables $x_1, x_2, \ldots, x_n$.

The Universal Bounds Property: For all f, $0 \leqq f \leqq 1$.

This is immediate from the definition of a switching function.

The Reflexive Property: For all f, $f \leqq f$.

Since, in fact, $f = f$, the weaker statement, $f \leqq f$, is certainly true.

The Antisymmetric Property: If $f \leqq g$ and $g \leqq f$, then $f = g$.

The antisymmetric property indicates that the relation $\leqq$ can never hold symmetrically between non-equivalent switching functions. The proof follows from the fact thet $g = 1$ for each combination that makes $f = 1$ and also $f = 1$ for each combination that makes $g = 1$. Thus f and g take on the value 1 for exactly the same combinations of values of the circuit variables. Hence, they are also both zero for the same combinations and therefore, having equal values for all combinations, they are in fact equal functions.

The Transitive Property: If $f \leqq g$ and $g \leqq h$, then $f \leqq h$.

Indeed, since $g = 1$ for each combination such that $f = 1$, and $h = 1$ for each combination such that $g = 1$, one has also $h = 1$ for each combination such that $f = 1$.

The Consistency Principle: For all f and g, $f \leqq g$ if and only if $f \cap g = f$. For all f and g, $f \leqq g$ if and only if $f \cup g = g$.

Each of these is in fact two statements of the physical situation.

In the first case suppose $f \leqq g$. Then if $f = 0$, $f \cap g = f$ becomes simply $0 = 0$. If on the other hand $f = 1$, then $g = 1$ also since $f \leqq g$, and hence $f \cap g = f$ becomes $1 = 1$. Thus when $f \leqq g$, one concludes $f \cap g = f$ always. Next suppose $f \cap g = f$ for all combinations of values of the circuit variables. Then for every combination such that $f = 1$, one must also have $g = 1$ in order to have $f \cap g = f$, i.e., $1 \cap g = 1$, here. Thus $f \leqq g$. One may verify the second principle in a similar fashion.

It should be noted that for given switching functions f and g, one need not necessarily have either $f \leqq g$ or $g \leqq f$. That is f and g are, in such a case, not comparable. The functions $f = (x \cap y) \cup (x' \cap y')$ and $g = (x \cap y') \cup (x' \cap y)$ provide an example. Since neither of these functions is identically zero and since they are never simultaneously 1, neither can be equal to or less than the other in the sense defined above.

What has been done is to show that by interpreting 0, 1, $\cup$, $\cap$, $'$, $=$, and $\leqq$ as indicated in prededing sections, and by comparing the properties of section 3 with the theorems of section 1, one sees that the set of all switching functions of n circuit variables can be interpreted as a Boolean algebra.

Since switching functions take on only the values 0 and 1, they provide a very special example of a Boolean algebra. It is worthwhile to reflect on how natural the postulates set up for the algebra of switching circuits appear to be, yet how they differ from those of the more familiar algebra of real and complex numbers. This will help to emphasize that there are algebras other than the familiar one. Also one must properly select and understand the basic postulates of the particular algebra he is working with in order to better understand the mathematics.

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ELEMENTARY BOOLEAN ALGEBRA

by

RICHARD EARL SIMS

B. S., Kansas State University, 1963

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This report concerns an introductory treatment of a Boolean algebra. Two different sets of postulates for the algebra are presented. The first set of postulates is given in terms of the operations of union and intersection. A Boolean algebra is defined as a special type of lattice in the second set of postulates. The report also includes an application of a Boolean algebra to elementary two-state switching circuits.

In the first section a Boolean algebra is defined by stating postulates concerning the operations of a Boolean algebra. Subsequently, theorems are developed which relate to these operations of a Boolean algebra and which concern particular elements of a Boolean algebra. The specific properties treated are those which are used later in the representation of two-state switching circuits by a Boolean algebra.

The second section is also a treatment of a Boolean algebra. An ordering relation, an ordered set, least upper bounds, and greatest lower bounds are defined in this section. From these definitions certain properties of ordered sets, least upper bounds, and greatest lower bounds are introduced. A lattice is defined as an ordered set in which each pair of elements has a least upper bound and a greatest lower bound. A general lattice and two special lattices are discussed. A Boolean algebra is a special type of lattice, a complemented distributive lattice.

In concluding the report it is shown that two-state switching circuits can be represented by a Boolean algebra. An analogy is drawn between the postulational development of a Boolean algebra and a system of two-state switching circuits. Certain restrictions are placed on the circuits so that systems of two-state switching circuits can be given as an application of a Boolean algebra. Finally, certain results concerning the application of a Boolean algebra to a system of two-state switching circuits are indicated.