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Published Version Information

Citation: Ramm, A. G. (2012). A problem in analysis. Analysis, 32(2), 107-110.

Digital Object Identifier (DOI): doi:10.1524/anly.2012.1152

Publisher's Link: http://www.oldenbourg-link.com/loi/anly

Analysis 32, 1001–1003 (2012) / **DOI** 10.1524/anly.2012.1152

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Received: Nov. 17, 2011

Summary: Assume that $D \subset \mathbb{R}^2$ is a bounded domain, diffeomorphic to a disc, star-shaped, with a $C^{1,\lambda}$ boundary C, $\lambda > 0$, which can be represented in polar coordinates as $r = f(\phi)$, where $f > 0$ is a smooth $2\pi-$periodic function. Let $\psi_{\pm n} := \psi_{\pm n}(\phi) := e^{\pm in\phi} f^n(\phi)$.
Theorem. *Assume that*

$$\int_0^{2\pi} \psi_{\pm n} f^2(\phi) d\phi = 0 \qquad n = 1, 2, \dots$$

Then $f = const$.

1 Formulation of the result

Assume that $D \subset \mathbb{R}^2$ is a bounded domain, diffeomorphic to a disc, star-shaped, with a $C^{1,\lambda}$ boundary C, $\lambda > 0$, which can be represented in polar coordinates as $r = f(\phi)$, where $f > 0$ is a smooth $2\pi-$periodic function. Let $\psi_{\pm n} := \psi_{\pm n}(\phi) := e^{\pm in\phi} f^n(\phi)$.

Theorem 1.1 *Assume that*

$$\int_0^{2\pi} \psi_{\pm n} f^2(\phi) d\phi = 0 \qquad n = 1, 2, \dots \tag{1.1}$$

Then $f = const$.

Remark 1.2 A similar result is true for $D \subset \mathbb{R}^m$, $m > 2$. Its proof is essentially the same.

Remark 1.3 The author raised the question, answered in Theorem 1.1, while thinking about the Pompeiu problem, see Chapter 11 in [1]. This question is of interest regardless of its relation to the Pompeiu problem since it gives an unusual result concerning completeness of a set of functions.

In Section 2 a proof is given.

AMS 2010 subject classification: 42C30
Key words and phrases: Completeness of a set of functions

2 Proof

Assumption (1.1) implies that

$$\int_D h_n dx = 0 \qquad n = 1, 2, \dots, \tag{2.1}$$

where $h_n := r^{|n|}e^{\pm in\phi}$ are harmonic functions regular at the origin, $x \in \mathbb{R}^2$, $x = (r, \phi)$, where (r, ϕ) are polar coordinates. To see that (1.1) is equivalent to (2.1), write the left-hand side of (2.1) in polar coordinates, integrate over r from 0 to $f(\phi)$, and get (1.1).

Let $y \in \mathbb{R}^2$, B_R be a ball (disc), centered at the origin and containing D inside, B'_R be its complement in $\mathbb{R}^2$, and $G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}$ be the fundamental solution of the Laplace equation in $\mathbb{R}^2$. Let

$$r := |x|, \qquad r' := |y|, \qquad x \cdot y = rr' \cos\theta.$$

Then, for $r > r'$, one has

$$2\pi G(x, y) = -\left[\ln r + \frac{1}{2}\left(\ln\left(1 - \frac{r'}{r}e^{i\theta}\right) + \ln\left(1 - \frac{r'}{r}e^{-i\theta}\right)\right)\right], \qquad r > r'. \tag{2.2}$$

Expanding $\ln(1 - \frac{r'}{r}e^{\pm i\theta})$ in Taylor series, which is possible since $\frac{r'}{r} < 1$, one gets

$$\ln\left(1 - \frac{r'}{r}e^{i\theta}\right) = -\sum_{n=1}^{\infty} \frac{h_n}{nr^{n+1}}, \qquad r > r', \quad h_n = (r')^n e^{\pm in\theta}. \tag{2.3}$$

We conclude from the assumption (2.1) and from (2.2)–(2.3) that

$$\int_D G(x, y)dy = -\frac{1}{2\pi}|D| \ln r, \qquad r > R, \tag{2.4}$$

where $|D|$ denotes area of D.

Using the method from [2] (see also [3]) we derive from (2.4) that D is a disc.

It follows from (2.4) that the harmonic in $D' = \mathbb{R}^2 \setminus D$ function

$$u(x) := \int_D G(x, y)dy = -\frac{1}{2\pi}|D| \ln r, \qquad r > R, \tag{2.5}$$

solves the equation

$$\Delta u(x) = -\eta\, |D|, \tag{2.6}$$

where η is the characteristic function of D, that is, $\eta = 1$ in D, and $\eta = 0$ in D'. Let C_R be the boundary of B_R. A harmonic in B_R function h satisfies the conditions

$$\int_{C_R} h_N ds = 0, \qquad \int_{C_R} h ds = 2\pi h(0). \tag{2.7}$$

It follows from (2.5) that the functions $u(x)$ and $u_N(x)$ are constant on C_R, since the normal $\mathbf{N}$ on C_R is directed along the radius. Multiply (2.6) by an arbitrary regular at the origin harmonic function $h = h_n$, integrate over a disc B_R, and use (2.7) to get

$$\int_D h dx = \int_{C_R} (uh_N - u_N h)ds = ch(0), \quad c = const. \tag{2.8}$$

If h is harmonic in B_R, then so is $h(gx)$, where g is a rotation by an arbitrary angle α around z-axis, the axis perpendicular to D. Since $h(g0) = h(0)$, one can replace $h(x)$ by $h(gx)$ in (2.8), differentiate with respect to α and then set $\alpha = 0$. This yields

$$\int_D \nabla h(x) \cdot [e_3, x] dx = 0, \tag{2.9}$$

where e_3 is a unit vector along z-axis, $\cdot$ stands for the scalar product, $[e_3, x]$ is the vector product in $\mathbb{R}^3$, and h is an arbitrary harmonic function in B_R, regular at the origin. One has

$$\nabla h(x) \cdot [e_3, x] = \nabla \cdot (h[e_3, x]), \tag{2.10}$$

because $\nabla \cdot [e_3, x] = 0$. Thus, integrating by parts in (2.9), one gets

$$\int_C (-N_1 s_2 + N_2 s_1) h ds = 0, \tag{2.11}$$

where N_j, $j = 1, 2$, are the components of the outer unit normal $\mathbf{N}$ to C. It is proved in [2] that *the set of restrictions of all harmonic functions in B_R, regular at the origin, onto a closed curve $C \subset B_R$, diffeomorphic to a circle, is dense in $L^2(C)$*. Therefore, (2.11) implies

$$-N_1 s_2 + N_2 s_1 = 0 \qquad \forall s \in C. \tag{2.12}$$

Let us derive from equation (2.12) that C is a circle. Geometrically equation (2.12) means that the radius-vector $\mathbf{r} := s_1 e_1 + s_2 e_2$ of the boundary C is parallel to the normal $\mathbf{N}$ to C, namely, $[\mathbf{r}, \mathbf{N}] = 0$. The unit tangential vector to C is $\mathbf{t} = d\mathbf{r}/ds$, where s is the arclength of C, and the normal $\mathbf{N}$ is directed along $d\mathbf{t}/ds$.

Since the normal $\mathbf{N}$ is orthogonal to $\mathbf{t}$, and $\mathbf{N}$ is parallel to $\mathbf{r}$ according to (2.12), it follows that $\mathbf{t} \cdot \mathbf{r} = 0$. Thus,

$$d\mathbf{r}/ds \cdot \mathbf{r} = 0 \qquad \forall s \in C. \tag{2.13}$$

Consequently,

$$\mathbf{r} \cdot \mathbf{r} = const \qquad \forall s \in C. \tag{2.14}$$

Therefore, C is a circle, and D is a disc.

Theorem 1.1 is proved. $\square$

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