COMPARISON OF ESTIMATORS FOR THE PARAMETERS OF THE POISSON-GAMMA MARGINAL DISTRIBUTION

by

BARBARA SUE KERR

B.S., University of Southern Colorado, 1977

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1980

Approved by:

Doris Grosh

Major Professor

This Report is dedicated to my parents, who taught me the important things.

ACKNOWLEDGMENTS

I wish to express my many thanks to Dr. Doris Grosh, my major professor. Working with her has not only been an educational experience but also a pleasure. I also wish to thank Dr. James Hess and Dr. George Milliken for their invaluable help. I would like to acknowledge Rei-Kung Sun who programmed the Newton-Raphson procedure which I incorporated into my simulation.

THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.

TABLE OF CONTENTS

SECTION 1: DERIVATIONS

Consider a Poisson process arising from an experiment where a component is observed for time T. During the duration of the experiment let X denote the number of "arrivals" (equipment failures and instantaneous repairs). Since this is a Poisson process, we assume an unknown arrival rate λ. The probability function for X = x is

$$[1.1] \qquad f(X|\lambda,T) = \frac{e^{-\lambda T}(\lambda T)^x}{x!} \qquad I_{\{0,1,2,\ldots\}}(x)$$

with $\quad E(X|\lambda,T) = \lambda T$

and $\quad Var(X|\lambda,T) = \lambda T$.

Using classical analysis the method of moments estimator of λ (also maximum likelihood estimator) is $\lambda = \frac{x}{T}$.

Thinking of λ in the Bayesian sense implies that the particular failure rate for an experiment is a realization of Λ, a random variable with some distribution, $g(\lambda|\underline{\theta})$. In this way we say that λ, the arrival rate of the Poisson process, is not a fixed value for every experiment; instead it may vary. In choosing a prior distribution $g(\lambda|\underline{\theta})$ it is important to choose one which gives positive probability to all possible values of λ and zero probability to those values not feasible. It is important to choose a distribution with flexible shape and one which is mathematically conformable to the conditional distribution $f(x|\lambda,T)$. For this case the gamma distribution fulfills all these requirements.

Assuming $\Lambda \sim gamma(\alpha,\beta)$ gives probability density function

$$[1.2] \qquad g(\lambda|\alpha,\beta) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{\lambda}{\beta}} \quad I_{[0,\infty)}(\lambda)$$

This distribution has as its support the interval $[0,\infty)$. Note that for all values of α and β $\lim_{\lambda\to\infty}(g(\lambda|\alpha,\beta))=0$. However, the shape of the distribution varies widely according to α. The parameter β is a scale parameter, thus providing sufficient flexibility of shape and scale.

Having assumed a gamma (α,β) prior distribution for λ it is necessary to know the particular distribution from which the λ is drawn. In other words, estimates of α and β are needed. The actual λ from each experiment is never observed, but is estimated by $\frac{x}{T}$. The estimates of α and β then will be based on the estimates of λ.

Recall we have conditional distribution

$$[1.1] \qquad f(x|\lambda,T) = \frac{e^{-\lambda T}(\lambda T)^x}{x!} \quad I_{\{0,1,2,...\}}(x)$$

and prior distribution

$$[1.2] \qquad g(\lambda|\alpha,\beta) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \, e^{-\frac{\lambda}{\beta}} \quad I_{[0,\infty)}(\lambda)$$

This gives a marginal distribution of X free of

$$h(x|\alpha,\beta) = \int_0^{\infty} \frac{e^{-\lambda T}(\lambda T)^x}{x!} \, \frac{\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} \, d\lambda = \frac{T^x\Gamma(\alpha+x)}{\Gamma(\alpha)x!\beta^{\alpha}(T+\frac{1}{\beta})^{\alpha+x}}$$

Now let $\frac{1}{\beta} = T'$.

$$[1.3] \qquad h(x|\alpha,\beta) = \binom{\alpha+x-1}{\alpha} \left(\frac{T'}{T+T'}\right)^{\alpha} \left(\frac{T'}{T+T'}\right)^{x} \quad I_{[0,1,2,...)}(x)$$

For integer α this is the negative binomial distribution with $x+\alpha$ trials until α successes, with probability of success $p = T'/T+T'$. This has

$$E(X) = \alpha(\frac{T}{T+T'})(\frac{T+T'}{T'}) = \alpha\beta T$$

and

$$\begin{aligned} Var(X) &= \alpha(\frac{T}{T+T'})(\frac{T+T'}{T'})^2 \\ &= \alpha\beta^2 T(T+T') \\ &= \alpha\beta^2 T^2 + \alpha\beta T \end{aligned}$$

Then $E(\hat{\lambda}) = E(\frac{x}{T}) = \alpha\beta$

and $Var(\hat{\lambda}) = Var(\frac{x}{T}) = \frac{\alpha\beta}{T} + \alpha\beta^2$

The $\hat{\lambda}$ is observable while λ is not. For this reason all estimation will be done from the marginal distribution and not the prior distribution.

The posterior distribution is

$$[1.4] \qquad k(\lambda|x,T) = \frac{e^{-\lambda(T+\frac{1}{\beta})} \lambda^{\alpha+x-1} (T+\frac{1}{\beta})^{\alpha+x}}{\Gamma(\alpha+x)} \qquad I_{[0,\infty)}(\lambda)$$

This is a gamma$(\alpha+x,(T'+T)^{-1})$ distribution where $T' = \frac{1}{\beta}$. It will be helpful to keep this in mind when interpreting α and β. We define x to be the number of arrivals and we can think of α as a "pseudo" number of arrivals. T is the experiment time and $T' = \frac{1}{\beta}$ is a "pseudo" experiment time.

The experiment is repeated n times resulting in a collection of times $T_1,\ldots,T_n$ and arrival counts $x_1,\ldots,x_n$. This gives n estimates $\hat{\lambda}_1,\ldots,\hat{\lambda}_n$. From these $\hat{\lambda}_i$ four methods of estimating α and β will be discussed. The first two estimators are method-of-moments estimators, the third is a maximum likelihood estimator and the fourth matches moments to the prior distribution.

SECTION 2: ESTIMATORS

2.1 Unweighted Method-of-Moments Estimation

In this case all experiments are considered equally informative and a λ_i from a short duration experiment is considered as important as one from a long duration experiment. The unweighted average is defined as

$$[2.1.1] \quad \bar{\hat{\lambda}}_u = \frac{1}{n} \sum_{i=1}^{n} \hat{\lambda}_i$$

and the unweighted variance as

$$[2.1.2] \quad S_{\hat{\lambda}_u} = \frac{1}{n-1} \sum_{i=1}^{n} (\hat{\lambda}_i - \bar{\hat{\lambda}}_u)^2$$

Note that

$$E(\bar{\hat{\lambda}}_u) = \frac{1}{n} E \sum_{i=1}^{n} (\hat{\lambda}_i) = \alpha\beta$$

and

$$E(S_{\hat{\lambda}_u}) = \frac{1}{n-1} E\Sigma(\hat{\lambda}_i - \bar{\hat{\lambda}}_u)^2 = \frac{1}{n-1} E(U)$$

where, using matrix notation for convenience,

$$U = \underline{\hat{\lambda}}'(I - \tfrac{1}{n}J)\underline{\hat{\lambda}} \quad \text{and} \quad \underline{\hat{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$$

I is the nxn identity and J is an nxn matrix of ones. From linear model theory

$$E(\underline{\hat{\lambda}}'(I - \tfrac{1}{n}J)\underline{\hat{\lambda}}) = E(\underline{\hat{\lambda}}'A\underline{\hat{\lambda}}) = tr(A\ddagger) + tr(A\underline{\mu}\underline{\mu}')$$

where $\ddagger$ is the variance-covariance matrix of $\underline{\hat{\lambda}}$ and $\underline{\mu}$ is the mean of $\underline{\hat{\lambda}}$. Here

$\underline{\mu}' = \alpha\beta(1\ldots1)$ and $\ddagger = [V_{ij}]_{nxn}$, with

$$V_{ij} = 0, \qquad i \neq j$$

$$V_{ij} = \alpha\beta^2 + \frac{\alpha\beta}{T_i}, \qquad i=j \;.$$

$\ddagger$ is a diagonal matrix since each experiment is independent of any other. Therefore

$$E(\underline{\hat{\lambda}}'(I-\frac{1}{n}J)\underline{\hat{\lambda}}) = (n-1)(\alpha\beta^2 + \frac{\alpha\beta}{n}\,\Sigma\frac{1}{T_i}) \;.$$

Hence

$$E(S^2_{\hat{\lambda}_u}) = \alpha\beta^2 + \frac{n\alpha\beta}{\Sigma T_i} \;.$$

Hence we have

$$E(\bar{\hat{\lambda}}_u) = \alpha\beta$$

and

$$E(S^2_{\bar{\hat{\lambda}}_u}) = \alpha\beta^2 + \frac{n\alpha\beta}{\sum_{i=1}^{n} T_i}$$

The unweighted method-of-moments estimators of α and β are

$$\hat{\alpha}_u = \frac{\bar{\hat{\lambda}}u^2}{S^2_{\hat{\lambda}_u} - \dfrac{n\bar{\hat{\lambda}}u}{\sum_{i=1}^{n}\frac{1}{T_i}}}$$

and

$$\hat{\beta}_u = \frac{S^2_{\hat{\lambda}_u}}{\bar{\hat{\lambda}}} - \frac{n}{\sum_{i=1}^{n}\frac{1}{T_i}}$$

2.2 Time-Weighted Method-of-Moments Estimation

It can be argued that those experiments with long duration provide more information than those which are short. Hence the concept of time-weighting arises. Define the time-weighted mean as

$$[2.2.1] \qquad \bar{\hat{\lambda}}_T = \frac{\sum_{i=1}^{n} T_i \hat{\lambda}_i}{\sum_{i=1}^{n} T_i} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} T_i}$$

and the time-weighted variance as

$$[2.2.2] \qquad S^2_{\hat{\lambda}_T} = \frac{\sum_{i=1}^{n} T_i (\hat{\lambda}_i - \bar{\hat{\lambda}}_T)^2}{\sum_{i=1}^{n} T_i} \quad .$$

Note that $E(\bar{\hat{\lambda}}_T) = \alpha\beta$.

We can then express the observed $\hat{\lambda}_i$ in terms of the linear model

$$\underline{\hat{\lambda}} = \underline{j}(\alpha\beta) + \underline{\varepsilon}$$

where $\underline{\hat{\lambda}}' = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$, $E(\hat{\lambda}_i) = \alpha\beta$, and $E(\underline{\varepsilon}) = \underline{0}$ with $\mathrm{Var}(\underline{\varepsilon}) = \Sigma = \mathrm{diag}(\alpha\beta^2 + \frac{\alpha\beta}{T_i})$.
Let $T = \mathrm{diag}(T_i)$ then

$$T^{\frac{1}{2}} \underline{\hat{\lambda}} = T^{\frac{1}{2}} \underline{j}(\alpha\beta) + (\underline{\varepsilon}').$$

The BLUE of $\alpha\beta$ then is

$$\hat{\alpha\beta} = (\underline{j}'T\underline{j})^{-1} \underline{j}'T\underline{\hat{\lambda}} = \frac{\sum_{i=1}^{n} T_i \hat{\lambda}_i}{\sum_{i=1}^{n} T_i} = \bar{\hat{\lambda}}_T$$

and the residual sum of squares is

$$\underline{\hat{\lambda}}'T^{\frac{1}{2}}(I-T^{\frac{1}{2}}\underline{j}(\underline{j}'T\underline{j})^{-1}\underline{j}'T^{\frac{1}{2}})T^{\frac{1}{2}}\underline{\hat{\lambda}}$$

$$= \underline{\hat{\lambda}}' \left(T-\frac{Tjj'T}{\sum_{i=1}^{n}T_i}\right)\underline{\hat{\lambda}} = \sum_{i=1}^{n} T_i(\hat{\lambda}_i-\bar{\hat{\lambda}}_T)^2$$

Let $$U = \left(T - \frac{Tjj'T}{\sum_{i=1}^{n}T_i}\right)$$

then

$$E(\underline{\hat{\lambda}}'U\underline{\hat{\lambda}}) = \sum_{i=1}^{n} T_i E(S\hat{\lambda}_T^2) = \mathrm{tr}(U\ddagger) + \underline{\mu}'U\underline{\mu}$$

where $\ddagger = \mathrm{Var}(\underline{\hat{\lambda}}) = \mathrm{diag}(\alpha\beta^2 + \frac{\alpha\beta}{T_i}) = \alpha\beta^2\underline{I} + \alpha\beta\underline{T}^{-1}$ and $\underline{\mu} = E(\underline{\hat{\lambda}}) = \underline{j}(\alpha\beta)$.

$$\mathrm{tr}(U\ddagger) + \underline{\xi}'U\underline{\xi} = \alpha\beta^2 \sum_{i=1}^{n} T_i - \alpha\beta^2 \frac{\sum_{i=1}^{n}T_i^2}{\sum_{i=1}^{n}T_i} + (n-1)\alpha\beta$$

Then

$$E(S_{\hat{\lambda}_T}^2) = \alpha\beta^2\left(1-\frac{\Sigma T_i^2}{(\Sigma T_i)^2}\right) + \frac{n-1}{T_i}\alpha\beta$$

The time-weighted method of moments estimators are

$$\hat{\alpha}_T = \frac{\bar{\hat{\lambda}}_T^2 \left[1- \dfrac{\sum_{i=1}^{n}T_i^2}{(\sum_{i=1}^{n}T_i)^2}\right]}{S_{\hat{\lambda}_T}^2 - \dfrac{(n-1)\bar{\hat{\lambda}}_T}{\sum_{i=1}^{n}T_i}}$$

and

$$\hat{\beta}_T = \frac{\bar{\hat{\lambda}}_T}{\hat{\alpha}_T}$$

These estimators differ from the unweighted method-of-moments estimators by a factor of

$$\left(1 - \frac{\sum_{i=1}^{n} T_i^2}{\left(\sum_{i=1}^{n} T_i\right)^2}\right)$$

and n is replaced by n-1. Note that if $T_i = T_j$, $\forall$ i, j = 1,...,n

$$\frac{\sum_{i=1}^{n} T_i^2}{\left(\sum_{i=1}^{n} T_i\right)^2} = \frac{1}{n}$$

and the time-weighted and unweighted method of moments estimators are identical.

2.3 Maximum Likelihood Estimation

Since the λ_i's are never observed, rather they are only estimated, the marginal distribution is used. Recall from equation [1.3] that the marginal distribution is negative binomial and can be written as

$$h(x|\alpha,\beta) = \frac{\beta^x T^x \Gamma(x+\alpha)}{\Gamma(\alpha)x!(1+\beta T)^{x+\alpha}}$$

The likelihood function is

$$[2.3.1] \qquad L = \prod_{i=1}^{n} \frac{T_i^{x_i}\beta^{x_i}\Gamma(x_i+\alpha)}{\Gamma(\alpha)x_i!(1+\beta T_i)^{x_i+\alpha}}$$

Let

$$K_i = \frac{T_i^{x_i}}{x_i!}$$

$$L = \prod_{i=1}^{n} \frac{K_i\beta^{x_i}\Gamma(x_i+\alpha)}{\Gamma(\alpha)(1+\beta T_i)^{x_i+\alpha}}$$

Then the log-likelihood function is

$$\mathcal{L} = \sum_{i=1}^{n} [(\ln K_i + x_i \ln\beta + \ln(\frac{\Gamma(x_i+\alpha)}{\Gamma(\alpha)}) - (x_i+\alpha)\ln(1+\beta T_i)]$$

Note that

$$\ln(\frac{\Gamma(x_i+\alpha)}{\Gamma(\alpha)}) = \ln(\frac{(x_i+\alpha-1)\ldots(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)}) = \sum_{z=1}^{x_i} \ln(\alpha-1+z)$$

so that

$$[2.3.2] \quad \mathcal{L} = \sum_{i=1}^{n} \ln K_i + \ln\beta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \sum_{z=1}^{x_i} \ln(\alpha+z-1) - \sum_{i=1}^{n} (x_i+\alpha)\ln(1+\beta T_i)$$

The likelihood equations are

$$[2.3.3] \quad f = \frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \sum_{i=1}^{x_i} \frac{1}{(\alpha+z-1)} - \sum_{i=1}^{n} \ln(1+\beta T_i)$$

and

$$[2.3.4] \quad g = \frac{\partial \mathcal{L}}{\partial \beta} = \frac{\sum_{i=1}^{n} x_i}{\beta} - \sum_{i=1}^{n} \frac{(x_i+\alpha)T_i}{1+\beta T_i}$$

No simple closed form solutions for $\hat{\alpha}$ and $\hat{\beta}$ can be found by setting the likelihood equations to zero, hence the Newton-Raphson iterative procedure is used to solve for the estimates. The Newton-Raphson procedure is derived and detailed in Section 5: Documentation.

2.4 Matching-Moments-to-the-Prior Estimation

As a point of interest the estimators

$$\hat{\alpha}_c = \frac{\bar{\hat{\lambda}}_u^2}{S_{\hat{\lambda}_u}^2}$$

and

$$\hat{\beta}_c = \frac{S_{\hat{\lambda}_u}^2}{\bar{\hat{\lambda}}_u}$$

are investigated. Recall the prior distribution of lambda is $\Lambda \sim$ gamma(α, β) which has classical method of moments estimators

[2.4.1]
[2.4.2]
$$\hat{\alpha} = \frac{\bar{\lambda}^2}{S^2} \quad \text{and} \quad \hat{\beta} = \frac{S^2}{\bar{\lambda}}$$

The unweighted and time-weighted method-of-moments estimators reduce to the matching-moments-to-the-prior estimators under special conditions. The two types of method-of-moments estimators are identical for equal experiment times. Consider the estimators if each experiment is continued infinitely long, that is, if for fixed n, it is true that

$$\sum_{i=1}^{n} T_i = \infty.$$

Then

$$\hat{\alpha} = \frac{\bar{\hat{\lambda}}^2}{S^2 - \frac{n\hat{\lambda}}{\sum_{i=1}^{n} T_i}} \rightarrow \frac{\bar{\hat{\lambda}}^2}{S^2}$$

and

$$\hat{\beta} = \frac{S^2}{\hat{\lambda}} - \frac{n}{\sum_{i=1}^{n} T_i} \rightarrow \frac{S^2}{\hat{\lambda}} \quad \text{as} \quad \sum_{i=1}^{n} T_i \rightarrow \infty .$$

This is consistent with the assumption of using the prior distribution, since if each experiment is continued infinitely long, the true arrival rate λ is observed.

These matching moments-to-the-prior estimators are not successful estimators of α and β, but they are useful as starting values in the Newton-Raphson procedure for finding the maximum likelihood estimators.

As sample size increases, that is, more experiments are used in averaging, the means of the matching-moments-to-the-prior estimators drift steadily away from the true value of the parameters being estimated. This result supports the use of the marginal distribution by emphasizing the fact that values from the prior distribution are not observed. These results became apparent early in this study and hence the matching moments with the prior estimators are not discussed further.

SECTION 3: BACKGROUND AND OBJECTIVES

Work on this problem began when the Nuclear Regulatory Commission provided data consisting of failure data on various types of pumps. Since the experiments were conducted on a number of days under various conditions and involved machines produced by different manufacturers it was reasonable to assume that the Poisson parameter λ was not fixed but for each experiment λ was a value of Λ, a random variable with distribution, in this case, gamma(α,β). The λ from the i^{th} experiment was never observed, so that the prior distribution is not available. The values which are observable are the estimates of λ, that is $\hat{\lambda} = \frac{x}{T}$. In order to use this available information to estimate the parameters of the gamma distribution the marginal distribution is used. We have no observations from the gamma prior distribution but each experiment gives an observation from the marginal distribution which, as shown in Section 1: Derivation, is the negative binomial distribution.

Values of α and β were selected and fixed for the simulation. The values chosen were $\alpha = 2$ and $\beta = \frac{1}{T_i} = .001$. Original plans included other values of α and β but the cost of further experimentation was prohibitive. With this in mind the resulting marginal distribution is

$$h(x\,|\,\alpha,\beta) = \binom{x_i+2-1}{2} \left(\frac{1000}{T_i+1000}\right)^2 \left(\frac{T_i}{T_i+1000}\right)^{x_i} .$$

For the purpose of simulating, experiment times T_i were generated randomly for Population One from a discrete uniform (0,9999) distribution by taking a random number in the interval (0,1), multiplying by 10,000 and retaining the integer part of the product. Upon visual examination these values of T_i did not "look" as widely spread as those observed in the Nuclear Regulatory

Commission data. Therefore another randomization is induced in Population Two. The multiplier is determined by another random number, RM, such that

$$\text{Multiplier} = \begin{cases} 1{,}000 \text{ if } RM < .3 \\ 10{,}000 \text{ if } .3 \leq RM \leq .7 \\ 100{,}000 \text{ if } .7 < RM \end{cases}$$

where RM $\sim$ uniform(0,1). This second randomization provides $0 < T_i < 100{,}000$. T_i is no longer from a uniform distribution. This is the sole difference between the generation of the two populations. The results obtained from the two populations vary in several ways as described below.

After the random experiment time is determined, a probability of a success in the marginal negative binomial distribution was calculated,

$$P_i = \frac{T'}{T_i + T'} = \frac{1000}{T_i + 1000} \quad .$$

Another random number, R, was generated and compared with P. If $R \leq P$, then it was deemed to be a success and if $R > P$ it was called a failure. The distribution gives the probability of x_i failures until the α^{th} success. Therefore $\alpha = 2$ = total successes and x_i = total failures. For each single experiment we have α "pseudo arrivals" in "pseudo experiment time" $T' = \frac{1}{\beta}$, and a simulated x_i arrivals in experiment time T_i.

From this information the estimate $\hat{\lambda}_i = \frac{x_i}{T_i}$ was computed. A sample of size ten consisted of ten single experiments and resulted in ten estimates $\hat{\lambda}_i$. Likewise for samples of size 20, 30, 40, ..., 100. The 10(20, 30, ..., 100) estimates $\hat{\lambda}_i$ were combined according to either the unweighted or the time-weighted methods described in subsections 2.1 and 2.2. These were the values used in the estimates of α and β as prescribed for the various estimators. Five hundred estimates of α and β of each type of estimator were generated for each sample size.

It was observed when working with small contrived samples (used to verify by hand calculation a computer program that was under development) that the unweighted and time-weighted method-of-moments estimates obtained were sometimes negative in value. Thus arose the first objective of this report. It was desired to determine if the method-of-moments estimates do, in real life situations, take on negative values, and if so, how often this occurs. Lacking a large body of real data, the method of simulation was selected to observe in a more realistic situation the values which might be obtained. The second objective of this report was to determine from the simulation some of the major characteristics of the empirical distributions of the unweighted and time-weighted method-of-moments estimators and the maximum likelihood estimators, and to compare various aspects of the estimators.

SECTION 4: RESULTS

Many interesting and unexpected results of the simulation were observed. Figures 4.1.1 to 4.1.6 are histograms of the unweighted method-of-moments estimates from Population One and Two and samples of size 10, 50 and 100. Figures 4.2.1 to 4.2.6 and Figures 4.3.1 to 4.3.6 are histograms of the time-weighted method-of-moments estimates and the maximum likelihood estimates from the same samples. It is easily noted that the estimates from Population Two with the wide spread experiment times display much more erratic behavior than those from Population One with experiment times from a uniform distribution. We now compare the advantages and disadvantages of the estimators.

4.1 Unweighted Method-of-Moments Estimators

It was noted that the unweighted method-of-moments estimates were often negative in value. As many as 41 percent of the 500 samples of size 100 from Population Two gave negative results. In general the number of negative values decreased as the sample size increased. The first 50 samples of size 10 are displayed in Table 4.1.1. These are included to give the reader a feel for the type of data generated by the simulation. Table 4.1.1a shows the samples from Population One and Table 4.1.1b shows the samples from Population Two. No readily apparent difference was discerned between samples which produced negative estimates and those which produced positive estimates. Noticably fewer negative estimates arose from Population One than from Population Two, indicating that a larger variance among the experiment times might contribute to a negative value of the unweighted method-of-moments estimate. A spot check of samples in Table 4.1.1 did not support this conclusion.

The ranges of values for $\hat{\alpha}_u$ and $\hat{\beta}_u$ decrease in Population One as the sample size increases. This pattern is repeated for $\hat{\beta}_u$ in Population Two

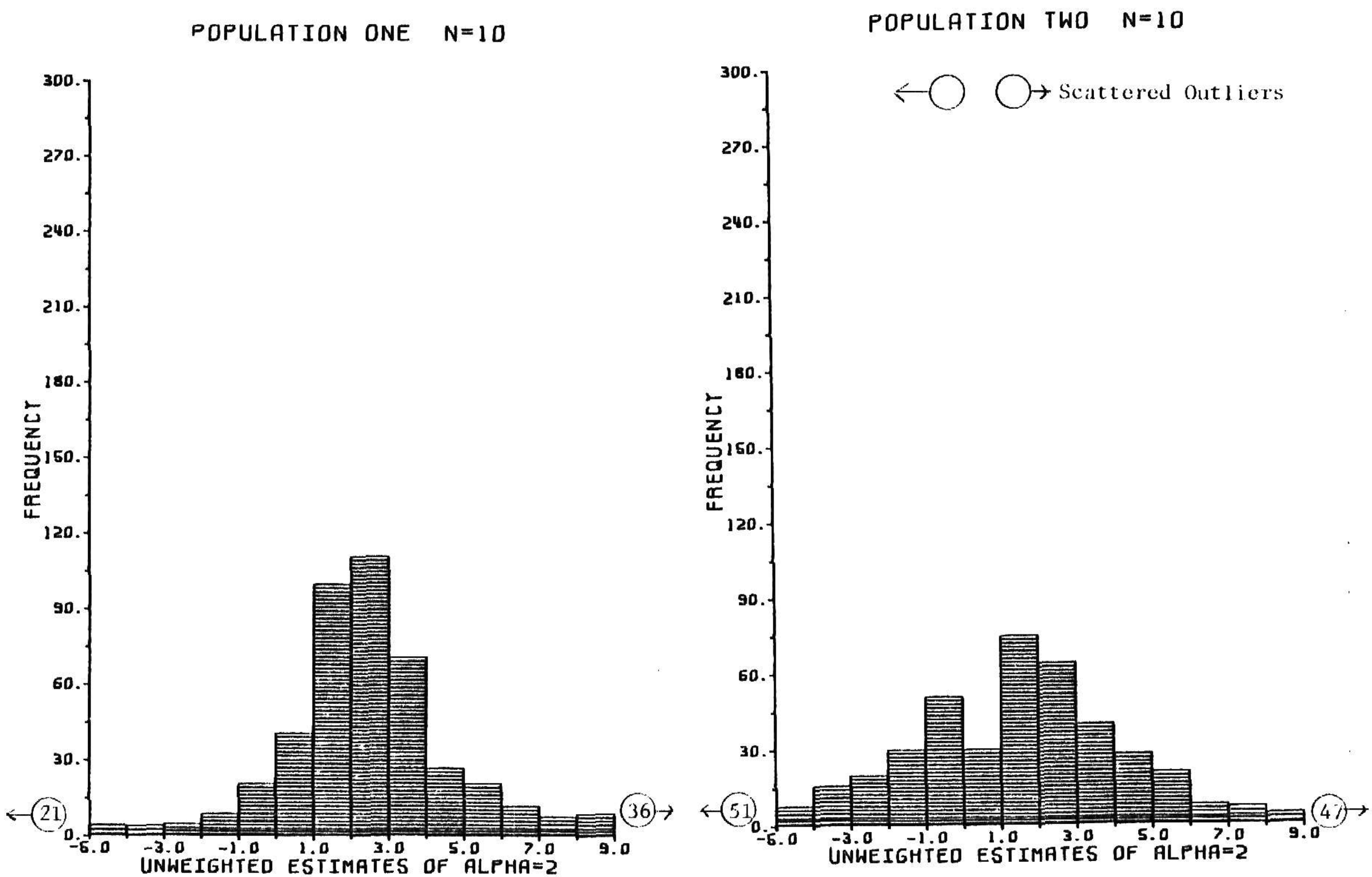
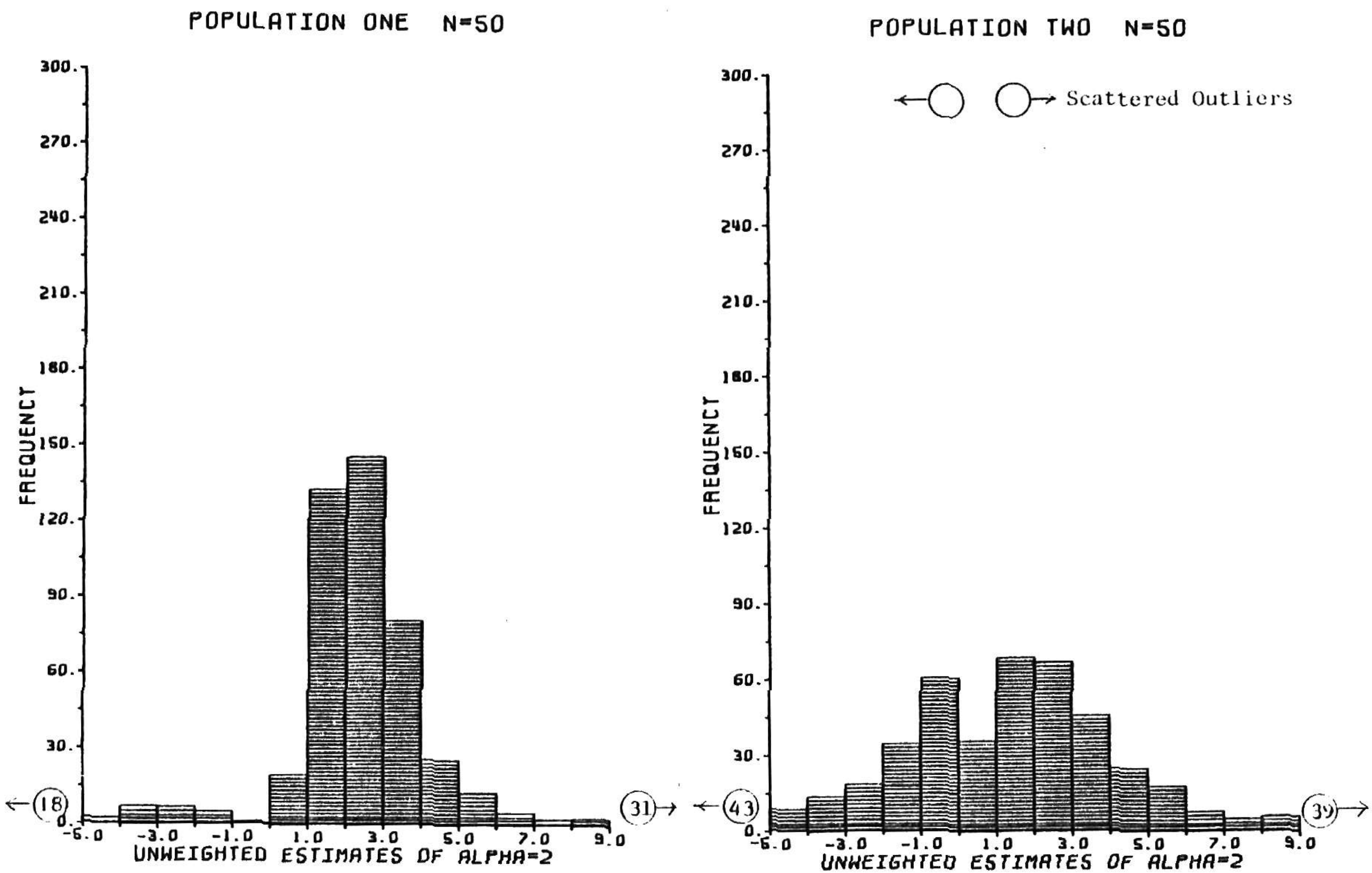

POPULATION ONE N=10
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
-6.0
-3.0
-1.0
1.0
3.0
5.0
7.0
9.0
UNWEIGHTED ESTIMATES OF ALPHA=2
21
36
POPULATION TWO N=10
Scattered Outliers
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
-6.0
-3.0
-1.0
1.0
3.0
5.0
7.0
9.0
UNWEIGHTED ESTIMATES OF ALPHA=2
51
47

Figure 4.1.1



POPULATION ONE N=50
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
-6.0
-3.0
-1.0
1.0
3.0
5.0
7.0
9.0
UNWEIGHTED ESTIMATES OF ALPHA=2
18
31
POPULATION TWO N=50
Scattered Outliers
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
-6.0
-3.0
-1.0
1.0
3.0
5.0
7.0
9.0
UNWEIGHTED ESTIMATES OF ALPHA=2
43
39

Figure 4.1.2

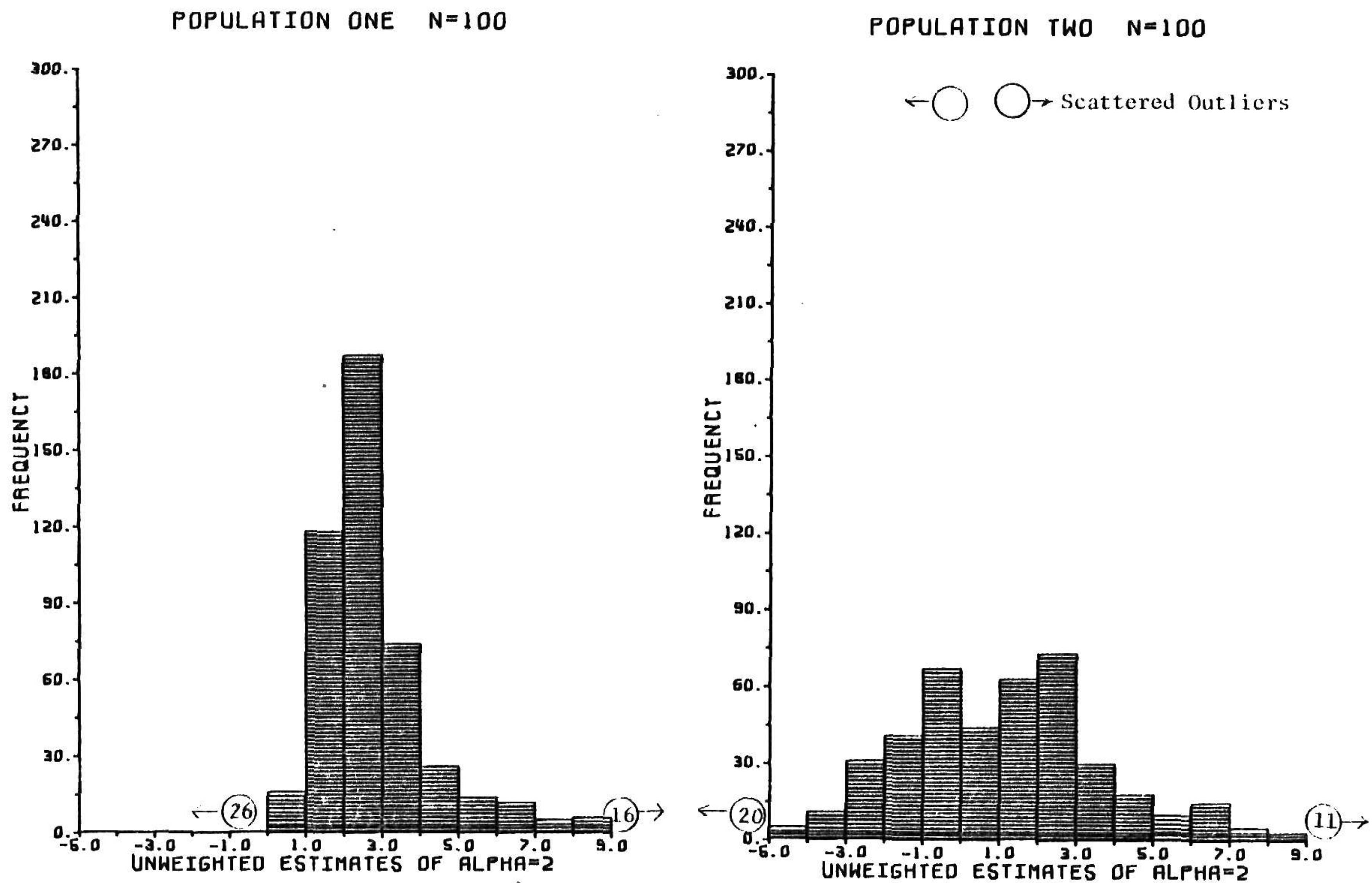

POPULATION ONE N=100
POPULATION TWO N=100
Scattered Outliers
FREQUENCY
UNWEIGHTED ESTIMATES OF ALPHA=2
26
16
20
11

Figure 4.1.3

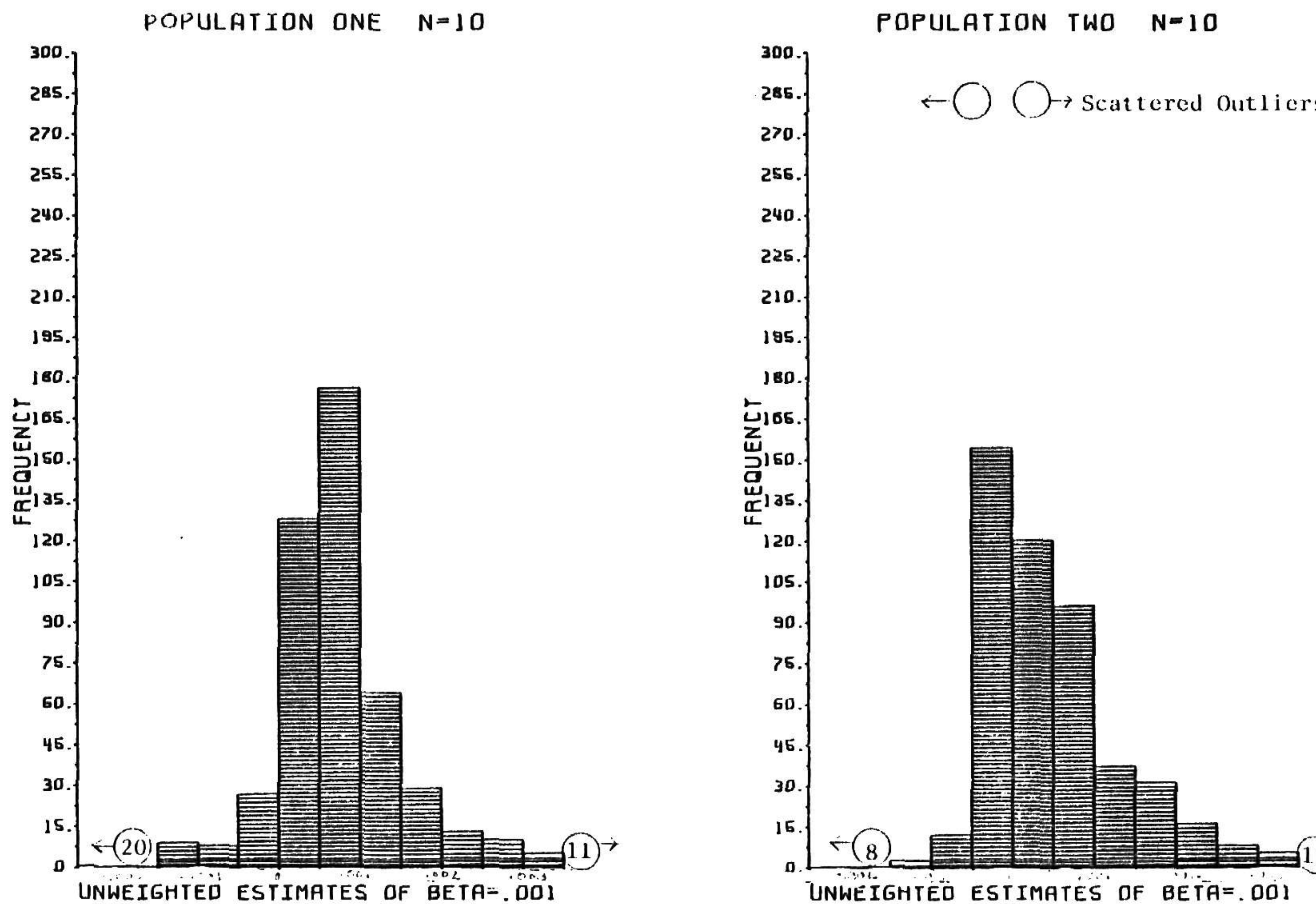

POPULATION ONE N=10
POPULATION TWO N=10
Scattered Outliers
FREQUENCY
300. 285. 270. 255. 240. 225. 210. 195. 180. 165. 150. 135. 120. 105. 90. 75. 60. 45. 30. 15. 0
20
11
8
11
UNWEIGHTED ESTIMATES OF BETA=.001
UNWEIGHTED ESTIMATES OF BETA=.001

Figure 4.1.4

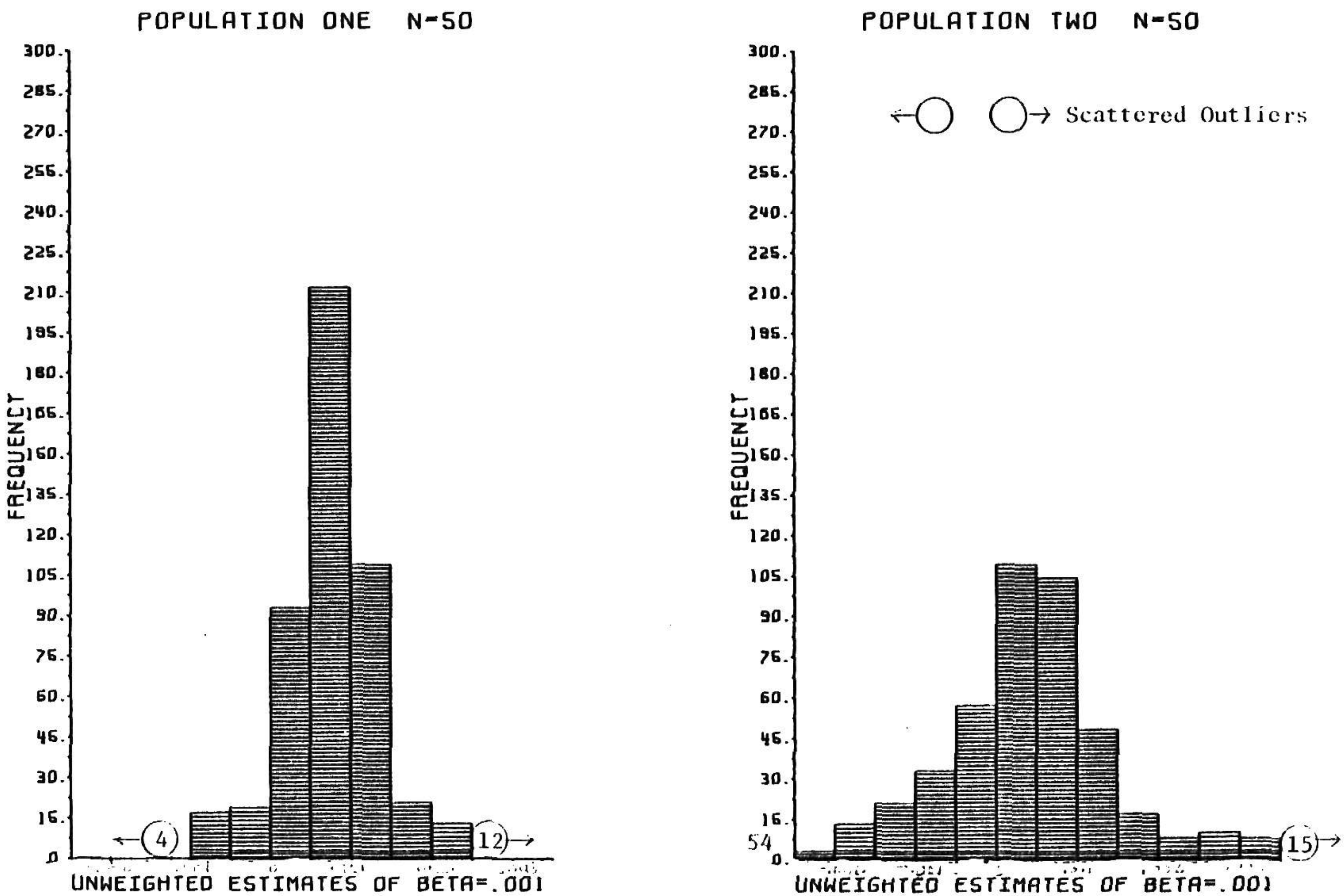

POPULATION ONE N=50
FREQUENCY
UNWEIGHTED ESTIMATES OF BETA=.001
4
12
POPULATION TWO N=50
Scattered Outliers
FREQUENCY
UNWEIGHTED ESTIMATES OF BETA=.001
54
15

Figure 4.1.5

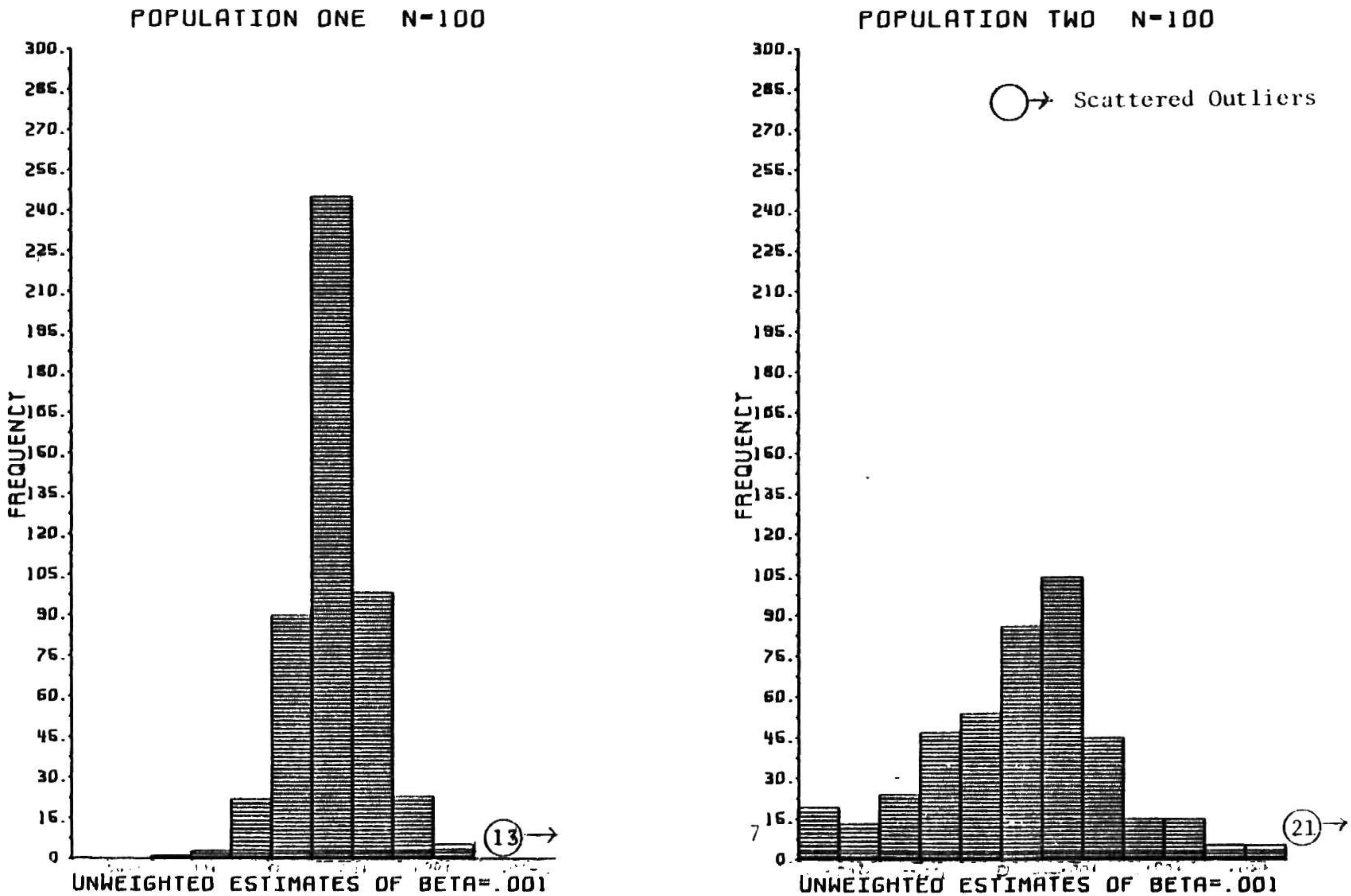

POPULATION ONE N=100
FREQUENCY
300.
285.
270.
255.
240.
225.
210.
195.
180.
165.
150.
135.
120.
105.
90.
75.
60.
45.
30.
15.
0
13
UNWEIGHTED ESTIMATES OF BETA=.001
POPULATION TWO N=100
Scattered Outliers
FREQUENCY
7
21
UNWEIGHTED ESTIMATES OF BETA=.001

Figure 4.1.6

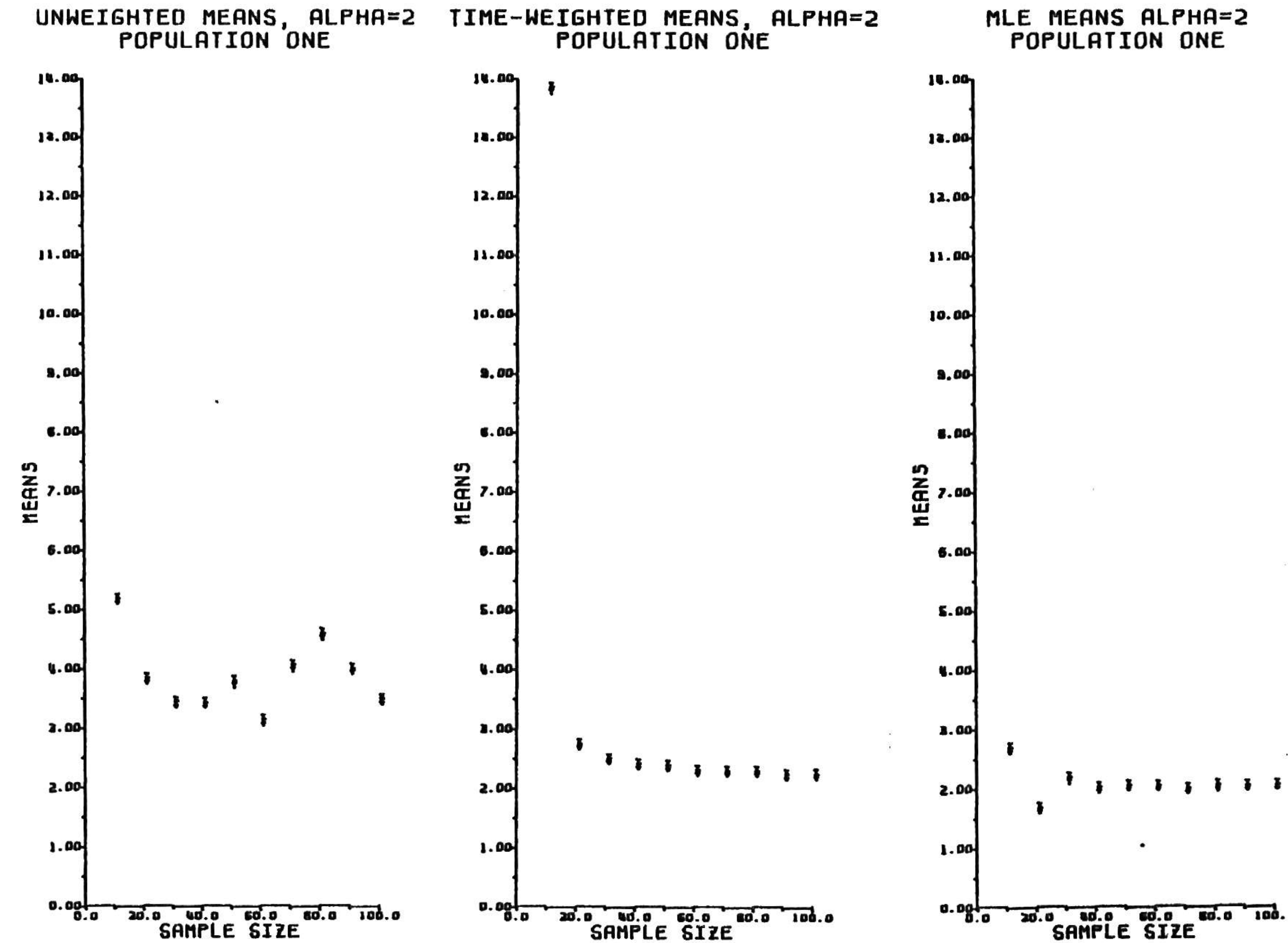

UNWEIGHTED MEANS, ALPHA=2
POPULATION ONE
TIME-WEIGHTED MEANS, ALPHA=2
POPULATION ONE
MLE MEANS ALPHA=2
POPULATION ONE
MEANS
SAMPLE SIZE

Figure 4.1.7

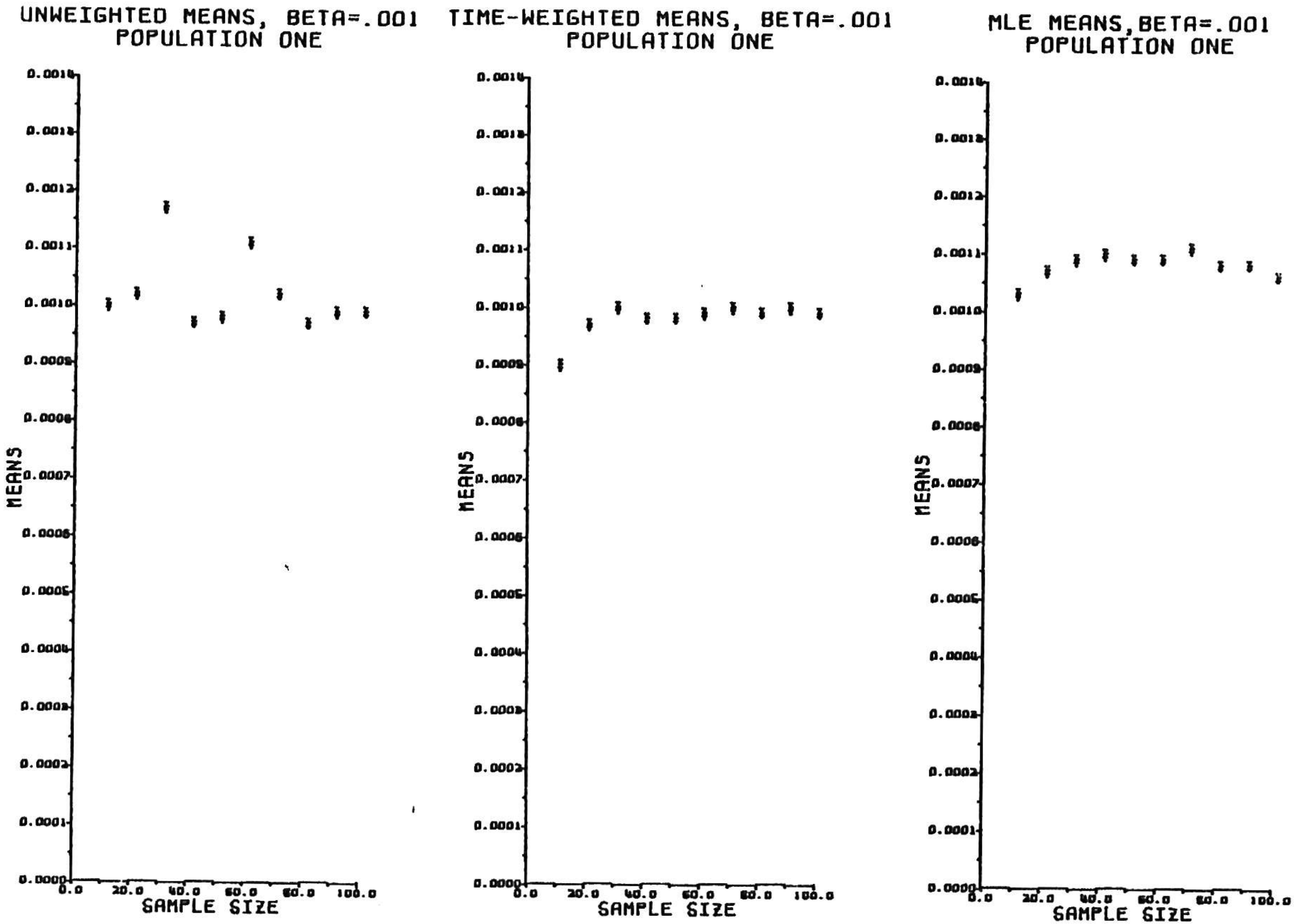

UNWEIGHTED MEANS, BETA=.001
POPULATION ONE
TIME-WEIGHTED MEANS, BETA=.001
POPULATION ONE
MLE MEANS,BETA=.001
POPULATION ONE
MEANS
SAMPLE SIZE
0.0000
0.0001
0.0002
0.0003
0.0004
0.0005
0.0006
0.0007
0.0008
0.0009
0.0010
0.0011
0.0012
0.0013
0.0014
0.0
20.0
40.0
60.0
80.0
100.0

Figure 4.1.8

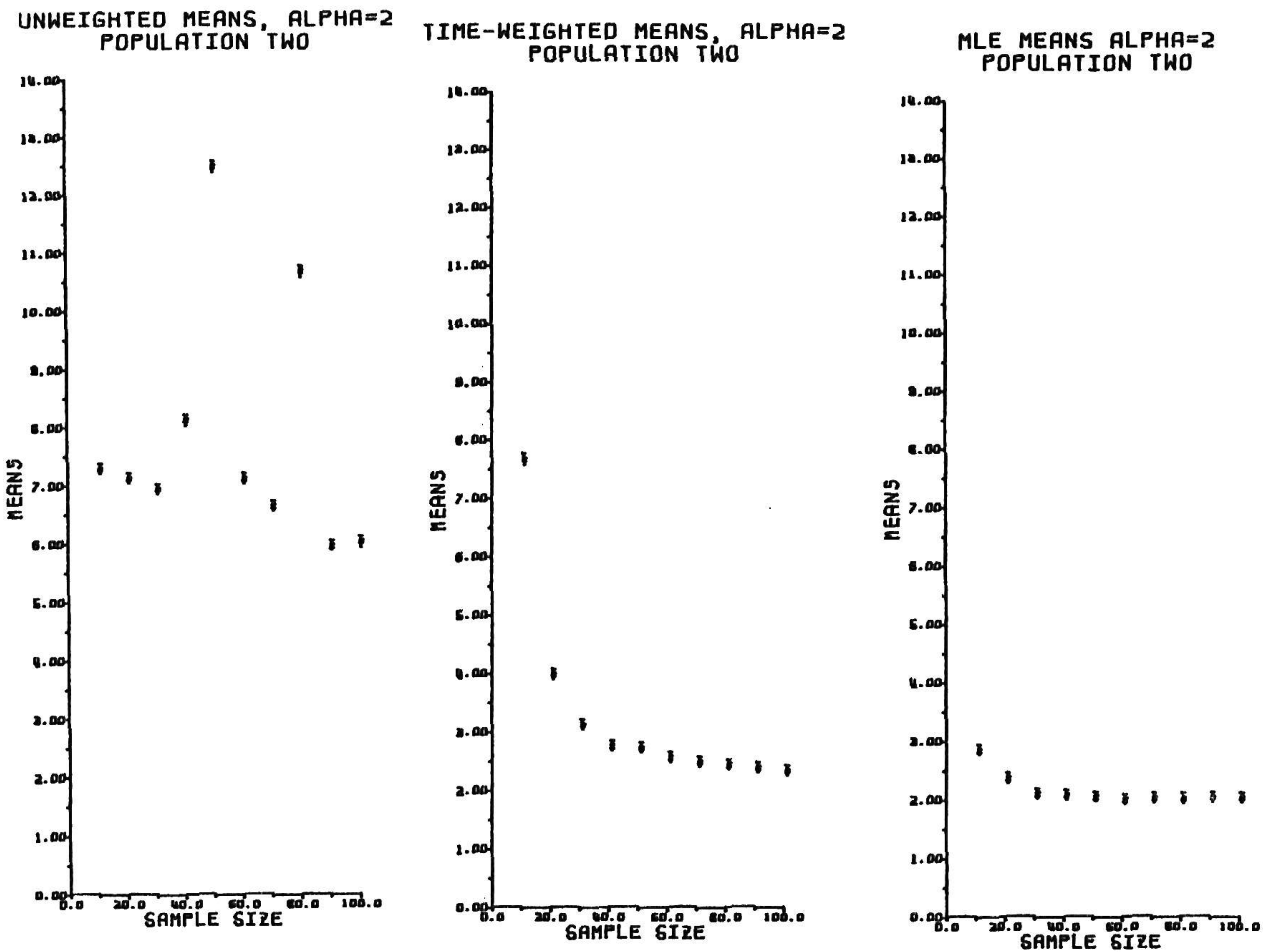

UNWEIGHTED MEANS, ALPHA=2
POPULATION TWO
MEANS
SAMPLE SIZE
TIME-WEIGHTED MEANS, ALPHA=2
POPULATION TWO
MEANS
SAMPLE SIZE
MLE MEANS ALPHA=2
POPULATION TWO
MEANS
SAMPLE SIZE

Figure 4.1.9

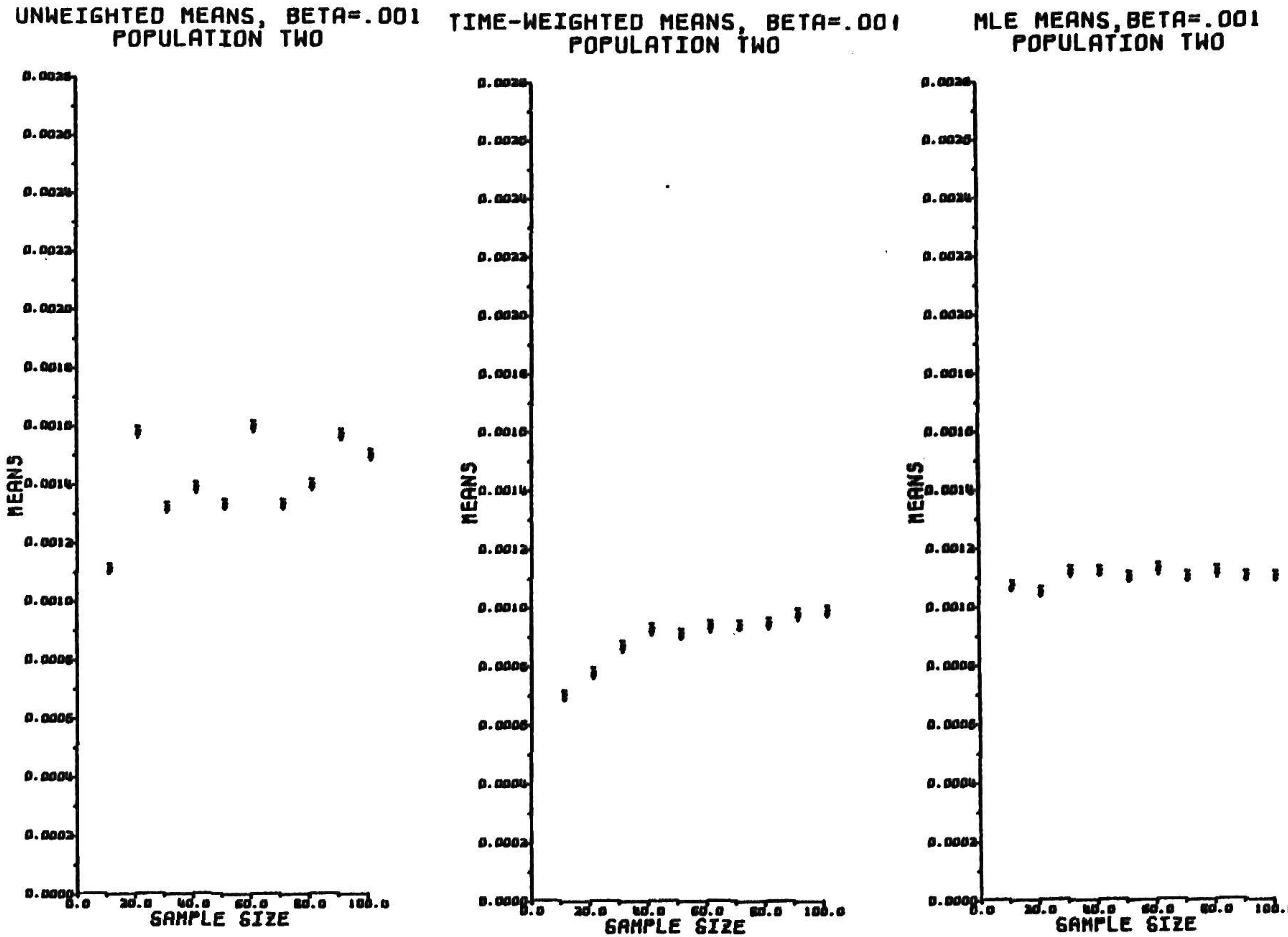

UNWEIGHTED MEANS, BETA=.001
POPULATION TWO
TIME-WEIGHTED MEANS, BETA=.001
POPULATION TWO
MLE MEANS, BETA=.001
POPULATION TWO
MEANS
0.0028
0.0026
0.0024
0.0022
0.0020
0.0018
0.0016
0.0014
0.0012
0.0010
0.0008
0.0006
0.0004
0.0002
0.0000
SAMPLE SIZE
0.0
20.0
40.0
60.0
80.0
100.0

Figure 4.1.10

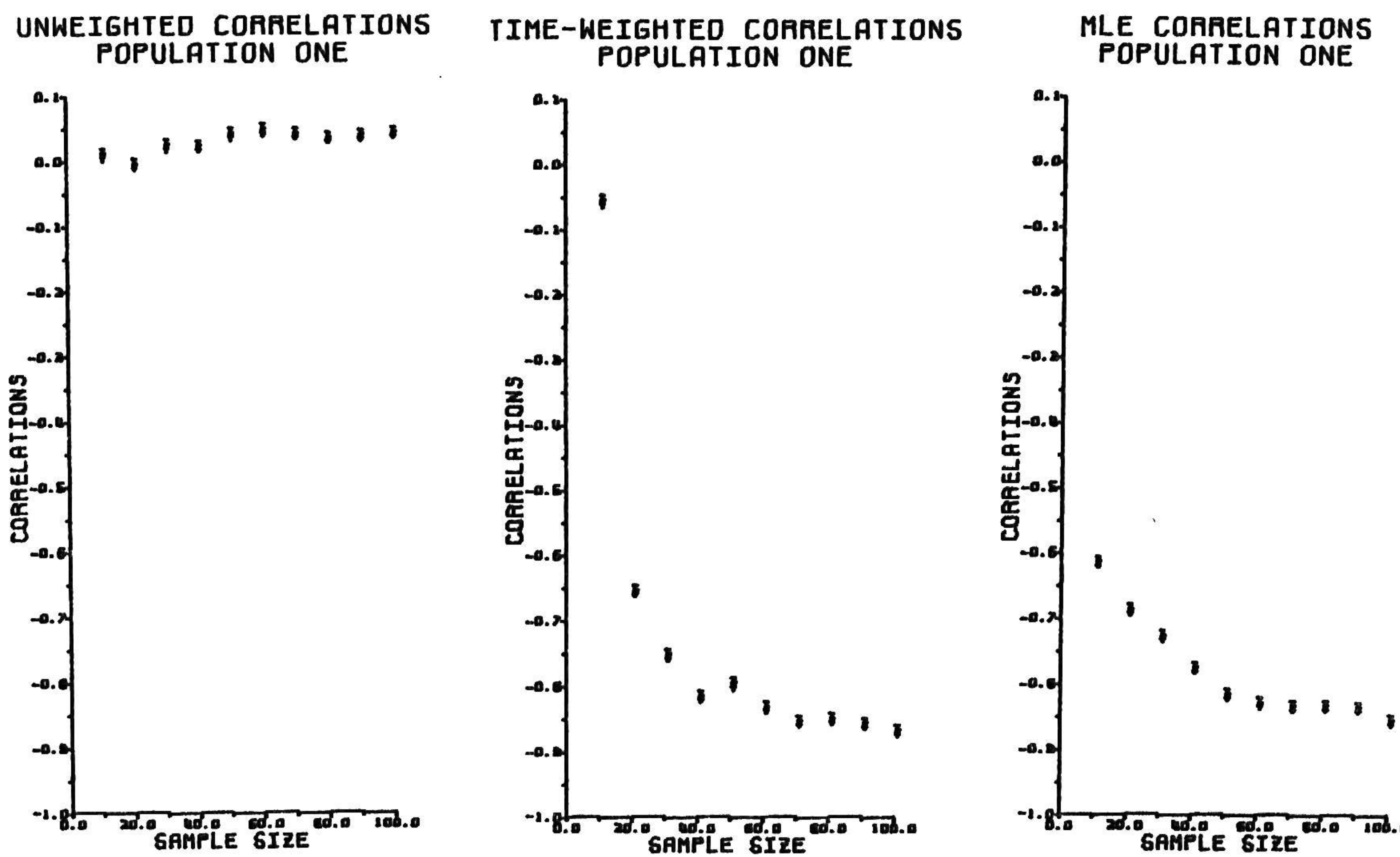

UNWEIGHTED CORRELATIONS
POPULATION ONE
TIME-WEIGHTED CORRELATIONS
POPULATION ONE
MLE CORRELATIONS
POPULATION ONE
CORRELATIONS
SAMPLE SIZE

Figure 4.1.11

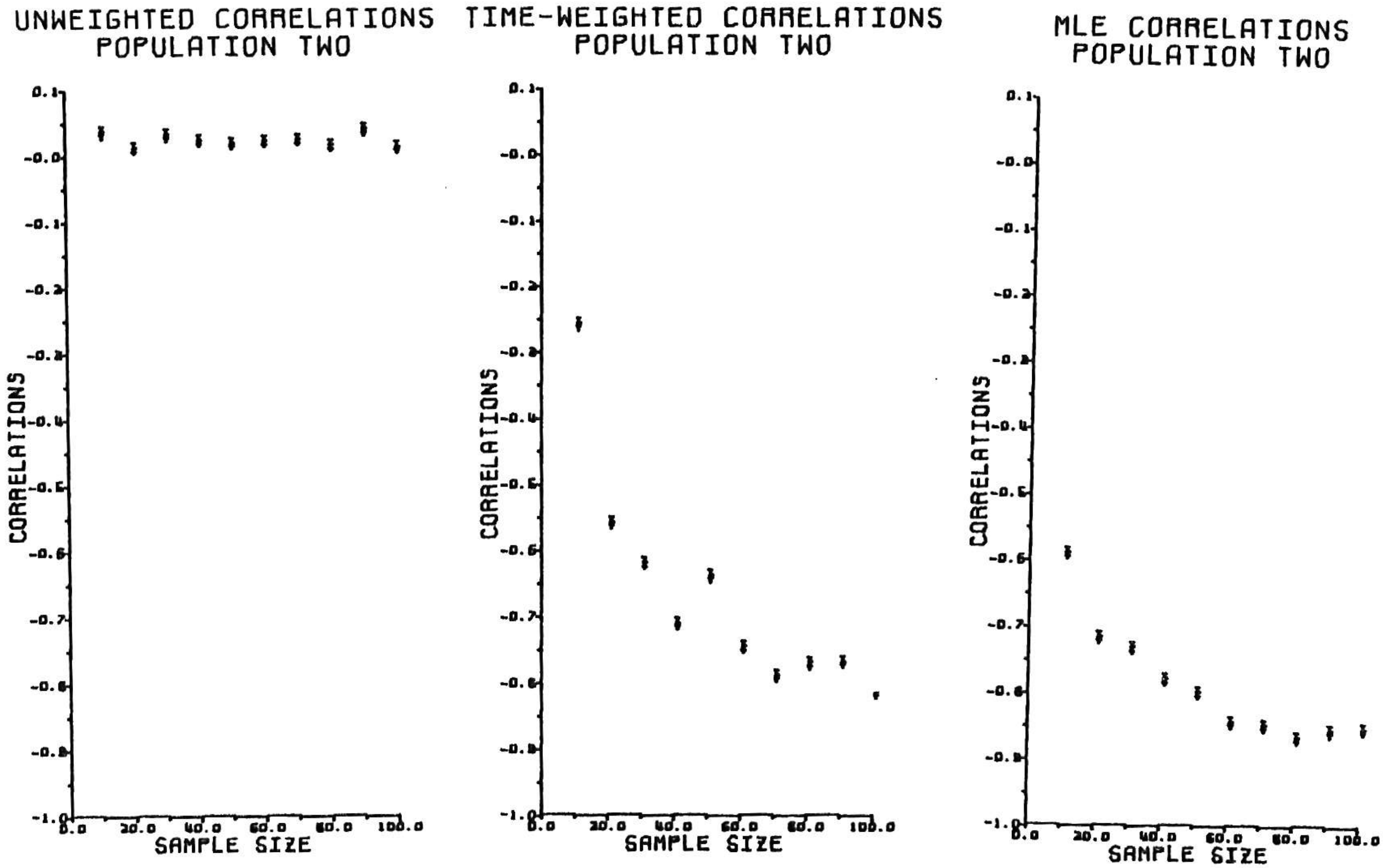

UNWEIGHTED CORRELATIONS
POPULATION TWO
TIME-WEIGHTED CORRELATIONS
POPULATION TWO
MLE CORRELATIONS
POPULATION TWO
CORRELATIONS
SAMPLE SIZE
0.0
20.0
40.0
60.0
80.0
100.0
0.1
-0.0
-0.1
-0.2
-0.7
-1.0

Figure 4.1.12

Table 4.1.1a Typical Samples (Size 10) From Population One

SAMPLE NUMBER	1	TIMES	COUNTS	SAMPLE NUMBER	6	TIMES	COUNTS
		2292	9			850	0
		1660	1			9013	22
		6176	10			5095	21
		6075	10			4302	5
		2128	15			2491	15
		5198	21			4133	12
		9858	27			6898	4
		5809	8			330	2
		2608	6			9542	55
		8347	28			4122	17
SAMPLE NUMBER	2	TIMES	COUNTS	SAMPLE NUMBER	7	TIMES	COUNTS
		6065	25			7645	14
		7064	16			3800	16
		2744	2			9904	13
		3821	12			9633	4
		107	0			3758	16
		6037	28			5898	7
		9235	24			3941	9
		2918	1			8013	29
		8232	2			5339	14
		6368	28			6385	2
SAMPLE NUMBER	3	TIMES	COUNTS	SAMPLE NUMBER	8	TIMES	COUNTS
		872	2			5620	4
		199	0			8428	15
		4143	12			7361	9
		7562	20			8106	3
		6207	14			5606	19
		3032	4			3842	7
		4199	14			2402	1
		1584	9			6472	5
		430	0			1524	5
		9742	22			3722	17
SAMPLE NUMBER	4	TIMES	COUNTS	SAMPLE NUMBER	9	TIMES	COUNTS
		984	0			8264	28
		154	0			120	0
		256	0			9476	24
		4958	11			3917	12
		7966	1			8403	12
		5693	4			9316	26
		9967	22			5690	9
		4351	16			4725	3
		1542	7			3538	10
		9074	28			5124	5
SAMPLE NUMBER	5	TIMES	COUNTS	SAMPLE NUMBER	10	TIMES	COUNTS
		6844	6			8210	11
		7924	4			8343	8
		1396	1			6810	47
		5291	7			2759	2
		3703	7			5758	12
		2897	6			3138	6
		3233	11			3928	5
		730	0			3627	9
		7456	35			6046	4
		6718	5			9777	8

Table 4.1.1a continued

SAMPLE NUMBER 11	TIMES	COUNTS
	4637	10
	6640	0
	4114	6
	4092	0
	1034	3
	8431	1
	5698	6
	5503	11
	9395	14
	8524	25

SAMPLE NUMBER 12	TIMES	COUNTS
	2438	2
	1586	3
	1508	0
	3517	2
	3000	9
	2351	9
	9568	24
	5218	11
	8510	26
	5516	1

SAMPLE NUMBER 13	TIMES	COUNTS
	4552	1
	645	2
	756	3
	4396	1
	1733	2
	6089	3
	7723	8
	2131	17
	5841	2
	5132	8

SAMPLE NUMBER 14	TIMES	COUNTS
	3848	11
	6075	4
	2489	2
	2533	13
	5066	17
	9295	11
	6583	9
	6787	13
	9897	19
	7669	13

SAMPLE NUMBER 15	TIMES	COUNTS
	819	2
	9438	11
	9368	28
	9688	6
	8306	38
	4425	7
	1456	4
	463	2
	6013	19
	4361	4

SAMPLE NUMBER 16	TIMES	COUNTS
	1020	2
	4467	13
	1157	2
	6754	5
	2956	8
	6753	5
	7927	21
	6412	31
	680	1
	1206	4

SAMPLE NUMBER 17	TIMES	COUNTS
	3340	12
	4538	6
	7470	13
	844	5
	8782	40
	6587	2
	4475	12
	6771	24
	2541	3
	7336	21

SAMPLE NUMBER 18	TIMES	COUNTS
	6754	3
	3205	4
	2448	18
	1831	4
	2111	3
	735	2
	1656	2
	4286	9
	5076	12
	3130	3

SAMPLE NUMBER 19	TIMES	COUNTS
	3147	8
	5903	7
	6002	5
	4582	2
	9708	41
	5182	7
	1332	0
	1233	3
	1222	0
	7109	17

SAMPLE NUMBER 20	TIMES	COUNTS
	5813	18
	7358	18
	3022	4
	7970	28
	2643	11
	3666	7
	7800	26
	9128	17
	5583	26
	4557	5

Table 4.1.1a continued

SAMPLE NUMBER	21	TIMES	COUNTS	SAMPLE NUMBER	26	TIMES	COUNTS
		8435	0			7941	6
		2946	7			5934	19
		9489	5			3221	5
		5417	2			9506	2
		1909	0			3890	14
		1886	2			6155	20
		625	2			9364	6
		6293	4			1683	0
		7500	22			9628	17
		5626	18			9908	17
SAMPLE NUMBER	22	TIMES	COUNTS	SAMPLE NUMBER	27	TIMES	COUNTS
		9487	14			520	0
		1089	2			8202	27
		5952	37			4604	2
		9788	15			2670	5
		8019	33			4385	4
		4556	7			3345	9
		6166	16			6383	19
		4866	7			5853	5
		1813	3			8075	20
		4133	6			374	0
SAMPLE NUMBER	23	TIMES	COUNTS	SAMPLE NUMBER	28	TIMES	COUNTS
		7367	22			4534	7
		218	0			7048	18
		1328	0			3032	9
		5046	15			1168	6
		5888	17			5654	5
		2294	5			8825	9
		4236	0			8804	13
		9054	44			8574	10
		9212	29			6596	3
		9454	12			6186	12
SAMPLE NUMBER	24	TIMES	COUNTS	SAMPLE NUMBER	29	TIMES	COUNTS
		3406	9			2534	1
		846	3			2894	9
		5907	3			145	0
		6250	16			5317	1
		8688	0			8470	6
		3541	17			2606	0
		827	4			7009	1
		6680	11			3221	9
		399	1			5376	7
		9622	12			8779	8
SAMPLE NUMBER	25	TIMES	COUNTS	SAMPLE NUMBER	30	TIMES	COUNTS
		404	1			6611	23
		474	0			2754	4
		236	0			7210	9
		1316	4			8965	18
		2438	2			416	0
		6294	3			529	3
		1320	0			5027	1
		8812	29			3328	19
		8508	29			1164	1
		1847	2			6338	4

Table 4.1.1a continued

SAMPLE NUMBER	TIMES	COUNTS	SAMPLE NUMBER	TIMES	COUNTS
31	5894	4	36	9261	11
	2734	1		1282	3
	8259	4		749	2
	3813	10		2095	2
	9586	21		3188	2
	1869	3		8415	6
	1766	3		2871	4
	4503	8		7421	6
	5983	7		99	0
	5860	11		8438	5
32	2433	2	37	2348	14
	999	2		7051	12
	8355	0		1486	2
	8379	53		3622	7
	4492	5		2599	3
	6854	25		3919	3
	5948	4		9244	13
	185	0		5981	11
	7327	10		9786	14
	7818	21		8296	2
33	4461	12	38	6509	6
	3135	8		6219	6
	4171	14		5690	2
	7741	21		6441	9
	9461	14		3417	6
	4955	1		1488	2
	4516	5		8188	21
	7540	4		9480	28
	4771	14		1732	9
	8818	10		8372	24
34	9653	8	39	7089	7
	1679	2		3636	7
	8352	36		1526	2
	2039	12		7421	9
	5300	7		4373	9
	5120	7		1073	1
	6845	3		1755	1
	167	0		2419	1
	8787	14		6317	11
	3598	7		468	0
35	3088	6	40	457	0
	2432	14		2975	8
	5356	25		1823	0
	42	0		9156	16
	7756	29		6745	13
	9051	12		1045	5
	3381	13		3403	20
	6415	6		2307	1
	4973	13		9352	13
	2015	1		7467	16

Table 4.1.1a continued

SAMPLE NUMBER 41	TIMES	COUNTS	SAMPLE NUMBER 46	TIMES	COUNTS
	1868	6		17	0
	4302	4		2548	4
	3933	4		6822	23
	3163	14		5570	4
	560	0		7419	12
	7449	3		498	1
	1052	1		6132	8
	843	0		8255	4
	1012	0		386	2
	5705	3		3378	16
SAMPLE NUMBER 42	TIMES	COUNTS	SAMPLE NUMBER 47	TIMES	COUNTS
	2057	4		9593	21
	5157	6		255	0
	7767	7		9662	26
	718	2		7034	11
	527	0		8905	9
	271	0		4452	4
	6454	9		8210	20
	5536	11		5295	11
	1287	6		5439	2
	8205	48		4154	6
SAMPLE NUMBER 43	TIMES	COUNTS	SAMPLE NUMBER 48	TIMES	COUNTS
	3678	4		310	2
	3903	10		2743	14
	8716	36		2435	18
	1054	1		5952	9
	1026	0		4158	0
	7188	15		1348	3
	3865	4		918	2
	4667	3		2782	7
	1875	0		9727	3
	9770	13		5103	1
SAMPLE NUMBER 44	TIMES	COUNTS	SAMPLE NUMBER 49	TIMES	COUNTS
	6123	13		8530	2
	9829	28		6058	10
	3794	11		7644	14
	1908	4		5422	2
	6357	15		6017	3
	7614	7		35	1
	5480	2		228	1
	8216	10		1692	6
	9711	25		4212	3
	2129	3		1487	3
SAMPLE NUMBER 45	TIMES	COUNTS	SAMPLE NUMBER 50	TIMES	COUNTS
	1712	1		7394	26
	8551	7		2713	8
	5145	18		6211	12
	6717	57		676	0
	3044	2		2336	2
	3122	8		8779	27
	4164	22		6419	2
	2073	7		3479	8
	347	2		4302	14
	7628	12		6578	6

Table 4.1.1b Typical Samples (Size 10) from Population Two

SAMPLE NUMBER 1 TIMES	COUNTS
22925	189
706	4
3537	7
2744	1
38218	78
694	2
9017	24
87	0
143	0
41437	11

SAMPLE NUMBER 2 TIMES	COUNTS
75621	188
907	5
98526	507
265	0
42739	98
1201	2
249	0
52183	258
22557	10
4114	5

SAMPLE NUMBER 3 TIMES	COUNTS
409	0
80364	253
15753	10
38484	15
216	0
38495	1
9828	35
4208	5
65831	62
8192	15

SAMPLE NUMBER 4 TIMES	COUNTS
9368	27
968	4
7873	38
44252	40
4361	3
1020	1
44678	12
1157	4
98281	81
304	2

SAMPLE NUMBER 5 TIMES	COUNTS
622	1
660	0
722	2
804	0
766	3
74706	115
7779	21
675	1
1155	1
9964	33

SAMPLE NUMBER 6 TIMES	COUNTS
735	1
16563	34
3147	7
5903	6
600	0
3269	4
5453	39
35	0
652	2
504	0

SAMPLE NUMBER 7 TIMES	COUNTS
182	2
178	0
7109	16
5813	17
7358	17
302	0
41764	67
9896	34
2607	6
482	2

SAMPLE NUMBER 8 TIMES	COUNTS
6403	9
5588	4
61	0
3432	5
835	0
607	0
1512	0
112	0
81089	21
2730	5

SAMPLE NUMBER 9 TIMES	COUNTS
2244	6
87811	130
455	1
64930	518
168	1
71790	14
99089	49
460	0
91	0
1571	4

SAMPLE NUMBER 10 TIMES	COUNTS
69389	72
8304	27
4262	5
68385	95
582	2
682	1
260	0
928	0
2722	7
5376	6

Table 4.1.1b continued

SAMPLE NUMBER 11	TIMES	COUNTS
	877	0
	2910	21
	9593	23
	8965	17
	41	0
	6788	19
	89668	312
	77412	180
	916	0
	732	2

SAMPLE NUMBER 12	TIMES	COUNTS
	19649	73
	39730	94
	2452	6
	1282	2
	7495	4
	5859	6
	320	0
	563	0
	13329	10
	99	0

SAMPLE NUMBER 13	TIMES	COUNTS
	650	1
	877	2
	10835	45
	39194	89
	34177	65
	17322	35
	7089	6
	3636	6
	15266	13
	43731	39

SAMPLE NUMBER 14	TIMES	COUNTS
	568	2
	35632	122
	98633	360
	40902	68
	4745	1
	435	1
	820	3
	949	0
	19815	79
	3122	7

SAMPLE NUMBER 15	TIMES	COUNTS
	4164	21
	2073	6
	3475	14
	335	0
	713	1
	8096	27
	251	0
	5446	12
	498	1
	7511	10

SAMPLE NUMBER 16	TIMES	COUNTS
	329	0
	97126	248
	85305	229
	43918	47
	74871	469
	2106	3
	90128	225
	858	1
	660	1
	360	4

SAMPLE NUMBER 17	TIMES	COUNTS
	581	5
	31130	37
	492	0
	973	4
	96950	317
	26799	64
	6999	9
	32892	46
	433	1
	659	2

SAMPLE NUMBER 18	TIMES	COUNTS
	732	1
	14230	23
	207	0
	500	1
	760	1
	988	0
	3008	2
	9445	7
	46047	20
	7079	10

SAMPLE NUMBER 19	TIMES	COUNTS
	203	0
	19436	46
	58514	258
	46788	71
	65271	143
	598	2
	8840	15
	67964	63
	8373	26
	26536	81

SAMPLE NUMBER 20	TIMES	COUNTS
	84	2
	9000	11
	337	2
	2375	14
	5071	3
	719	0
	9212	7
	27071	55
	775	0
	70442	57

Table 4.1.1b continued

SAMPLE NUMBER 21

TIMES	COUNTS
3859	3
588	0
983	1
986	4
734	2
5302	4
20710	38
72045	70
2310	6
1369	2

SAMPLE NUMBER 22

TIMES	COUNTS
797	0
32	0
385	0
822	0
47050	56
2445	9
6206	18
1535	3
7762	26
53232	61

SAMPLE NUMBER 23

TIMES	COUNTS
527	4
4031	28
123	0
7517	15
957	1
975	1
671	1
128	1
72192	133
3941	6

SAMPLE NUMBER 24

TIMES	COUNTS
6945	6
50	0
679	1
4534	8
269	1
40354	39
27697	7
80538	318
2084	3
778	2

SAMPLE NUMBER 25

TIMES	COUNTS
85500	180
328	0
358	0
768	2
173	0
91008	114
8069	28
5253	16
772	2
57851	147

SAMPLE NUMBER 26

TIMES	COUNTS
928	5
2524	29
39771	99
7626	7
8528	12
3690	13
5477	16
494	0
9414	17
9022	17

SAMPLE NUMBER 27

TIMES	COUNTS
295	0
962	2
612	1
617	1
102	0
1931	1
397	1
962	1
78207	92
17590	13

SAMPLE NUMBER 28

TIMES	COUNTS
9155	11
1030	1
8761	16
9166	13
36867	64
16338	30
50431	108
1063	2
8393	16
54308	70

SAMPLE NUMBER 29

TIMES	COUNTS
5561	5
807	4
2531	1
63753	98
1921	5
9136	10
4232	3
8746	5
272	0
763	6

SAMPLE NUMBER 30

TIMES	COUNTS
625	2
21405	14
8080	2
992	1
63368	39
7364	17
842	0
71213	39
86918	183
4140	8

SAMPLE NUMBER 31

TIMES	COUNTS
594	2
494	0
2055	8
3987	16
656	1
27220	24
521	0
7401	43
62746	394
28026	29

SAMPLE NUMBER 32

TIMES	COUNTS
3965	1
8832	10
5798	8
129	0
2787	2
781	2
93	0
158	0
14932	12
4481	9

SAMPLE NUMBER 33

TIMES	COUNTS
572	5
6878	12
664	3
7234	13
19396	45
306	1
1011	2
36280	30
86942	93
77222	164

SAMPLE NUMBER 34

TIMES	COUNTS
3889	2
3178	3
55118	273
41888	85
233	0
3729	2
776	0
96329	158
3677	5
9381	11

SAMPLE NUMBER 35

TIMES	COUNTS
54910	95
4661	4
795	0
429	1
87465	111
146	0
875	1
526	0
27872	10
9591	11

SAMPLE NUMBER 36

TIMES	COUNTS
623	0
1289	11
57655	51
390	0
9180	22
55989	89
9135	8
61041	49
6240	4
209	0

SAMPLE NUMBER 37

TIMES	COUNTS
5834	6
8782	42
811	1
104	0
31134	27
685	0
42335	159
751	3
287	0
70862	31

SAMPLE NUMBER 38

TIMES	COUNTS
9162	40
19849	30
6166	5
417	0
8387	6
835	6
75539	53
900	0
4772	7
1664	4

SAMPLE NUMBER 39

TIMES	COUNTS
36387	82
679	1
12638	25
192	0
144	1
6215	4
7705	48
8422	19
69	2
990	3

SAMPLE NUMBER 40

TIMES	COUNTS
27999	130
1465	0
6986	14
713	2
653	1
7647	25
6949	5
16137	8
55891	72
4188	2

Table 4.1.1b continued

SAMPLE NUMBER 41

TIMES	COUNTS
8796	22
185	0
297	1
383	0
56800	107
227	1
2036	7
7362	8
65539	61
5909	17

SAMPLE NUMBER 42

TIMES	COUNTS
40495	62
250	0
98555	45
6523	0
32569	53
884	3
299	0
47301	87
272	1
6089	8

SAMPLE NUMBER 43

TIMES	COUNTS
5015	7
49652	16
16514	44
105	0
1379	0
933	1
64729	119
21604	44
29188	102
3936	12

SAMPLE NUMBER 44

TIMES	COUNTS
7593	7
3591	1
135	0
1878	0
73297	178
83070	262
9151	8
908	1
517	3
288	3

SAMPLE NUMBER 45

TIMES	COUNTS
292	1
366	0
416	0
34359	83
9551	25
670	3
29131	12
94264	136
8	0
42876	142

SAMPLE NUMBER 46

TIMES	COUNTS
723	2
80600	272
67922	8
89728	268
2681	3
9568	18
315	0
62341	157
556	2
8175	25

SAMPLE NUMBER 47

TIMES	COUNTS
915	0
59987	39
601	1
784	0
5214	10
17	0
2113	5
2782	1
8675	26
435	0

SAMPLE NUMBER 48

TIMES	COUNTS
90551	147
807	1
396	0
3993	1
1695	1
9206	7
903	0
145	0
311	0
749	2

SAMPLE NUMBER 49

TIMES	COUNTS
73689	228
83479	399
96132	12
96452	249
5229	5
27572	61
9746	37
33164	14
538	4
895	3

SAMPLE NUMBER 50

TIMES	COUNTS
663	1
6458	13
3893	6
2642	3
500	0
676	4
95591	61
7574	18
858	4
57527	104

Table 4.1.2 Summary Statistics for $\hat{\alpha}$ and $\hat{\beta}$ for all Cases.

VARIABLE		N	MEAN	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE
					POPN=1	N=10
AUN	$(\hat{\alpha}_u)$	500	2.86831495	21.14356147	-377.71875438	120.47584447
AMLE	$(\hat{\alpha}_{MLE})$	385	2.57044330	1.75020573	0.54103562	15.30937720
ATW	$(\hat{\alpha}_T)$	500	13.73877788	244.77927948	-471.12075688	5437.72022431
AC	$(\hat{\alpha}_c)$	500	1.78880671	0.99360734	0.25897269	8.99774759
BUN	$(\hat{\beta}_u)$	500	0.00066747	0.00157634	-0.01023562	0.01339748
BMLE	$(\hat{\beta}_{MLE})$	385	0.00102044	0.00056952	0.00012054	0.00369205
BTW	$(\hat{\beta}_T)$	500	0.00089145	0.00064762	-0.00009987	0.00584707
BC	$(\hat{\beta}_c)$	500	0.00143184	0.00127644	0.00022942	0.01591968
					POPN=1	N=20
AUN	$(\hat{\alpha}_u)$	500	2.68974141	7.23345482	-64.55432561	72.50353339
AMLE	$(\hat{\alpha}_{MLE})$	383	2.19237651	1.09218448	0.81499530	11.22754453
ATW	$(\hat{\alpha}_T)$	500	2.62805670	1.50394850	0.62024799	13.28019968
AC	$(\hat{\alpha}_c)$	500	1.57296626	0.60614224	0.17818306	4.93644141
BUN	$(\hat{\beta}_u)$	500	0.00079580	0.00135681	-0.00443949	0.01672417
BMLE	$(\hat{\beta}_{MLE})$	383	0.00106169	0.00040853	0.00015294	0.00242917
BTW	$(\hat{\beta}_T)$	500	0.00096453	0.00054552	0.00013448	0.00418562
BC	$(\hat{\beta}_c)$	500	0.00152732	0.00130292	0.00039950	0.02135745
					POPN=1	N=30
AUN	$(\hat{\alpha}_u)$	500	1.62712646	11.88926061	-205.10704120	60.37376985
AMLE	$(\hat{\alpha}_{MLE})$	362	2.09429297	0.84162310	0.84120049	9.67324318
ATW	$(\hat{\alpha}_T)$	500	2.37829785	1.13079438	0.69615111	11.56585040
AC	$(\hat{\alpha}_c)$	500	1.46199523	0.51216395	0.09591127	3.32870813
BUN	$(\hat{\beta}_u)$	500	0.00099369	0.00225004	-0.00237670	0.04460259
BMLE	$(\hat{\beta}_{MLE})$	362	0.00107527	0.00035879	0.00018054	0.00256728
BTW	$(\hat{\beta}_T)$	500	0.00098850	0.00042664	0.00014840	0.00305167
BC	$(\hat{\beta}_c)$	500	0.00170166	0.00236389	0.00051600	0.04837636
					POPN=1	N=40
AUN	$(\hat{\alpha}_u)$	500	2.25434716	7.69585810	-117.20745603	42.48550417
AMLE	$(\hat{\alpha}_{MLE})$	342	1.92303906	0.46182491	1.15701181	4.06159035
ATW	$(\hat{\alpha}_T)$	500	2.30470062	0.84313145	0.81301577	7.68972455
AC	$(\hat{\alpha}_c)$	500	1.47394636	0.45493956	0.15553903	3.19460875
BUN	$(\hat{\beta}_u)$	500	0.00083483	0.00111106	-0.00157724	0.01996949
BMLE	$(\hat{\beta}_{MLE})$	342	0.00109434	0.00025659	0.00053006	0.00192846
BTW	$(\hat{\beta}_T)$	500	0.00096886	0.00033441	0.00025443	0.00215898
BC	$(\hat{\beta}_c)$	500	0.00152230	0.00111330	0.00058112	0.02192685
					POPN=1	N=50
AUN	$(\hat{\alpha}_u)$	500	-0.31707720	64.98230548	-1437.52300977	110.60665112
AMLE	$(\hat{\alpha}_{MLE})$	299	1.94618576	0.42687501	1.07292574	3.70269778
ATW	$(\hat{\alpha}_T)$	500	2.27233467	0.72082036	0.62428273	6.30578566
AC	$(\hat{\alpha}_c)$	500	1.44130233	0.41347668	0.15259209	2.96667660
BUN	$(\hat{\beta}_u)$	500	0.00084636	0.00108832	-0.00214688	0.01730302
BMLE	$(\hat{\beta}_{MLE})$	299	0.00108134	0.00024793	0.00055075	0.00196805
BTW	$(\hat{\beta}_T)$	500	0.00096836	0.00035261	0.00030713	0.00344424
BC	$(\hat{\beta}_c)$	500	0.00155941	0.00105695	0.00065955	0.01368581

Table 4.1.2 (cont).

VARIABLE		N	MEAN	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE
					PCPN=1	N=60
ALN	$(\hat{\alpha}_u)$	500	1.29054370	10.53146769	-130.35017431	73.91826896
AMLE	$(\hat{\alpha}_{MLE})$	280	1.95398125	0.45100984	1.15330654	4.39083237
ATW	$(\hat{\alpha}_T)$	500	2.18766434	0.60014622	0.88756181	5.04663996
AC	$(\hat{\alpha}_c)$	500	1.39110927	0.36815960	0.11603380	2.62909953
BUN	$(\hat{\beta}_u)$	500	0.00095278	0.00134127	-0.00220243	0.02321298
BMLE	$(\hat{\beta}_{MLE})$	280	0.00107526	0.00025195	0.00043821	0.00191717
BTW	$(\hat{\beta}_T)$	500	0.00097824	0.00030776	0.00030455	0.00238714
BC	$(\hat{\beta}_c)$	500	0.00168889	0.00135992	0.00071403	0.02444499
					PCPN=1	N=70
ALN	$(\hat{\alpha}_u)$	500	0.62357246	32.44718825	-469.65394587	173.77352272
AMLE	$(\hat{\alpha}_{MLE})$	237	1.90267874	0.37934961	1.23822030	3.54757691
ATW	$(\hat{\alpha}_T)$	500	2.18490014	0.61039014	0.77703827	5.10550269
AC	$(\hat{\alpha}_c)$	500	1.41188813	0.35187696	0.13773689	2.73721781
BUN	$(\hat{\beta}_u)$	500	0.00091965	0.00141330	-0.00117306	0.02000257
BMLE	$(\hat{\beta}_{MLE})$	237	0.00109746	0.00021661	0.00049530	0.00171493
BTW	$(\hat{\beta}_T)$	500	0.00098666	0.00029064	0.00034661	0.00237990
BC	$(\hat{\beta}_c)$	500	0.00162542	0.00150135	0.00075368	0.02275115
					PCPN=1	N=80
ALN	$(\hat{\alpha}_u)$	500	-2.93540747	151.21815558	-3346.54007746	414.83029657
AMLE	$(\hat{\alpha}_{MLE})$	220	1.95822467	0.37378138	1.28029094	3.44056935
ATW	$(\hat{\alpha}_T)$	500	2.17751326	0.54637655	0.87893410	4.24736719
AC	$(\hat{\alpha}_c)$	500	1.38862416	0.32711119	0.12715540	2.42748749
BUN	$(\hat{\beta}_u)$	500	0.00090720	0.00123420	-0.00068870	0.02302795
BMLE	$(\hat{\beta}_{MLE})$	220	0.00106950	0.00020636	0.00059061	0.00191669
BTW	$(\hat{\beta}_T)$	500	0.00097687	0.00025901	0.00043548	0.00207420
BC	$(\hat{\beta}_c)$	500	0.00160050	0.00123077	0.00085301	0.02391324
					PCPN=1	N=90
ALN	$(\hat{\alpha}_u)$	500	-2.83993113	133.15607358	-2959.97601727	247.70722155
AMLE	$(\hat{\alpha}_{MLE})$	210	1.94615422	0.34104616	1.16804507	3.18288250
ATW	$(\hat{\alpha}_T)$	500	2.11797467	0.49548865	1.12144317	3.68745836
AC	$(\hat{\alpha}_c)$	500	1.35666222	0.31540550	0.13357697	2.26403216
BUN	$(\hat{\beta}_u)$	500	0.00093754	0.00124449	-0.00067556	0.02154488
BMLE	$(\hat{\beta}_{MLE})$	210	0.00107136	0.00019934	0.00064014	0.00159667
BTW	$(\hat{\beta}_T)$	500	0.00099167	0.00024665	0.00048652	0.00185908
BC	$(\hat{\beta}_c)$	500	0.00164163	0.00131164	0.00076390	0.02389578
					PCPN=1	N=100
ALN	$(\hat{\alpha}_u)$	500	1.63492874	11.56378995	-129.03171368	72.73812809
AMLE	$(\hat{\alpha}_{MLE})$	195	1.96347302	0.34103539	1.20916636	3.40118558
ATW	$(\hat{\alpha}_T)$	500	2.13292721	0.49384102	0.97690597	4.54784515
AC	$(\hat{\alpha}_c)$	500	1.36234510	0.31548719	0.16316701	2.33434227
BUN	$(\hat{\beta}_u)$	500	0.00091653	0.00129708	-0.00106843	0.01622258
BMLE	$(\hat{\beta}_{MLE})$	195	0.00104781	0.00018389	0.00054983	0.00153527
BTW	$(\hat{\beta}_T)$	500	0.00098237	0.00024817	0.00037344	0.00215046
BC	$(\hat{\beta}_c)$	500	0.00164413	0.00130765	0.00080834	0.01727645

Table 4.1.2 (cont.)

VARIABLE		N	MEAN	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE
					PCPN=2	N=10
ALN	($\hat{\alpha}_u$)	500	2.67240299	16.47586603	-73.70930344	185.66401644
AMLE	($\hat{\alpha}_{MLE}$)	331	2.74919700	2.24016790	0.41753001	18.56444747
ATW	($\hat{\alpha}_T$)	500	7.57701804	20.09302921	-24.68177198	289.93515789
AC	($\hat{\alpha}_c$)	500	1.42074804	1.13977417	0.19772239	19.58776863
BUN	($\hat{\beta}_u$)	500	-0.00064227	0.00843137	-0.09975146	0.01237043
BMLE	($\hat{\beta}_{MLE}$)	381	0.00104654	0.00062476	0.00003495	0.00383732
BTW	($\hat{\beta}_T$)	500	0.00067778	0.00058587	-0.00011771	0.00417362
BC	($\hat{\beta}_c$)	500	0.00186041	0.00153123	0.00003719	0.01403740
					PCPN=2	N=20
ALN	($\hat{\alpha}_u$)	500	-3.90707869	108.59959981	-2240.18308693	323.76701430
AMLE	($\hat{\alpha}_{MLE}$)	388	2.26500146	1.08275913	0.87677253	10.59110818
ATW	($\hat{\alpha}_T$)	500	3.39012449	3.38973428	0.41791478	41.87340826
AC	($\hat{\alpha}_c$)	500	1.13468369	0.50217090	0.10345166	4.76171014
BUN	($\hat{\beta}_u$)	500	-0.00026530	0.00697683	-0.05003494	0.06793155
BMLE	($\hat{\beta}_{MLE}$)	388	0.00103431	0.00042118	0.00021350	0.00286962
BTW	($\hat{\beta}_T$)	500	0.00075502	0.00049808	0.00003212	0.00308741
BC	($\hat{\beta}_c$)	500	0.00244611	0.00434658	0.00049566	0.07460930
					PCPN=2	N=30
ALN	($\hat{\alpha}_u$)	500	1.73334759	24.18215923	-226.19957624	334.46973295
AMLE	($\hat{\alpha}_{MLE}$)	375	2.00023117	0.68383453	0.92594557	3.23637788
ATW	($\hat{\alpha}_T$)	500	3.03237388	2.01287791	0.72230347	17.39169132
AC	($\hat{\alpha}_c$)	500	1.07035986	0.39545110	0.08617075	2.90548079
BUN	($\hat{\beta}_u$)	500	-0.00025827	0.00460251	-0.03956371	0.04291246
BMLE	($\hat{\beta}_{MLE}$)	375	0.00110320	0.00034673	0.00022540	0.00239395
BTW	($\hat{\beta}_T$)	500	0.00085050	0.00046519	0.00010625	0.00338379
BC	($\hat{\beta}_c$)	500	0.00229261	0.00265270	0.00063056	0.04710504
					PCPN=2	N=40
ALN	($\hat{\alpha}_u$)	500	2.94785615	29.81229066	-179.55587305	373.74358538
AMLE	($\hat{\alpha}_{MLE}$)	360	1.98404040	0.55331186	0.84873664	4.54635117
ATW	($\hat{\alpha}_T$)	500	2.66195436	1.29749324	0.68463362	9.49738372
AC	($\hat{\alpha}_c$)	500	1.04570501	0.36161255	0.09466636	2.71769632
BUN	($\hat{\beta}_u$)	500	0.00017560	0.00399585	-0.03033700	0.03475907
BMLE	($\hat{\beta}_{MLE}$)	360	0.00110009	0.00032626	0.00041544	0.00291709
BTW	($\hat{\beta}_T$)	500	0.00091051	0.00044112	0.00017459	0.00297385
BC	($\hat{\beta}_c$)	500	0.00241093	0.00277974	0.00077683	0.03894991
					PCPN=2	N=50
ALN	($\hat{\alpha}_u$)	500	5.45859236	91.64778091	-130.99287936	2017.82093735
AMLE	($\hat{\alpha}_{MLE}$)	330	1.95463671	0.51001135	0.89757734	4.47302972
ATW	($\hat{\alpha}_T$)	500	2.63485538	1.24019365	0.59829274	13.83034742
AC	($\hat{\alpha}_c$)	500	1.01253606	0.34044151	0.08904025	2.32499252
BUN	($\hat{\beta}_u$)	500	-0.00022950	0.00445794	-0.02389609	0.04785729
BMLE	($\hat{\beta}_{MLE}$)	330	0.00108517	0.00028619	0.00041319	0.00285437
BTW	($\hat{\beta}_T$)	500	0.00088602	0.00043546	0.00015375	0.00565953
BC	($\hat{\beta}_c$)	500	0.00243133	0.00312078	0.00078016	0.05211494

Table 4.1.2 (cont).

VARIABLE		N	MEAN	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE
					POPN=2	N=60
AUN	($\hat{\alpha}_u$)	500	-0.70207798	40.73474439	-429.47395779	496.86734692
AMLE	($\hat{\alpha}_{MLE}$)	288	1.91056051	0.44099009	1.05290069	3.74037729
ATW	($\hat{\alpha}_T$)	500	2.46876283	0.92336922	0.73423858	6.07325954
AC	($\hat{\alpha}_c$)	500	1.01287347	0.31891510	0.07064940	2.09796808
BUN	($\hat{\beta}_u$)	500	0.00003194	0.00432342	-0.01729585	0.05027137
BMLE	($\hat{\beta}_{MLE}$)	288	0.00110966	0.00025777	0.00049460	0.00208458
BTW	($\hat{\beta}_T$)	500	0.00092382	0.00039647	0.00024507	0.00350355
BC	($\hat{\beta}_c$)	500	0.00250384	0.00343596	0.00074040	0.05363724
					POPN=2	N=70
AUN	($\hat{\alpha}_u$)	500	0.32525361	37.96921741	-671.97467425	226.98107548
AMLE	($\hat{\alpha}_{MLE}$)	275	1.93360965	0.41190704	1.01083616	3.53242220
ATW	($\hat{\alpha}_T$)	0	.	.	.	.
AC	($\hat{\alpha}_c$)	500	0.97964273	0.28085205	0.07612608	1.84272779
BUN	($\hat{\beta}_u$)	500	-0.00006926	0.00390585	-0.01820543	0.04206101
BMLE	($\hat{\beta}_{MLE}$)	275	0.00108476	0.00024892	0.00048094	0.00190705
BTW	($\hat{\beta}_T$)	0	.	.	.	.
BC	($\hat{\beta}_c$)	500	0.00247184	0.00296271	0.00102729	0.04860958
					POPN=2	N=80
AUN	($\hat{\alpha}_u$)	500	-1.31025036	117.18061909	-1732.28297691	1669.77397294
AMLE	($\hat{\alpha}_{MLE}$)	251	1.92672903	0.40183824	1.13867107	3.34617159
ATW	($\hat{\alpha}_T$)	0	.	.	.	.
AC	($\hat{\alpha}_c$)	500	0.96151176	0.29209458	0.07377236	2.04593202
BUN	($\hat{\beta}_u$)	500	0.00005723	0.00370163	-0.01491859	0.03876876
BMLE	($\hat{\beta}_{MLE}$)	251	0.00109735	0.00024286	0.00059475	0.00184025
BTW	($\hat{\beta}_T$)	0	.	.	.	.
BC	($\hat{\beta}_c$)	500	0.00256122	0.00308321	0.00081028	0.04603445
					POPN=2	N=90
AUN	($\hat{\alpha}_u$)	500	0.79136739	17.10305637	-212.41874904	152.12301841
AMLE	($\hat{\alpha}_{MLE}$)	236	1.93972801	0.36821192	1.23509280	3.12548430
ATW	($\hat{\alpha}_T$)	0	.	.	.	.
AC	($\hat{\alpha}_c$)	500	0.96222123	0.29118987	0.06857482	1.76925460
BUN	($\hat{\beta}_u$)	500	0.00022749	0.00355566	-0.01141393	0.04031046
BMLE	($\hat{\beta}_{MLE}$)	236	0.00107807	0.00021394	0.00063213	0.00176903
BTW	($\hat{\beta}_T$)	0	.	.	.	.
BC	($\hat{\beta}_c$)	500	0.00260207	0.00303117	0.00095090	0.04464179
					POPN=2	N=100
AUN	($\hat{\alpha}_u$)	800	-27.31097817	538.30635596	-10744.0525210	402.02191508
AMLE	($\hat{\alpha}_{MLE}$)	335	1.92843721	0.36918159	1.2962321	3.23082133
ATW	($\hat{\alpha}_T$)	600	-17.72665534	439.67079275	-10744.0525210	44.91091688
AC	($\hat{\alpha}_c$)	800	0.94252753	0.27275381	0.0881537	1.64688011
BUN	($\hat{\beta}_u$)	800	0.00000578	0.00341113	-0.0147885	0.03301532
BMLE	($\hat{\beta}_{MLE}$)	335	0.00107845	0.00020111	0.0006256	0.00178264
BTW	($\hat{\beta}_T$)	600	1.13725436	1.22667378	0.0003746	4.65687376
BC	($\hat{\beta}_c$)	800	0.00258518	0.00271429	0.0011213	0.03824761

but the range of $\hat{\alpha}_u$ fluctuates. The distributions of the estimates appear in the histograms to become more concentrated around the true value as the sample size increases; however, the variance fluctuates in both populations. Table 4.1.2 gives summary statistics of all the estimates from every sample size and both populations. Since the negative values of $\hat{\alpha}_u$ and $\hat{\beta}_u$ are unacceptable they are omitted from the calculations of the means illustrated in Figures 4.1.7 to 4.1.10 and also from the statistics in Table 4.1.2. Table 4.1.3 is an ordered listing of $\hat{\alpha}_u$ from samples of size 100 from Population Two. The large number of negatives can be observed.

The empirical distributions of $\hat{\alpha}_u$ and $\hat{\beta}_u$ were tested for normality using SAS procedure Univariate. This procedure uses the Kolmogorov-Smirnov D statistic. In testing

H_0: The estimates come from a normal (μ,σ^2) distribution <u>vs</u>

H_1: The obvious alternative hypothesis

the hypothesis is rejected for large values of D. In every case, the null hypothesis of a normal distribution of $\hat{\alpha}_u$ and $\hat{\beta}_u$ was rejected. A shifted lognormal or a shifted gamma is suggested as a trial distribution for future study. It was expected that $\hat{\alpha}_u$ and $\hat{\beta}_u$ would show high correlation as had been observed in [1]. However, Figures 4.1.11 and 4.1.12 show little correlation between $\hat{\alpha}_u$ and $\hat{\beta}_u$.

4.2 Time-Weighted Method-of-Moments Estimators

The time-weighted method-of-moments estimators behaved less erratically than the unweighted method-of-moments estimators. This was unexpected for at least two reasons. According to the Bayesian philosophy it should not be desirable to weight the estimates of λ_i by the experiment times. It can be argued that nature chooses the values of λ_i for each experiment. Therefore,

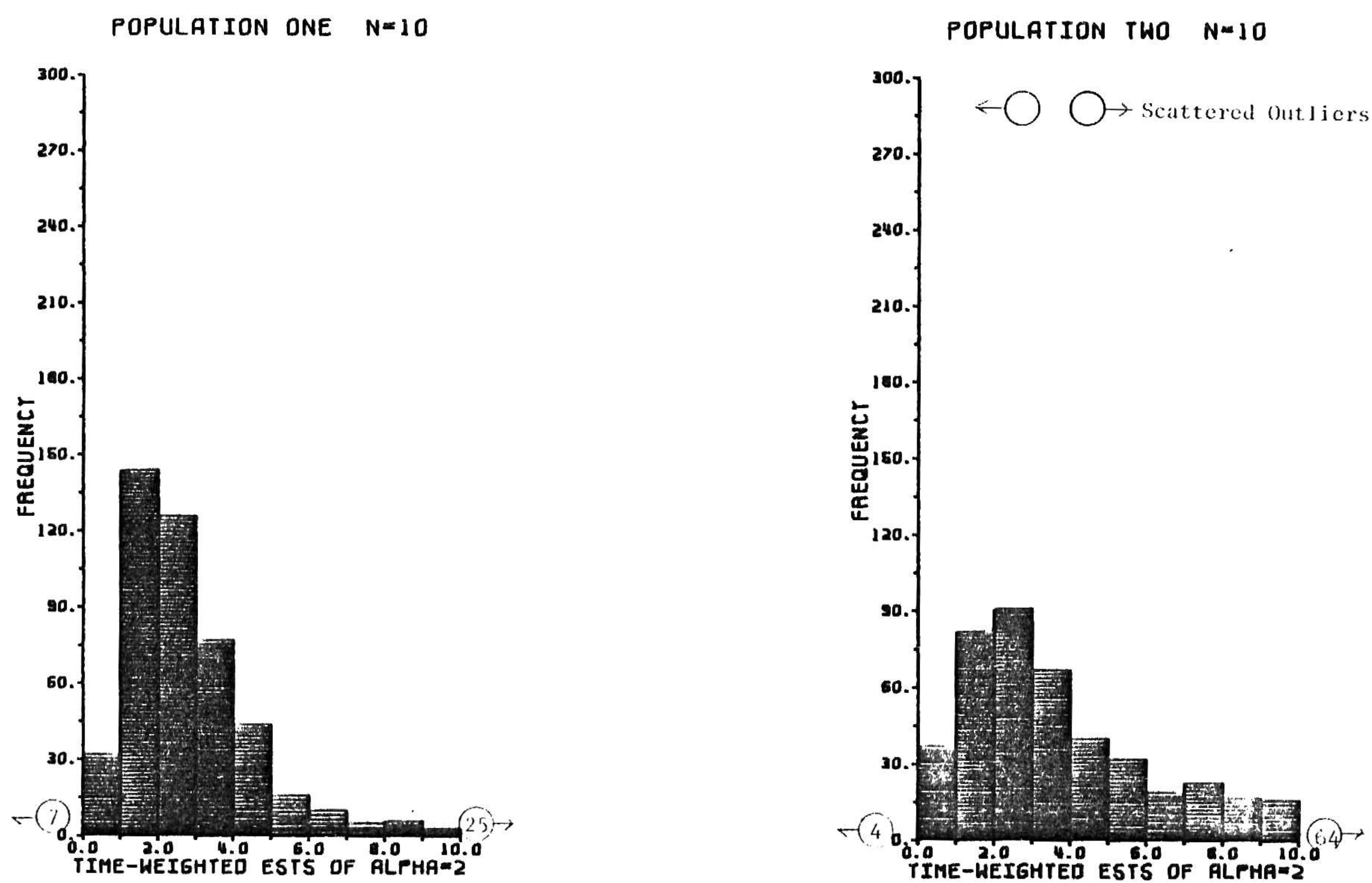
Table 4.1.3 Ordered values of $\hat{\alpha}_{11}$ for n=100 from Population Two

AUN	AUN	AUN	AUN	AUN	AUN	AUN	AUN	AUN	AUN
-10744	-3.5841	-1.4907	-0.73518	0.51710	1.54775	2.22148	3.02947	5.7200	37.907
-612	-3.4557	-1.4816	-0.63100	0.55677	1.55028	2.23070	3.03128	5.7912	40.612
-265	-3.4050	-1.4708	-0.63050	0.60162	1.55340	2.25879	3.04571	6.0429	44.911
-115	-3.3911	-1.4579	-0.61350	0.63031	1.57613	2.27146	3.13774	6.0443	46.910
-56	-3.3346	-1.4467	-0.60483	0.63703	1.61075	2.29115	3.14705	6.0479	402.022
-53	-3.2870	-1.4233	-0.60260	0.65757	1.61418	2.30616	3.16679	6.1583	
-47	-3.2585	-1.4138	-0.58192	0.67120	1.64878	2.30812	3.23207	6.1930	
-46	-3.1972	-1.3890	-0.57291	0.73100	1.65211	2.31257	3.23719	6.3912	
-28	-3.1519	-1.3533	-0.57026	0.75688	1.68783	2.31841	3.24273	6.4828	
-26	-3.0594	-1.3320	-0.55954	0.78830	1.70443	2.37166	3.24367	6.5288	
-25	-2.9812	-1.3159	-0.52555	0.79284	1.71947	2.38471	3.25280	6.6809	
-25	-2.8827	-1.3120	-0.51457	0.80253	1.72033	2.39504	3.25644	6.7081	
-24	-2.8803	-1.2952	-0.50342	0.82002	1.72348	2.41952	3.31564	6.7522	
-19	-2.8479	-1.2679	-0.45760	0.83713	1.72499	2.44132	3.32519	6.8268	
-17	-2.8073	-1.2504	-0.45616	0.86941	1.76786	2.45799	3.38980	6.9688	
-17	-2.8054	-1.2391	-0.44550	0.87557	1.77419	2.46278	3.42321	6.9861	
-17	-2.7553	-1.2313	-0.43474	0.87683	1.78107	2.48150	3.43664	6.9874	
-15	-2.7328	-1.1808	-0.43387	0.88478	1.79441	2.48180	3.49233	7.2302	
-15	-2.7271	-1.1610	-0.36887	0.89517	1.80023	2.48278	3.52172	7.2764	
-15	-2.7162	-1.1275	-0.36071	0.89710	1.80631	2.49782	3.58876	7.3799	
-14	-2.6462	-1.0955	-0.31353	0.92222	1.80711	2.51602	3.59379	7.4371	
-14	-2.6452	-1.0576	-0.29853	0.92617	1.81632	2.51774	3.61334	7.7263	
-14	-2.6230	-1.0420	-0.27477	0.92817	1.82042	2.54740	3.68044	8.3999	
-14	-2.5919	-1.0324	-0.27269	0.93886	1.84170	2.57343	3.81962	8.4031	
-11	-2.5410	-1.0259	-0.25167	0.95628	1.85981	2.59138	3.88430	8.6157	
-11	-2.5309	-1.0247	-0.22676	0.98864	1.86429	2.63396	3.91084	9.3313	
-9	-2.5257	-1.0244	-0.21707	0.98872	1.86444	2.63488	3.92864	9.3827	
-9	-2.5147	-0.9996	-0.20935	0.99028	1.86744	2.65091	3.94387	9.6352	
-8	-2.5024	-0.9692	-0.20773	1.00306	1.86944	2.65850	3.96534	9.6686	
-8	-2.4772	-0.9400	-0.20537	1.00769	1.89197	2.65890	4.03725	9.7790	
-7	-2.4274	-0.9187	-0.19667	1.02033	1.89546	2.73135	4.07198	9.8282	
-7	-2.4112	-0.9014	-0.19003	1.06927	1.91182	2.73607	4.13308	9.9052	
-7	-2.3514	-0.8974	-0.18925	1.07858	1.93261	2.73766	4.18619	9.9336	
-7	-2.3093	-0.8662	-0.18732	1.10265	1.96226	2.73767	4.31207	10.0207	
-6	-2.2651	-0.8255	-0.18527	1.13098	1.97898	2.73771	4.33343	10.5466	
-6	-2.2518	-0.8227	-0.17848	1.14733	1.98306	2.76476	4.38097	11.1939	
-6	-2.2197	-0.8222	-0.15989	1.19736	2.02431	2.77060	4.72625	11.4370	
-6	-2.1554	-0.8129	-0.14860	1.21586	2.02456	2.77072	4.74432	11.7808	
-6	-2.0672	-0.8014	-0.11987	1.24172	2.04980	2.79960	4.77676	12.1629	
-5	-2.0450	-0.8011	0.10212	1.28751	2.06005	2.80750	4.79587	13.5106	
-5	-2.0035	-0.7627	0.12312	1.29096	2.06792	2.81897	4.80041	13.5818	
-5	-1.9066	-0.7506	0.14326	1.30389	2.07125	2.83226	4.83611	14.3298	
-5	-1.8864	-0.7367	0.18169	1.32256	2.07468	2.84242	4.85660	15.2772	
-5	-1.8798	-0.7357	0.19770	1.34198	2.08430	2.84764	4.87457	15.6488	
-5	-1.8680	-0.7333	0.25102	1.39273	2.11593	2.85003	4.93762	15.7080	
-5	-1.8572	-0.7196	0.26713	1.39304	2.13549	2.85133	4.96169	16.8080	
-5	-1.7886	-0.7136	0.26831	1.45538	2.13988	2.85463	4.98376	17.5919	
-5	-1.7432	-0.7121	0.27387	1.46145	2.15824	2.86810	5.07897	19.7689	
-5	-1.6352	-0.6998	0.29576	1.48223	2.16001	2.92071	5.30972	20.7154	
-4	-1.6239	-0.6952	0.31667	1.48762	2.17287	2.93471	5.42458	21.3808	
-4	-1.5873	-0.6821	0.34597	1.50437	2.19240	2.95471	5.42699	22.4867	
-4	-1.5616	-0.6777	0.38727	1.50490	2.19312	2.98707	5.48722	24.3352	
-4	-1.5605	-0.6585	0.47299	1.51273	2.19848	2.99003	5.48731	31.1155	
-4	-1.5329	-0.6524	0.51225	1.51891	2.20810	2.99777	5.55857	33.2482	
-3	-1.5204	-0.6394	0.51311	1.54774	2.22133	3.01274	5.59726	35.2365	

each is as important as the next and the estimate $\hat{\lambda}_i$ from a short experiment is as valuable to a statistician as an estimate $\hat{\lambda}_i$ from a long experiment. Additionally, it has been demonstrated by Grosh [2] that the $\bar{\bar{\lambda}}_u$ usually have smaller variance than the $\bar{\bar{\lambda}}_T$, and hence it could be expected that $\hat{\alpha}_u$ and $\hat{\beta}_u$ would have smaller variances than $\hat{\alpha}_T$ and $\hat{\beta}_T$. In spite of this, the simulation results showed the time-weighted estimators to be better in several respects than the unweighted estimators. The sample variances of the values obtained by $\hat{\alpha}_T$ and $\hat{\beta}_T$ generally decreased as the sample size increased in both populations while the range fluctuated. The variances were less for the time-weighted method-of-moments estimates than for the unweighted method-of-moments estimates in every case except when estimating α from samples of size ten. Table 4.1.2 documents these results. Also note that the sample variance was smaller for estimates of α from Population One than for Population Two and larger for estimates of β.

The time-weighted method-of-moments estimator did not produce such a discouraging number of negative values. In fact, although it can take on those unsatisfactory values, none were observed for samples of size twenty or more. Figures 4.2.1 to 4.2.6 show the histograms of the values obtained by $\hat{\alpha}_T$ and $\hat{\beta}_T$ in Population One and Population Two for samples of size 10, 50 and 100. The tendency of the estimates to cluster more and more toward the true value is apparent as the sample size increases.

Figures 4.1.7 to 4.1.10 illustrate the means of the 500 estimates $\hat{\alpha}_T$ and $\hat{\beta}_T$ drawn from each of the sample sizes. The time-weighted method-of-moments estimators are clearly biased. The bias tends toward zero as the sample size increased in both populations. The correlation between $\hat{\alpha}_u$ and $\hat{\beta}_u$ increased in a negative direction as the sample size increased. Figures 4.1.11 and 4.1.12 illustrate this fact. Recalling that the gamma distribution



POPULATION ONE N=10
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
7
25
0.0
2.0
4.0
6.0
8.0
10.0
TIME-WEIGHTED ESTS OF ALPHA=2
POPULATION TWO N=10
Scattered Outliers
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
4
64
0.0
2.0
4.0
6.0
8.0
10.0
TIME-WEIGHTED ESTS OF ALPHA=2

Figure 4.2.1

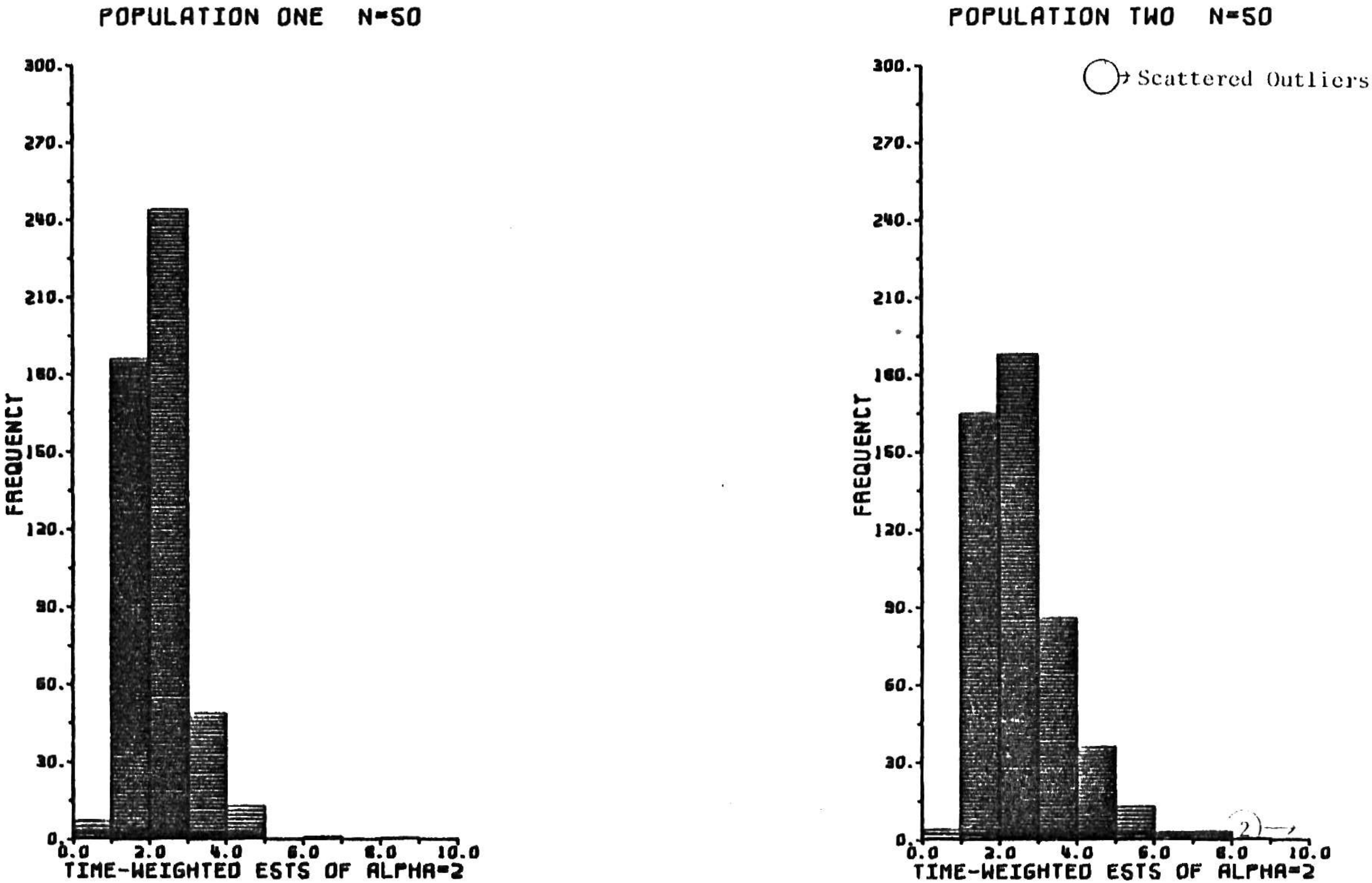
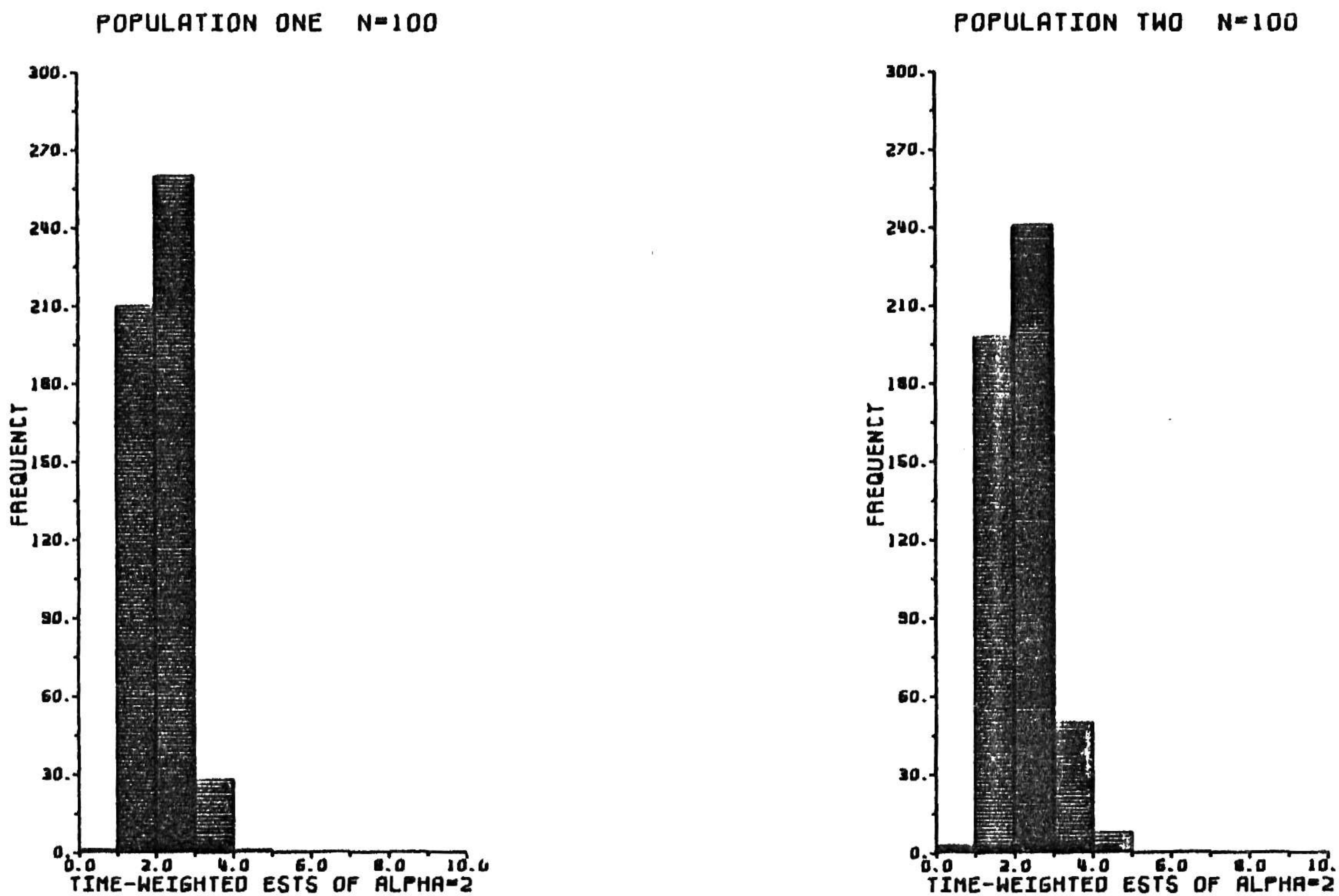

POPULATION ONE N=50
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
FREQUENCY
0.0
2.0
4.0
6.0
8.0
10.0
TIME-WEIGHTED ESTS OF ALPHA=2
POPULATION TWO N=50
Scattered Outliers
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
FREQUENCY
2
0.0
2.0
4.0
6.0
8.0
10.0
TIME-WEIGHTED ESTS OF ALPHA=2

Figure 4.2.2



POPULATION ONE N=100
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
0.0
2.0
4.0
6.0
8.0
10.0
TIME-WEIGHTED ESTS OF ALPHA=2
POPULATION TWO N=100
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
0.0
2.0
4.0
6.0
8.0
10.0
TIME-WEIGHTED ESTS OF ALPHA=2

Figure 4.2.3

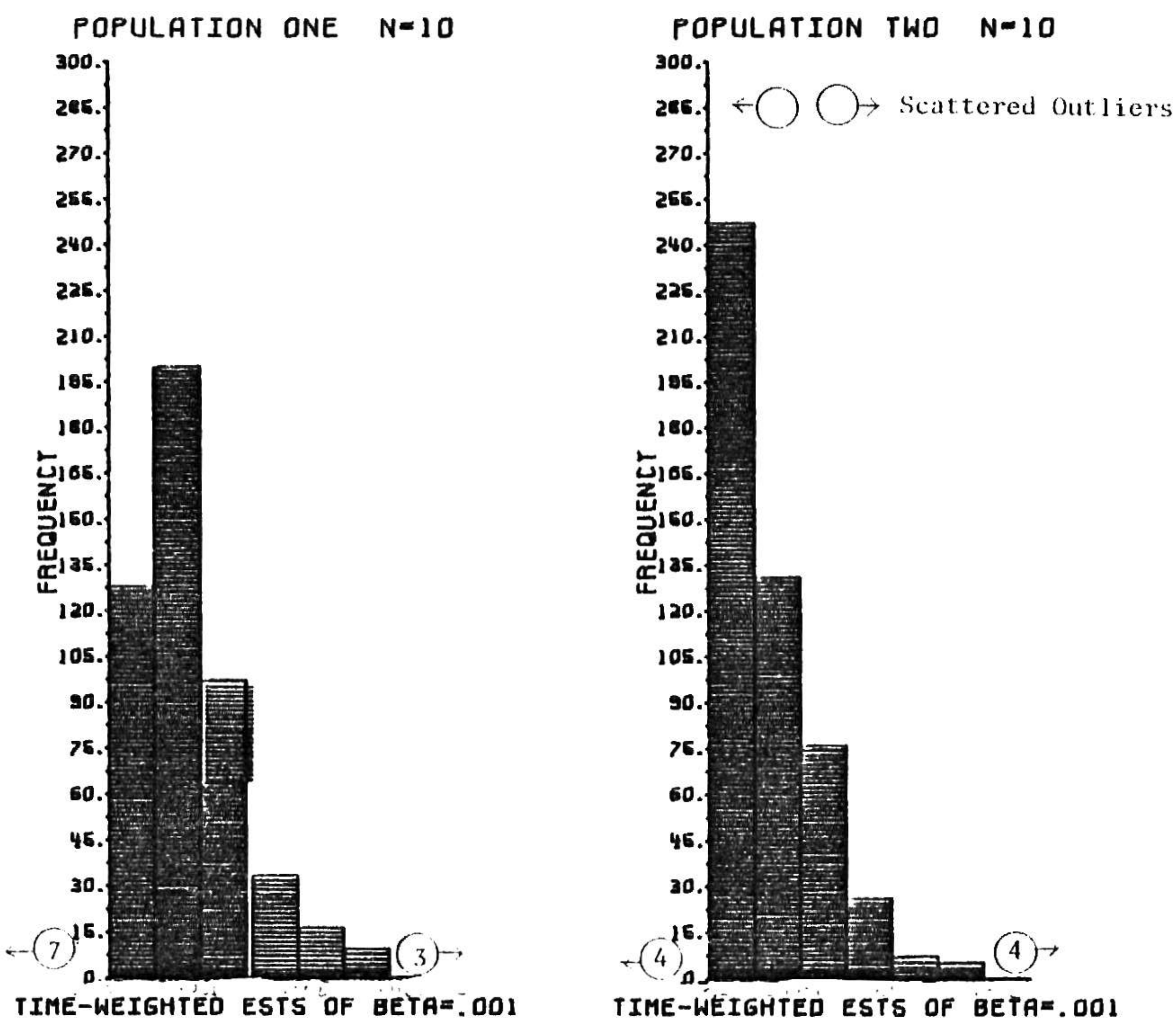

POPULATION ONE N=10
FREQUENCY
TIME-WEIGHTED ESTS OF BETA=.001
7
3
POPULATION TWO N=10
Scattered Outliers
FREQUENCY
TIME-WEIGHTED ESTS OF BETA=.001
4
4

Figure 4.2.4

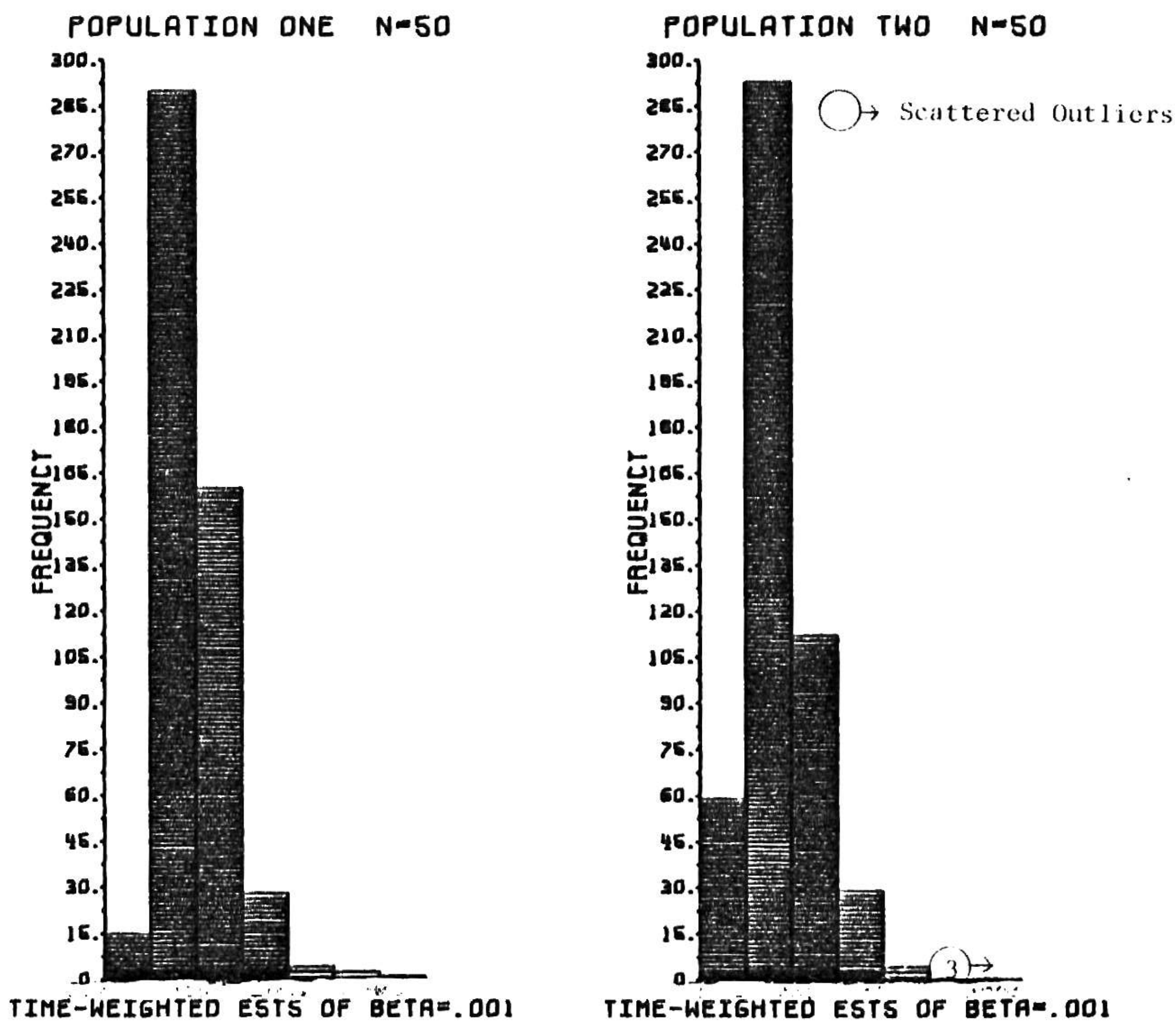

POPULATION ONE N=50
FREQUENCY
TIME-WEIGHTED ESTS OF BETA=.001
POPULATION TWO N=50
Scattered Outliers
3
FREQUENCY
TIME-WEIGHTED ESTS OF BETA=.001

Figure 4.2.5

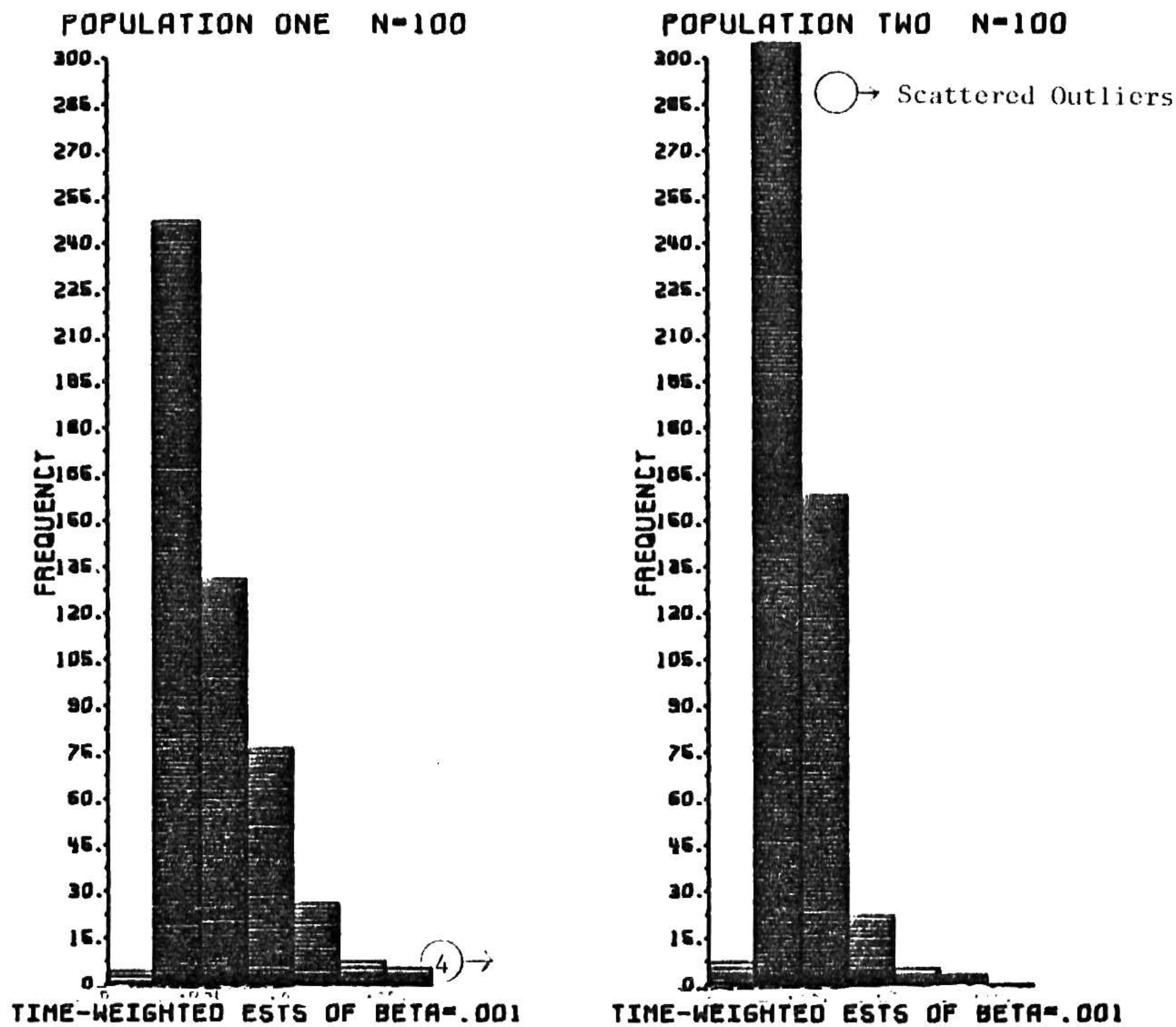

POPULATION ONE N=100
FREQUENCY
TIME-WEIGHTED ESTS OF BETA=.001
4
POPULATION TWO N=100
Scattered Outliers
FREQUENCY
TIME-WEIGHTED ESTS OF BETA=.001

Figure 4.2.6

has $E(\lambda) = \alpha\beta$, we expected $\hat{\alpha}$ or $\hat{\beta}$ to decrease as the other increases so the negative correlation was expected.

4.3 Maximum Likelihood Estimators

The maximum likelihood estimators cannot produce any negative values since the Newton-Raphson procedure used in this study is constrained to positive values only (see Section 6: Documentation for details used in programming the Newton-Raphson procedure).

Figures 4.3.1 to 4.3.6 show the comparison of the histograms of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$. As the sample size increased the histograms became narrower and more spiked in the vicinity of the true values, $\alpha = 2$ and $\beta = .001$. The ranges of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ were dramatically smaller than the ranges of $\hat{\alpha}_u$ and $\hat{\beta}_u$ and noticably narrower than those of $\hat{\alpha}_T$ and $\hat{\beta}_T$.

Table 4.3.1 is an ordered list of $\hat{\beta}_{MLE}$ for samples of size 50 from Population One. This shows the narrow range and also illustrates the major difficulty of the maximum likelihood estimators. For economy the Newton-Raphson procedure was limited in this study to twenty iterations. If the Newton-Raphson procedure did not converge within the twenty allowed iterations, it was denoted by $\hat{\alpha}_{MLE} = \hat{\beta}_{MLE} =$ '.'. SAS treats '.' as a missing value, and it was ignored in calculations of the statistics summarized in Table 4.1.2.

The ranges and variances of the $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ generally decreased as the sample size increased, and both are less for Population One than for Population Two. The means of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ are plotted in Figures 4.1.7 to 4.1.10. As the sample size increases the means of the maximum likelihood estimates began poorly, tended toward the parameter value and then "overshot" it. It is possible that the means may be beginning a trend back to the true values, but larger samples would need to be simulated to investigate this behavior further. A possible explanation of this odd behavior is the

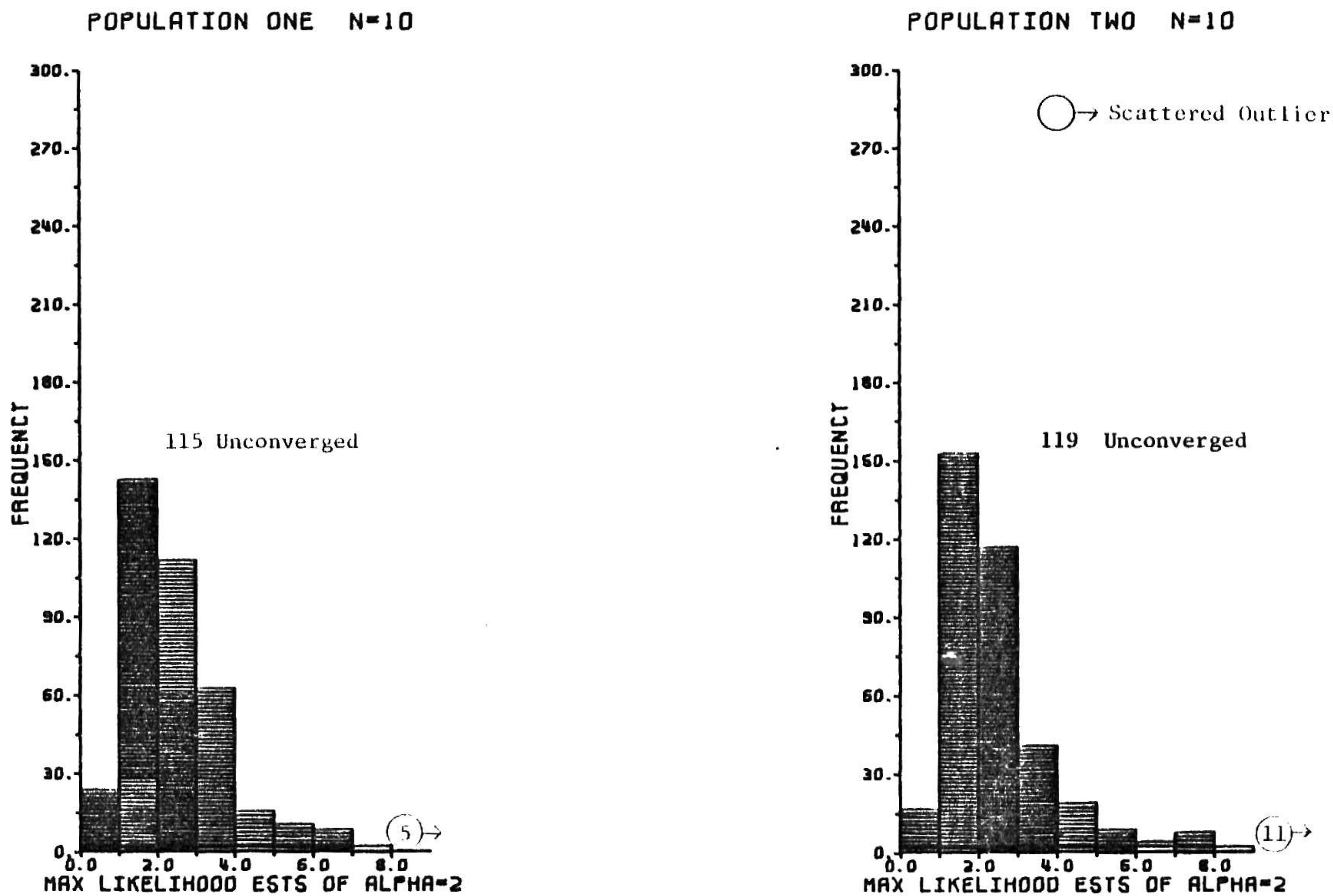

POPULATION ONE N=10
FREQUENCY
MAX LIKELIHOOD ESTS OF ALPHA=2
115 Unconverged
5
POPULATION TWO N=10
FREQUENCY
MAX LIKELIHOOD ESTS OF ALPHA=2
Scattered Outliers
119 Unconverged
11

Figure 4.3.1

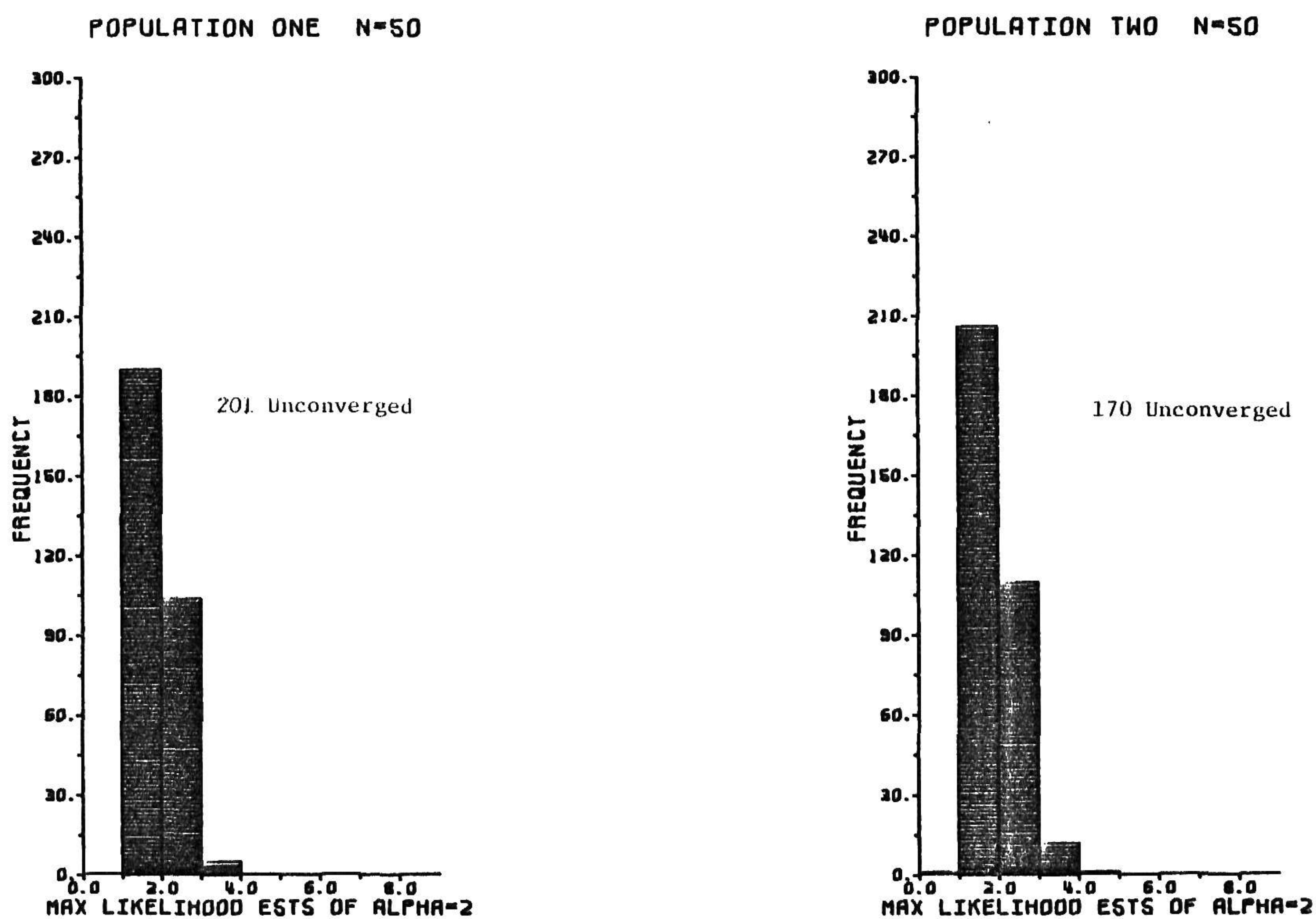

POPULATION ONE N=50
201 Unconverged
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
0.0
2.0
4.0
6.0
8.0
MAX LIKELIHOOD ESTS OF ALPHA=2
POPULATION TWO N=50
170 Unconverged
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
0.0
2.0
4.0
6.0
8.0
MAX LIKELIHOOD ESTS OF ALPHA=2

Figure 4.3.2

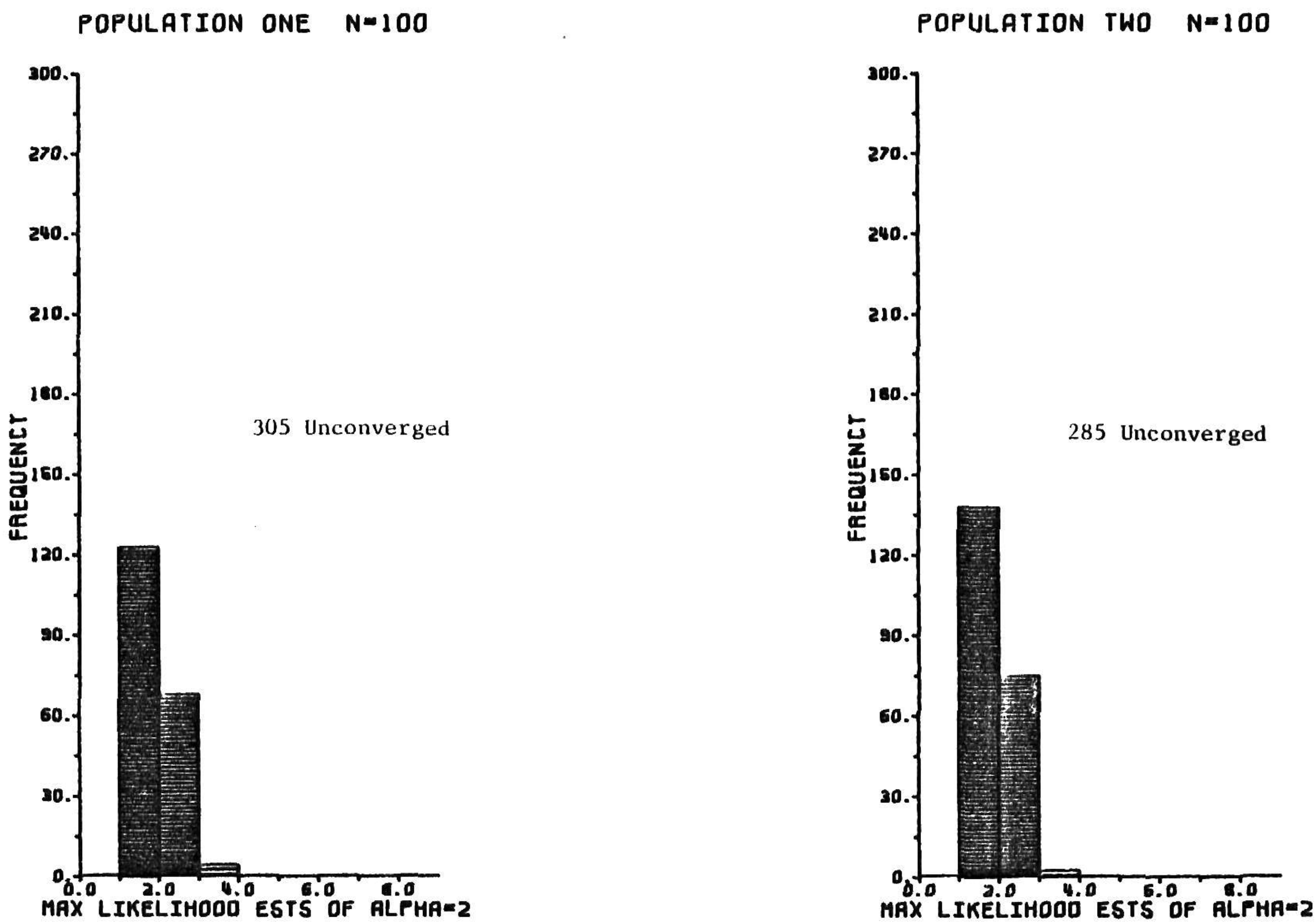

POPULATION ONE N=100
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
305 Unconverged
0.0
2.0
4.0
6.0
8.0
MAX LIKELIHOOD ESTS OF ALPHA=2
POPULATION TWO N=100
FREQUENCY
300.
270.
240.
210.
180.
150.
120.
90.
60.
30.
0.
285 Unconverged
0.0
2.0
4.0
6.0
8.0
MAX LIKELIHOOD ESTS OF ALPHA=2

Figure 4.3.3

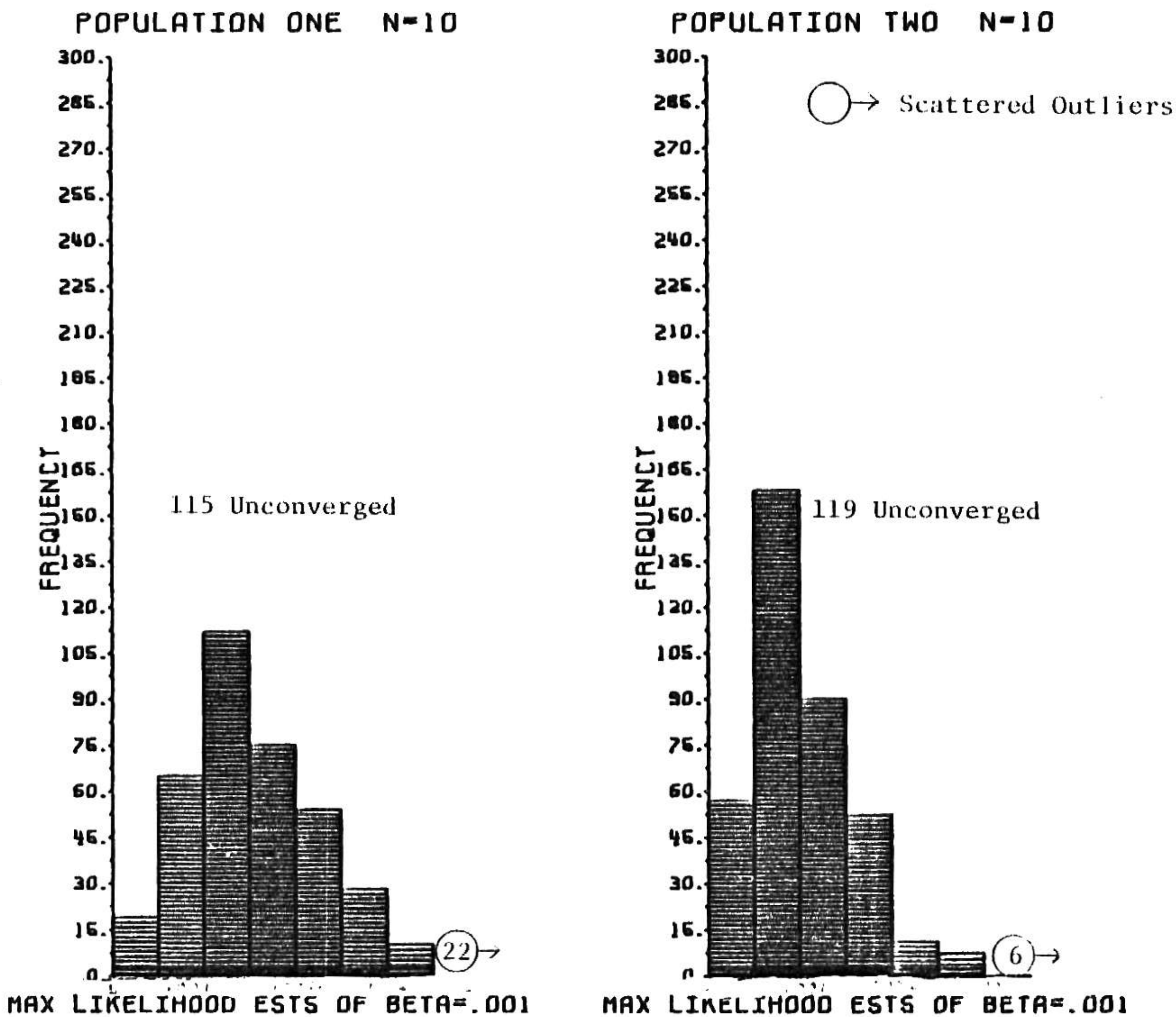

POPULATION ONE N=10
POPULATION TWO N=10
FREQUENCY
300.
285.
270.
255.
240.
225.
210.
195.
180.
165.
150.
135.
120.
105.
90.
75.
60.
45.
30.
15.
0.
115 Unconverged
119 Unconverged
Scattered Outliers
22
6
MAX LIKELIHOOD ESTS OF BETA=.001

Figure 4.3.4

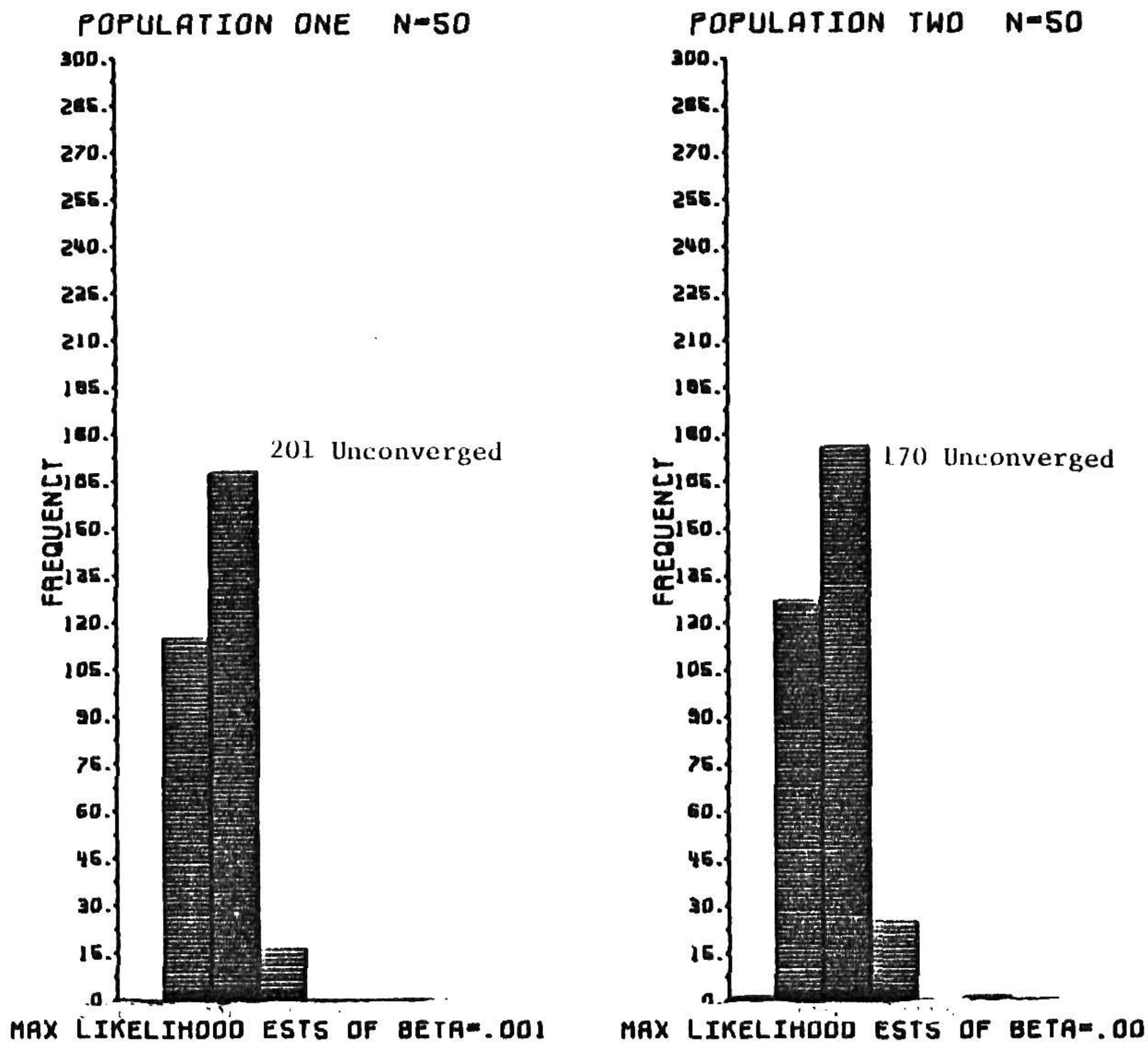

POPULATION ONE N=50
FREQUENCY
300.
285.
270.
255.
240.
225.
210.
195.
180.
165.
150.
135.
120.
105.
90.
75.
60.
45.
30.
15.
0
201 Unconverged
MAX LIKELIHOOD ESTS OF BETA=.001
POPULATION TWO N=50
FREQUENCY
300.
285.
270.
255.
240.
225.
210.
195.
180.
165.
150.
135.
120.
105.
90.
75.
60.
45.
30.
15.
0
170 Unconverged
MAX LIKELIHOOD ESTS OF BETA=.001

Figure 4.3.5

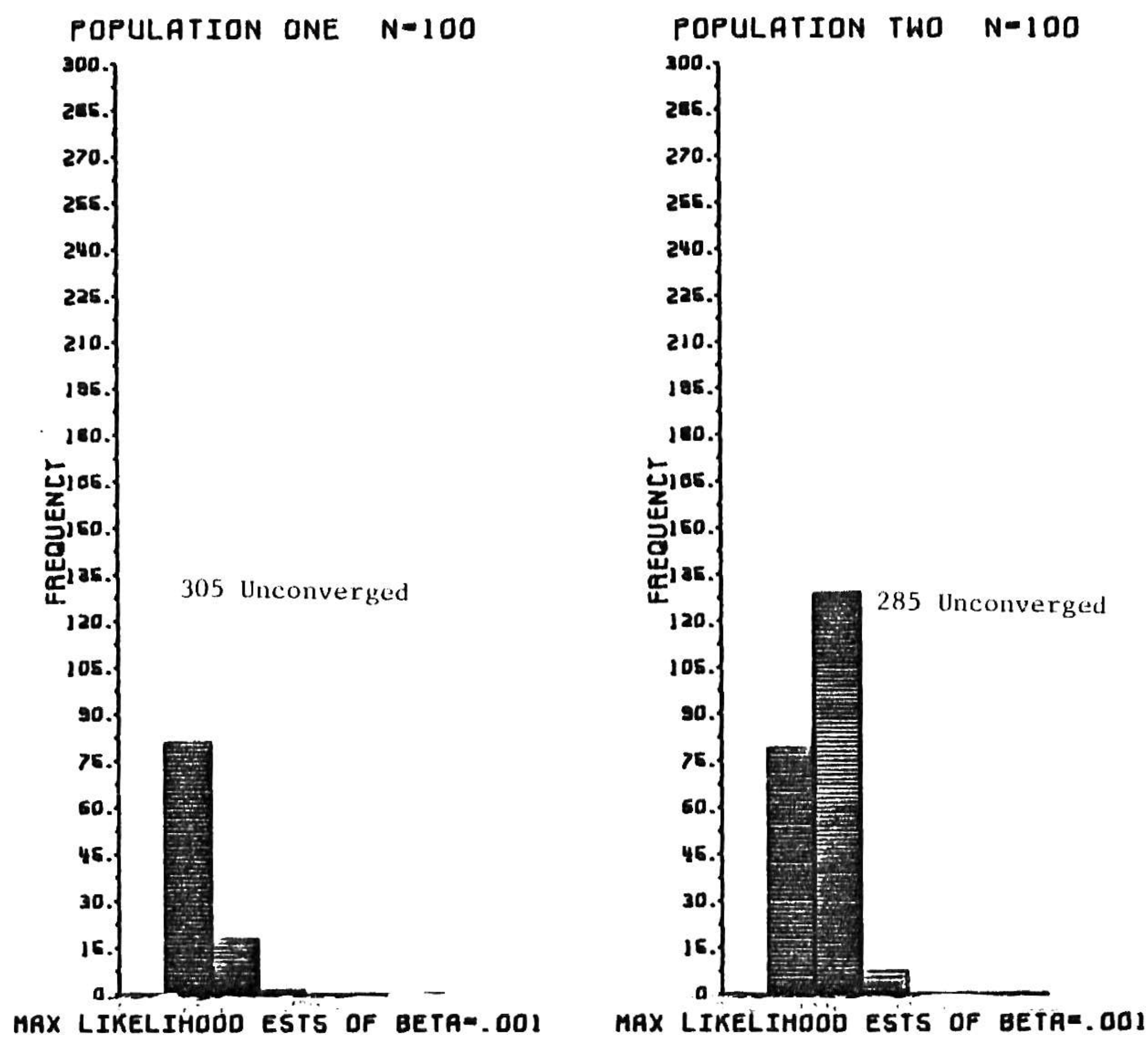

POPULATION ONE N=100
FREQUENCY
300.
285.
270.
255.
240.
225.
210.
195.
180.
165.
150.
135.
120.
105.
90.
75.
60.
45.
30.
15.
0
305 Unconverged
MAX LIKELIHOOD ESTS OF BETA=.001
POPULATION TWO N=100
FREQUENCY
300.
285.
270.
255.
240.
225.
210.
195.
180.
165.
150.
135.
120.
105.
90.
75.
60.
45.
30.
15.
0
285 Unconverged
MAX LIKELIHOOD ESTS OF BETA=.001

Figure 4.3.6

Table 4.3.1 Ordered List of $\hat{\beta}_{MLE}$ for n=50 (Population One)

BMLE	BMLE	BMLE	BMLE	BMLE	BMLE	BMLE	BMLE	BMLE	BMLE
.	.	.	.	0.000712054	0.000971159	0.00103340	0.00113225	0.00128401	0.00166151
.	.	.	.	0.000717721	0.000971459	0.00103656	0.00113676	0.00129169	0.00168907
.	.	.	.	0.000731675	0.000971809	0.00103901	0.00113695	0.00129190	0.00179641
.	.	.	.	0.000731777	0.000971827	0.00104262	0.00114086	0.00129513	0.00192699
.	.	.	.	0.000732476	0.000972090	0.00104267	0.00114087	0.00130615	0.00196805
.	.	.	.	0.000736961	0.000972155	0.00104297	0.00114214	0.00131270	
.	.	.	.	0.000743375	0.000972210	0.00104319	0.00114317	0.00131322	
.	.	.	.	0.000745179	0.000972217	0.00104338	0.00114520	0.00132177	
.	.	.	.	0.000757453	0.000972613	0.00104487	0.00114587	0.00132642	
.	.	.	.	0.000764814	0.000972732	0.00104824	0.00114869	0.00132849	
.	.	.	.	0.000767043	0.000973080	0.00105454	0.00115052	0.00132943	
.	.	.	.	0.000768167	0.000973207	0.00105680	0.00115381	0.00132970	
.	.	.	.	0.000775516	0.000973566	0.00105781	0.00115988	0.00133421	
.	.	.	.	0.000786611	0.000973700	0.00106120	0.00116242	0.00133465	
.	.	.	.	0.000795714	0.000973848	0.00106242	0.00116596	0.00133540	
.	.	.	.	0.000800406	0.000974390	0.00106571	0.00117630	0.00133681	
.	.	.	.	0.000802503	0.000974947	0.00106787	0.00118681	0.00133701	
.	.	.	.	0.000803416	0.000975073	0.00106844	0.00118721	0.00133858	
.	.	.	.	0.000804368	0.000975297	0.00106887	0.00118816	0.00134897	
.	.	.	.	0.000812454	0.000975442	0.00107129	0.00118868	0.00134979	
.	.	.	.	0.000814155	0.000975454	0.00107319	0.00118969	0.00135443	
.	.	.	.	0.000815411	0.000975845	0.00107462	0.00119437	0.00135579	
.	.	.	.	0.000816928	0.000975885	0.00107487	0.00120110	0.00135622	
.	.	.	.	0.000817411	0.000975967	0.00107622	0.00120220	0.00136672	
.	.	.	.	0.000819572	0.000976166	0.00107771	0.00120331	0.00137512	
.	.	.	.	0.000823407	0.000976463	0.00107917	0.00120375	0.00137711	
.	.	.	.	0.000828905	0.000976721	0.00107944	0.00120411	0.00138015	
.	.	.	.	0.000829215	0.000977040	0.00108905	0.00120555	0.00138112	
.	.	.	.	0.000833779	0.000977082	0.00108986	0.00120673	0.00138174	
.	.	.	.	0.000835374	0.000977346	0.00109124	0.00121242	0.00138805	
.	.	.	.	0.000842014	0.000977376	0.00109274	0.00121344	0.00138917	
.	.	.	.	0.000842114	0.000978072	0.00109536	0.00121689	0.00139217	
.	.	.	.	0.000845279	0.000978393	0.00109671	0.00121692	0.00140116	
.	.	.	.	0.000847014	0.000978859	0.00109884	0.00121698	0.00140745	
.	.	.	.	0.000848978	0.000978874	0.00110089	0.00121726	0.00140918	
.	.	.	.	0.000851712	0.000979222	0.00110184	0.00122386	0.00141005	
.	.	.	0.000550750	0.000854315	0.000979373	0.00110213	0.00122737	0.00141233	
.	.	.	0.000550942	0.000858743	0.000979557	0.00110297	0.00122765	0.00141361	
.	.	.	0.000570524	0.000860080	0.000979578	0.00110412	0.00123440	0.00142171	
.	.	.	0.000597177	0.000865154	0.000979582	0.00110620	0.00123553	0.00144544	
.	.	.	0.000611014	0.000870761	0.000979903	0.00110702	0.00123661	0.00145051	
.	.	.	0.000621607	0.000873321	0.00100477	0.00110751	0.00124019	0.00145700	
.	.	.	0.000622343	0.000879941	0.00100630	0.00111187	0.00124314	0.00148268	
.	.	.	0.000623665	0.000884750	0.00100751	0.00111270	0.00124351	0.00149678	
.	.	.	0.000625350	0.000885134	0.00100777	0.00111397	0.00124431	0.00150143	
.	.	.	0.000641867	0.000886177	0.00100846	0.00111467	0.00125307	0.00151031	
.	.	.	0.000641935	0.000886544	0.00101142	0.00111522	0.00125457	0.00151446	
.	.	.	0.000661826	0.000886824	0.00101276	0.00111695	0.00125469	0.00152491	
.	.	.	0.000666708	0.000886898	0.00101826	0.00111757	0.00125921	0.00155786	
.	.	.	0.000691901	0.000887128	0.00102107	0.00111758	0.00126065	0.00157120	
.	.	.	0.000696687	0.000895677	0.00102210	0.00112028	0.00126755	0.00160168	
.	.	.	0.000700632	0.000896144	0.00102741	0.00112214	0.00127274	0.00160746	
.	.	.	0.000706468	0.000898047	0.00102786	0.00112779	0.00127748	0.00161409	
.	.	.	0.000707584	0.000901163	0.00102928	0.00112829	0.00127843	0.00161807	
.	.	.	0.000708598	0.000907022	0.00103086	0.00112848	0.00127991	0.00166129	

number of unconverged estimates. The values which the maximum likelihood estimators for these samples would eventually attain if the Newton-Raphson procedure were continued through enough iterations might effect the means. It was expected that the means would show a bias which would tend toward zero for large sample sizes.

More samples from Population One than from Population Two yielded maximum likelihood estimates, except when using samples of size ten. The Newton-Raphson procedure is sensitive to the starting value used and this may contribute to the larger number of unconverged maximum likelihood estimates for larger sample sizes. Recall (from 2.4) that the means of the matching-moments-to-the-prior estimates drifted away from the true values as the sample size increased. This value was used as a starting value and that may serve to explain the decreased number of maximum likelihood estimates in the large sample sizes. The unweighted and time-weighted method-of-moments estimates were tried as starting values for samples of size ten from Population One. The resulting number of unconverged maximum likelihood estimates was roughly twice the number observed when the matching-moments-to-the-prior estimates were used. The number of converged maximum likelihood estimates may increase greatly for the larger sample sizes if another estimate is used as a starting value, thus regaining the lost information from the small sample size. This possibility was not checked.

It was noticed that when ordering the unweighted and time-weighted method-of-moments estimates that the samples which yielded extreme values often also had unconverged maximum likelihood estimates. A Chi-squared test of independence was run to check this tendency. The unweighted method-of-moments estimators were classified as negative or non-negative, and the maximum likelihood estimators were classified as converged or unconverged. The hypothesis:

H_o: $\hat{\alpha}_u$ and $\hat{\beta}_u$ (classified as above) are independent of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ (classified as above), vs

H_1: not independent

was rejected five of ten times in Population One, and seven of ten times in Population Two. These tests looked only at one end of the scale in which the unweighted method-of-moments estimators could obtain extreme values, but they seem to indicate a relationship between the "bad" estimates from the different types of estimators. Since only samples of size ten produced negative time-weighted method-of-moments estimates the Chi-squared test was not run using $\hat{\alpha}_T$ and $\hat{\beta}_T$ with $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$.

It was desired to find the economic feasibility of the maximum likelihood estimators compared with the two method-of-moments estimators. Five hundred samples of size ten from Population One were run once including the calculations of the maximum likelihood estimates and once excluding them. That showed the maximum likelihood estimators to be quite expensive in comparison. The program with the maximum likelihood estimates ran in 42.57 seconds while it ran in 5.00 seconds without the maximum likelihood estimates. Cost also increased sharply as the number of iterations of the Newton-Raphson procedure increased. Thus we see the cost of computing many maximum likelihood estimates is high, but need not deter the experimenter with only a few data sets from which he wants estimates of α and β.

The empirical distributions of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ were tested for normality. In both populations samples of size 50 or more and Population One sample size 40 showed the distribution $\hat{\beta}_{MLE}$ could be approximated by the normal distribution. Also the distributions of $\hat{\alpha}_{MLE}$ from sample size 80 from both populations were normal. It is well known that maximum likelihood estimators are asymptotically normally distributed under certain constraints.

The correlation between $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ was significantly negative and increased with the sample size in both populations. Figures 4.1.11 and 4.1.12 illustrate the correlation. The correlation between $\hat{\alpha}_{MLE}$ and $\hat{\alpha}_{MLE}$ was slightly higher than that of $\hat{\alpha}_T$ and $\hat{\beta}_T$. This was especially true in Population Two.

SECTION 5: DOCUMENTATION

The purpose of this program is to simulate the failure count from a Poisson process whose failure rate has a specific prior distribution. The prior distribution is chosen so that $X \sim$ Poisson(λT) and $\Lambda \sim$ gamma(α=2,β=.001). Recall from equation [1.3] that the marginal distribution can be written as

$$h(x|\alpha,\beta) = \binom{x+\alpha-1}{\alpha} \left(\frac{T'}{T+T'}\right)^{\alpha} \left(\frac{T}{T+T'}\right)^{x}$$

where $T' = \frac{1}{\beta}$.

Since α and β are fixed for the simulation we can generate an experiment time and simulate a failure count. These failure counts and times are used to compute the estimates.

The steps used to generate the experiment time are determined by which population is being generated. If we are interested in Population One then ICODE = 1. A random number U is generated from a uniform (0,1) distribution, multiplied by 10,000 and the integer part of the product is the experiment time IT(I). To generate Population Two we set ICODE = 2 and a second randomization is introduced. The random number U is generated. Instead of multiplying by 10,000 the multiplier is determined by another random number RM so that

$$\text{IT(I) equals the integer part of} \quad \begin{cases} U \times 1{,}000 \text{ if } RM < .3 \\ U \times 10{,}000 \text{ if } .3 \le RM \le .7 \\ U \times 100{,}000 \text{ if } .7 < RM \end{cases}$$

where RM $\sim$ uniform (0,1). If IT(I) equals zero it is ignored and another U is generated to begin the process.

Having the experiment time, the probability of an arrival in the negative binomial marginal distribution is

$$P = \frac{T'}{T+T'} = \frac{1000}{IT(I)+1000} .$$

We are ready to begin simulating the arrivals and failure counts. We count the number of failures until the α^{th} (in this case 2) arrival. The balance of needed arrivals is denoted IALBAL. Another random number R is generated from a uniform (0,1) distribution, and compared to P the probability of an arrival. If R is an arrival (i.e., $R \leq P$) then IALBAL is decreased by one. If R is a failure (i.e., $R > P$) then IX(I) the failure count is increased by one. When IALBAL equals zero IX(I) represents the simulated failure count in IT(I) hours. When experiments are simulated the failure counts and experiment times are used to estimate λ_i by $\hat{\lambda}_i = \frac{IX(I)}{IT(I)}$. This process is repeated N times for N = 10, 20, 30, ..., 100. These N values of λ_i are averaged according to the procedures described in 2.1 Unweighted Averaging and 2.2 Time-Weighted Averaging, and the unweighted method-of-moments estimates, the time-weighted method-of-moments estimates and the matching-moments-to-the-prior estimates are computed as specified in equations [2.1.1], [2.1.2], [2.2.1], [2.2.2], [2.4.1] and [2.4.2]. The Newton-Raphson procedure follows and then the four types of estimates of α and β are recorded. This completes one sample of size N. The entire process is repeated until 500 samples of size N have been generated.

The Newton-Raphson procedure is an iterative approximation to the root(s) of one or more simultaneous equations. The two dimensional case is of interest in this study. Solutions α^* and β^* to $f(\alpha,\beta) = 0$ and $g(\alpha,\beta) = 0$ are desired such that $f(\alpha^*,\beta^*) = g(\alpha^*,\beta^*) = 0$. Using (α_o,β_o) as an initial expansion point, the first order Taylor series approximation to $f(\alpha,\beta) = 0$ and $g(\alpha,\beta) = 0$ are

$$[5.1] \qquad f(\alpha,\beta) \doteq f(\alpha_o,\beta_o) + \left(\frac{\partial f}{\partial \alpha}\right)_o (\alpha-\alpha_o) + \left(\frac{\partial f}{\partial \beta}\right)_o (\beta-\beta_o) = 0$$

$$[5.2] \qquad g(\alpha,\beta) \sim g(\alpha_o,\beta_o) + \left(\frac{\partial g}{\partial \alpha}\right)_o (\alpha-\alpha_o) + \left(\frac{\partial g}{\partial \beta}\right)_o (\beta-\beta_o) = 0$$

For ease of notation we let

$$f^o = f(\alpha_o,\beta_o) \qquad g^o = g(\alpha_o,\beta_o)$$

$$f^o_\alpha = \left(\frac{\partial f}{\partial \alpha}\right)_o \qquad g^o_\beta = g\left(\frac{\partial g}{\partial \alpha}\right)_o$$

$$f^o_\alpha = \left(\frac{\partial f}{\partial \beta}\right)_o \qquad g^o_\beta = g\left(\frac{\partial g}{\partial \beta}\right)_o$$

In matrix notation equations [5.1] and [5.2] can be written as

$$[5.3] \qquad \begin{bmatrix} f^o \\ g^o \end{bmatrix} + \begin{bmatrix} f^o_\alpha & f^o_\beta \\ g^o_\alpha & g^o_\beta \end{bmatrix} \begin{bmatrix} \alpha-\alpha_o \\ \beta-\beta_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for (α,β)

$$[5.4] \qquad \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_o \\ \beta_o \end{bmatrix} - \begin{bmatrix} f^o_\alpha & f^o_\beta \\ g^o_\alpha & g^o_\beta \end{bmatrix}^{-1} \begin{bmatrix} f^o \\ g^o \end{bmatrix}$$

where α_1 and β_1 are improved estimates of α^* and β^*. This procedure is continued interactively replacing (α_i,β_i) with $(\alpha_{i+1},\beta_{i+1})$ until $|f(\alpha_{i+1},\beta_{i+1})| < \varepsilon$ and $|g(\alpha_{i+1},\beta_{i+1})| < \varepsilon$ for a sufficiently small ε.

The likelihood equations from equations [2.3.3] and [2.3.4] are

$$f = \sum_{i=1}^{n} \sum_{z=1}^{x_i} \frac{1}{\alpha+z-1} - \sum_{i=1}^{n} \ln(1+\beta T_i) = \text{SUMF1} - \text{SUMF2}$$

$$g = \frac{\sum_{i=1}^{n} x_i}{\beta} = \sum_{i=1}^{n} \frac{(x_i+\alpha)T_i}{1+\beta T_i} = \text{SUMG1} - \text{SUMG2}$$

The derivatives of the likelihood equations are

$$f_\alpha^o = \sum_{i=1}^{n} \sum_{z=1}^{x_i} \frac{-1}{(\alpha+z-1)^2} = AA$$

$$g_\beta^o = \frac{\sum_{i=1}^{n} x}{\beta^2} + \sum_{i=1}^{n} \frac{(x_i+\alpha)T_i^{\,2}}{(1+\beta T_i)^2} = CC$$

and

$$f_\beta^o = g_\alpha^o = \sum_{i=1}^{n} \frac{-T_i}{(1+\beta T_i)} = BB$$

Then equation [5.4] becomes

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_o \\ \beta_o \end{bmatrix} - \frac{1}{(AA)(CC)-(BB)^2} \begin{bmatrix} CC & -BB \\ -BB & AA \end{bmatrix} \begin{bmatrix} f^o \\ g^o \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_o \\ \beta_o \end{bmatrix} - \frac{1}{DET} \begin{bmatrix} CCf^o-BBg^o \\ -BBf^o+AAg^o \end{bmatrix}$$

Hence

$$\alpha_1 = \alpha_o - \frac{(CCf^o-BBg^o)}{DET} = \alpha_o - Y1$$

$$\beta_1 = \beta_o - \frac{(AAg^o-BBf^o)}{DET} = \beta_o - Y2$$

The initial expansion point is taken as the matching-moments-to-the-prior estimate. The program tests for negative values using

```
IF(AL1.LE.0.)GO TO 715
IF(BE1-0.)715,715,725
```

If α_1 or β_1 is not a positive value then the program in step 715 reduces the step size by one-half, recomputes α_1 and β_1 and checks again for negative values. For positive values of α_1 and β_1, $f(\alpha_1,\beta_1)$ and $g(\alpha_1,\beta_1)$ are computed and checked for convergence. If $|f(\alpha_1,\beta_1)|$ or $|g(\alpha_1,\beta_1)|$ is not less than ε, (EPSL = .01), and the maximum number of iterations MAITER=20 has not been reached then another iteration is carried out by returning to step 250 and using (α_1,β_1) as the new expansion point. If $|f(\alpha_1,\beta_1)| < \varepsilon$ and $|g(\alpha_1,\beta_1)| < \varepsilon$ or if twenty iterations have been completed then all estimates are recorded and another sample is generated. If MAITER is reached, we set AL=999.0 and BE=999.0 which is interpreted as a sample with unconverged maximum likelihood estimates.

Notation used in the program

AA $= f_\alpha^o$

AC $= \hat{\alpha}_{mmp}$

AL $= \hat{\alpha}_{MLE}$

AL1 = α trial value after iteration step

AUN $= \hat{\alpha}_u$

ATW $= \hat{\alpha}_T$

BB $= f_\beta^o = g_\alpha^o$

BC $= \hat{\beta}_{mmp}$

BE $= \hat{\beta}_{MLE}$

BE1 = β trial value after iteration step

BET1 $= 1 + \beta T_i$

BUN $= \hat{\beta}_u$

BTW $= \hat{\beta}_T$

CC $= g_\beta^o$

DET = $\det\begin{bmatrix} AA & BB \\ BB & CC \end{bmatrix} = (AA)(CC)-(BB)^2$

EPSL = $\varepsilon = .01$

F = $\frac{\partial L}{\partial \alpha}$

G = $\frac{\partial L}{\partial \beta}$

H = $\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T_i}$ = inverse harmonic mean of experiment times

IA = $\alpha = 2$

IALBAL = number of arrivals still needed in an experiment

ICODE = designation of population

ISUMT = $\sum_{i=1}^{n} T_i$

ISUMX = $\sum_{i=1}^{n} x_i$

IT(I) = T_i one experiment time

IX(I) = x_i failure count for one experiment

LAMBDA = $\hat{\lambda}_i$

M = 500 = total samples of each sample size

MAITER = 20 = maximum iterations for Newton-Raphson procedure

N = sample size 10, 20, 30, ..., 100

P = probability of an arrival $\frac{T'}{T+T'}$

R = random event in negative binomial, either an arrival ($R \leq P$) or a failure ($R > P$).

RM = random number determining multipliers in Population Two

SEED = seed value for "super duper" random number generator

SLL = $\sum_{i=1}^{n} \hat{\lambda}_i^2$

STL = $\sum_{i=1}^{n} T_i \hat{\lambda}_i$

$$\text{STLL} = \sum_{i=1}^{n} T_i \hat{\lambda}_i^2$$

$$\text{STT} = \sum_{i=1}^{n} T_i^2$$

$$\text{SUMC2} = \sum_{i=1}^{n} \frac{(x_i+\alpha)T_i^2}{(1+\beta T_i)^2}$$

$$\text{SUMF1} = \sum_{i=1}^{n} \sum_{z=1}^{x_i} \frac{1}{(\alpha+z-1)}$$

$$\text{SUMF2} = \sum_{i=1}^{n} \ln(1+\beta T_i)$$

$$\text{SUMG1} = \frac{\sum_{i=1}^{n} x_i}{\beta}$$

$$\text{SUMG2} = \sum_{i=1}^{n} \frac{(x_i+\alpha)T_i}{(1+\beta T_i)}$$

$$\text{SUMLAM} = \sum_{i=1}^{n} \hat{\lambda}_i$$

$$\text{TPRIME} = T' = \frac{1}{\beta}$$

U = random number used in experiment time

$$\text{UBARL} = \bar{\hat{\lambda}}_u$$

$$\text{UVARL} = S^2_{\hat{\lambda}_u}$$

$$\text{WBARL} = \bar{\hat{\lambda}}_T$$

$$\text{WVARL} = S^2_{\hat{\lambda}_T}$$

$$\text{Y1} = \frac{CCf^o - BBg^o}{DET} = \text{increment (step size)}$$

$$\text{Y2} = \frac{AAg^o - BBf^o}{DET} = \text{increment (step size)}$$

COMPUTER PROGRAM

```
      DIMENSION IX(100),IT(100)
      REAL*8 LAMBDA,SUMLAM,UBARL,SLL,UVARL,WBARL,P,TLL,TL,STLL,STL,WVARL
     C,AUN,AWEI,BUN,BWEI,TBAR,H,AC,BC,AL,BE,SUMF1,SUMF2,SUMG1,SUMG2,
     CAA,BB,CC,SUMC2,BET1,DET,F,G,Y1,Y2,AL1,EPSL
C
C         500 SAMPLES
      M=500
C
C         MAXIMUM ITERATIONS FOR NEWTON-RAPHSON PROCEDURE
      MAITER=20
C
C         EPSILON FOR N-R
      EPSL=.01D0
C
C         ALPHA
      IA=2
C
C         SEED VALUE FOR "SUPER-DUPER" RANDOM NUMBER GENERATOR
      SEED=0
C
C         T'=1/BETA
      TPRIME=1000.D0
C
C         POPULATION ONE OR TWO
      ICODE=1
C
C         SAMPLE SIZE (10, 20, 30, ...., 100)
      N=10
C
C         SET ALPHA BALANCE = ALPHA
      IALBAL=IA
C
C         BEGIN 500 SAMPLES
      DO 999 K=1,M
C
C         INITIALIZE SUMS
      ISUMX=0
      SLL=0
      ISUMT=0
      STT=0
      SUMLAM=0
      STLL=0
      STL=0
      H=0
C
C         GENERATE N EXPERIMENTS
      DO 998 I=1,N
C
C         INITIALIZE FAILURE COUNT FOR THE EXPERIMENT
      IX(I)=0
C
C
C
C     GENERATE EXPERIMENT TIME
C
C
C         RANDOM NUMBER FOR EXPERIMENT TIME
    7 U=UNI(SEED)
C
C         IF TIME = 0 GET ANOTHER TIME
      IF(U.LT..00099999)GO TO 7
C
C         IF POPULATION ONE, TIME = U*10000
      IF(ICODE.EQ.1)GO TO 5
C
C         IF POPULATION TWO, DO SECOND RANDOMIZATION
C
C                            1000 IF RM < .3
C         TIME = U *        10000 IF .3 <= RM <= .7
C                          100000 IF .7 < RM
C
C         WHERE RM IS A RANDOM NUMBER BETWEEN 0 AND 1
C
```

```
      RM=UNI(SEED+2)
      IF(RM.LT..3)GO TO 3
      IF(RM.GT..7)GO TO 4
      GO TO 5
    3 IT(I)=INT(U*1000)
      GO TO 6
    4 IT(I)=INT(U*100000)
      GO TO 6
    5 IT(I)=INT(U*10000)
C
C         PROBABILITY OF SUCCESS IN THE NEGATIVE BINOMIAL DISTRIBUTION IS
C                         P = T'/(T + T')
    6 P=TPRIME/(IT(I)+TPRIME)
C
C         INVERSE HARMONIC MEAN OF TIMES IS (1/N)*(SUM(1/T))
      H=H+(1./(N*DFLOAT(IT(I))))
C
C
C    ONE EXPERIMENT
C
C
C         GENERATE A RANDOM NUMBER, CHECK IT AGAINST P.  IF IT IS A SUCCESS,
C              (LESS THAN P) THEN REDUCE ALPHA BALANCE BY ONE.  IF IT IS A
C               FAILURE, INCREASE THE FAILURE COUNT BY BY ONE.
C               REPEAT UNTIL ALPHA SUCCESSES
C
  100 R=UNI(SEED+1)
      IF (R.LE.P) GO TO 500
      IX(I)=IX(I)+1
      GO TO 100
  500 IALBAL=IALBAL-1
      IF (IALBAL.EQ.0) GO TO 800
      GO TO 100
C
C
C         CALCULATE LAMBDA HAT FOR THE EXPERIMENT
  800 LAMBDA=DFLOAT(IX(I))/DFLOAT(IT(I))
C
C         VALUES INVOLVED IN CALCULATING ESTIMATES OF ALPHA AND BETA
      ISUMX=ISUMX+IX(I)
      ISUMT=ISUMT+IT(I)
      SUMLAM=SUMLAM+LAMBDA
      SLL=SLL+LAMBDA**2
      STLL=STLL+DFLOAT(IT(I))*LAMBDA**2
      STL=STL+DFLOAT(IT(I))*LAMBDA
      STT=STT+IT(I)*IT(I)
C
C         REINITIALIZE ALPHA BALANCE
      IALBAL=IA
C
C         BACK TO NEXT EXPERIMENT
  998 CONTINUE
C
C
C
C
C         UNWEIGHTED AND TIME WEIGHTED AVERAGES
C
      UBARL=SUMLAM/N
      TBAR=DFLOAT(ISUMT)/DFLOAT(N)
      UVARL=(SLL-(SUMLAM**2)/N)/(N-1.)
      WBARL=DFLOAT(ISUMX)/DFLOAT(ISUMT)
      WVARL=(STLL-2.*WBARL*STL+WBARL**2*ISUMT)/DFLOAT(ISUMT)
C
C
C
C         UNWEIGHTED METHOD OF MOMENTS
   13 AUN=UBARL**2/(UVARL-(UBARL*H))
      BUN=(UVARL/UBARL)-H
C
C
C         TIME WEIGHTED METHOD OF MOMENTS
      ATW=(WBARL**2)*C/(WVARL-(WBARL*DFLOAT(N-1)/DFLOAT(ISUMT)))
      BTW=((WVARL/WBARL)-DFLOAT(N-1)/DFLOAT(ISUMT))/C
C
C
```

```
C          MATCHING MOMENTS WITH THE PRIOR
      AC=(UBARL**2)/UVARL
      BC=UVARL/UBARL
C
C
C
C   MAXIMUM LIKELIHOOD ESTIMATION USING NEWTON-RAPHSON PROCEDURE
C
C
C          STARTING VALUE FOR N-R
      AL=AC
      BE=BC
C
C          INITIALIZE ITERATION
      NITER=0
C
C          INCREMENT ITERATION
  250 NITER=NITER+1
C
C          INITIALIZE SUMS
      SUMF1=0
      AA=0
      SUMF2=0
      SUMG2=0
      BB=0
      SUMC2=0
C
C
C          CALCULATE THE LIKELIHOOD FUNCTIONS, F AND G
C
      DO 350 I=1,N
      BET1=1.+BE*IT(I)
      J=IX(I)
      IF(J.EQ.0)GO TO 450
      DO 550 L=1,J
      SUMF1=SUMF1+1./(AL+L-1.)
  550 AA=AA-1./(AL+L-1.)**2
  450 SUMG2=SUMG2+(IX(I)+AL)*IT(I)/BET1
      SUMF2=SUMF2+DLOG(BET1)
      BB=BB-IT(I)/BET1
      SUMC2=SUMC2+(IX(I)+AL)*((DFLOAT(IT(I))/BET1)**2)
  350 CONTINUE
      SUMG1=DFLOAT(ISUMX)/BE
      CC=DFLOAT(-ISUMX)/BE**2+SUMC2
      DET=AA*CC-BB**2
      F=SUMF1-SUMF2
      G=SUMG1-SUMG2
C
C
C          CHECK THE NUMBER OF ITERATIONS.  IF BEYOND THE LIMIT SET MLE'S = 999.0
C               AND RECORD VALUES FOR ALL TYPES OF ESTIMATORS
      IF(NITER.LE.MAITER)GO TO 650
      AL=999.D0
      BE=999.D0
      GO TO 751
C
C          CHECK CONVERGENCE
C           IF CONVERGED RECORD ALL TYPES OF ESTS
  650 IF(DABS(G)-EPSL)651,651,705
  651 IF(DABS(F).LE.EPSL)GO TO 751
C
C          CALCULATE STEP SIZE
  705 Y1=(1./DET)*(CC*F-BB*G)
      Y2=(1./DET)*(BB*F-AA*G)
C
C          NEW ESTIMATES OF ALPHA AND BETA
  710 AL1=AL-Y1
      BE1=BE+Y2
C
C
```

```
C
C         CHECK FOR NEGATIVE VALUES.  IF NEGATIVE REDUCE STEP SIZE AND CALCULATE
C              NEW ESTIMATES OF ALPHA AND BETA
      IF(AL1.LE.0.)GO TO 715
      IF(BE1-0.)715,715,725
  715 Y1=0.5*Y1
      Y2=0.5*Y2
      GO TO 710
C
C
C         NEW STARTING VALUE FOR NEXT ITERATION
  725 AL=AL1
      BE=BE1
C
C         RETURN FOR NEXT ITERATION
      GO TO 250
  751 WRITE(8)K,N,ICODE,AUN,AWEI,AC,AL,BUN,BWEI,BC,BE
C
C         NEXT SAMPLE OF SIZE N
  999 CONTINUE
      END
/*
```

REFERENCES

1. Grosh, D. L., "The Distribution of Method-of-Moments Estimators for Beta Parameters Based on a Simulation Study." Unpublished manuscript.

2. Grosh, D. L., "Marginal Distribution Estimators of the Gamma Prior Parameters for a Group of Poisson Processes." Unpublished manuscript.

COMPARISON OF ESTIMATORS FOR THE PARAMETERS OF THE

POISSON-GAMMA MARGINAL DISTRIBUTION

by

BARBARA SUE KERR

B.S., University of Southern Colorado, 1977

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1980

Work began on this study when the Nuclear Regulatory Commission provided data consisting of failure counts and experiment times for pumps. This data could be considered to come from a Poisson process with failure rate λ where $\Lambda \sim \text{gamma}(\alpha,\beta)$. The resulting marginal distribution is (for integer α) the negative binomial

$$h(x|\alpha,\beta) = \binom{\alpha+x-1}{\alpha} \left(\frac{T'}{T+T'}\right)^{\alpha}\left(\frac{T'}{T+T'}\right)^{x} I_{\{0,1,2,,,\}}(x)$$

where $T' = \frac{1}{\beta}$.

Estimates of α and β were desired but no observations from the prior distribution were available since the λ_i were never observed, only estimated by $\hat{\lambda}_i = \frac{X_i}{T_i}$. The values which were observable were failure counts and experiment times, which were from the marginal distribution.

Four types of estimators of α and β, namely the unweighted method-of-moments estimators, the time-weighted method-of-moments estimators, the maximum likelihood estimators and the matching-moments-to-the-prior estimators were derived and compared. Lacking a large body of data, a simulation was done as a basis of comparison. Original plans included several values of α and β, however, cost of computing proved prohibitive, and all simulation was done with $\alpha = 2$ and $\beta = .001$.

The estimators were compared by histograms, means and bias, variance among the estimates and by correlation between $\hat{\alpha}$ and $\hat{\beta}$ of each type. The empirical distribution of every estimator was tested for normality, with only the maximum likelihood estimates fitting the normal distribution for sufficiently large sample sizes. Summary statistics for every estimator were also included.