SUMMABILITY PROCEDURES APPLIED TO FOURIER SERIES

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Introduction

In this report there is a discussion of summability of series in general, followed by an introduction to summability of Fourier Series. In that the definition of "sum" of an infinite series is merely that - a definition - since it is physically impossible to add up an infinite number of terms, other more general definitions of a "sum" can hardly be faulted.

In addition, in many of the applications of series, it is not necessary that the series be convergent. Often merely being summable in some sense is sufficient and, indeed, some workers have used strictly divergent series with a fair degree of success.

It is necessary to begin by citing a few fundamental definitions and theorems. Not all theorems are proved and many important concepts and theorems have to be omitted due to lack of space.

In dealing with many of the concepts and ideas, particular cases are used. Thus, when $|a_n - L|$ is referred to, one usually thinks of real (or complex) numbers. However, in general, the definition or theorem usually is valid in other instances, where $|a_n - L|$ refers to the "distance between" two elements a_n and L. In the next few paragraphs, the elements are numbers, in the real (or complex) domain.

A sequence is defined as a function whose domain is the set of non-negative integers, or portion thereof. If the function is denoted by f, its value at n is given by f(n). The sequence itself is the set $\{(n, f(n)): n = 0, 1, 2, \ldots\}$, the set of all pairs (n, f(n)), with n a non-negative integer. Since the domain is usually the same, it is customary to shorten the notation and just write $\{f(n)\}$ instead of $\{(n, f(n))\}$. Thus the sequence $\{(n, \frac{1}{n}):n\ 1,\ 2,\ \ldots\}$ would be written simply as $\frac{1}{n}$. Also, $\{a_n\}$ is used to denote the sequence $\{(n, a_n)\}$.

A sequence $\{a_n\}$ may have different values a_n for different values of n. Suppose that as n increases the different a_n's tend to cluster around some fixed number L. If there is a number L such that $|a_n - L|$ can be made arbitrarily small for all sufficiently large n, the sequence $\{a_n\}$ is said to converge to L. Quantitatively, if, given $\epsilon > 0$, there is an N such that, for all $n > N$, $|a_n - L| < \epsilon$, then $\{a_n\}$ converges to L. This can be written as $\lim_{n\to\infty} a_n = L$. If no such limit exists, the sequence is said to diverge.

A very important way of creating a sequence $\{s_n\}$ is by addition. Suppose the elements to be added are $u_1, u_2, \ldots, u_n, \ldots$. Let

$$
\begin{aligned}
s_1 &= u_1, \\
s_2 &= u_1 + u_2, \\
&\ \vdots \\
s_n &= u_1 + u_2 + \ldots + u_n = \sum_{k=1}^{n} u_k, \\
&\ \vdots
\end{aligned}
\tag{1}
$$

Consider $\sum_{k=1}^{\infty} u_k$, that is consider the sum s_n as n increases without bound. The limit of s_n, as n increases without bound, is called an infinite series.

The sequence $\{s_n\}$ given by (1) is called the sequence of partial sums of the series. The n^{th} partial sum is s_n. If $\{s_n\}$ converges to a limit S as n increases without bound, $\lim_{n\to\infty} s_n = S$, then "the series converges to its sum S". This is written as

$$\sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} \left(\sum_{k=1}^{n} u_k \right) = S.$$

Theorem 1: A necessary condition for the convergence of $\sum_{k=1}^{\infty} u_k$ is that $\lim_{k\to\infty} u_k = 0$. A series is said to be divergent if its sequence of partial sums is divergent.

Although a series may be divergent, it may still be useful. Much can be done with a divergent series if it is handled by certain methods called summability procedures. A summability process is defined as a method of assigning a "sum" to a series. This report will be restricted to those methods for which the "sum" of a convergent series is the same as the sum in the ordinary sense of convergence. Methods with this property are called "regular". In order to preserve the analogy between convergence and summability, a summability process should satisfy the following conditions:

I. If $\sum_{k=0}^{\infty} u_k = S$, then $\sum_{k=1}^{\infty} u_k = S - u_0$, and conversely.

II. If $\sum_{k=0}^{\infty} u_k = S$, $\sum_{k=0}^{\infty} v_k = T$, then $\sum_{k=0}^{\infty} (u_k + v_k) = S + T$.

III. If $\sum_{k=0}^{\infty} u_k = S$, then $\sum_{k=0}^{\infty} \alpha u_k = \alpha S$, where α is any constant.

In addition,

IV. The process must be regular.

Cesàro Summability

Consider the following series,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots , \tag{1}$$

and let

$$\left.\begin{aligned} s_n &= a_0 + a_1 + \cdots + a_n \\ \sigma_n &= \frac{s_o + s_1 + \cdots + s_n}{n+1} \end{aligned}\right\} \quad (n = 0, 1, 2, \ldots)$$

and

If $\lim_{n\to\infty} \sigma_n = A$ we say that the series given by (1) is Cesàro summable to A or summable (C, 1) to A. (Cesàro summability (C, α) can be defined for general α, $\alpha > -1$). As an example consider the series

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots . \tag{2}$$

This series diverges, but $s_o = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$, ... so that

$$\sigma_n = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{2} + (2n+2)^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Hence the $\lim_{n\to\infty} \sigma_n = \frac{1}{2}$ and the series given by (2) is Cesàro summable to $\frac{1}{2}$.

Theorem 1: Cesàro summability is a regular method.

Proof: Suppose that a given series is convergent, with sum A and, therefore, $\lim_{n\to\infty} s_n = A$. Then for any $\varepsilon > 0$, there exists a number m such that $|s_n - A| < \frac{\varepsilon}{2}$ whenever $n \geq m$. Now consider

$$\sigma_n - A = \frac{s_o + s_1 + \cdots + s_n - (n+1)A}{n+1} = \frac{1}{n+1} \sum_{i=0}^{n} (s_i - A).$$

For $n > m$,

$$\sigma_n - A = \frac{1}{n+1} \sum_{i=0}^{m-1} (s_i - A) + \frac{1}{n+1} \sum_{i=m}^{n} (s_i - A)$$

and hence

$$|\sigma_n - A| \leq \frac{1}{n+1} \sum_{i=0}^{m-1} |s_i - A| + \frac{1}{n+1} \sum_{i=m}^{n} |s_i - A|. \tag{3}$$

Since m is a fixed number

$$\frac{1}{n+1} \sum_{i=0}^{m-1} |s_i - A| < \frac{\varepsilon}{2} \tag{4}$$

for all sufficiently large n, $n > M$, say. But since $|s_n - A| < \frac{\varepsilon}{2}$ for $n \geq m$

$$\frac{1}{n+1} \sum_{i=m}^{n} (s_i - A| < \frac{n - m + 1}{n+1} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} \quad . \tag{5}$$

Combining equations (3), (4), and (5), $|\sigma_n - A| < \varepsilon$ provided $n > m + M$, which proves the theorem.

Summability (C, 1) can be generalized to summability (C, k) as follows: Given a sequence $\{a_n\} = a_0, a_1, a_2, \ldots, a_n \ldots$, let

$$s_n^{(0)} = a_0 + a_1 + \ldots + a_n,$$

$$s_n^{(1)} = s_0^{(0)} + s_1^{(0)} + \ldots + s_n^{(0)},$$

and, in general, for any given positive integer k,

$$s_n^{(k)} = s_0^{(k-1)} + s_1^{(k-1)} + \dots + s_n^{(k-1)} \ . \tag{6}$$

The sums $s_n^{(k)}$ are linearly expressible in terms of the sums $s_n^{(0)}$. The number of these sums which occur is $\binom{n+k}{k}$. This clearly holds if $k = 0$, and by induction it appears that the right side of (6) contains

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

terms of $s_n^{(0)}$. ([5], page 90) We now state the following definition:

Definition: If $C_n^{(k)} = S_n^{(k)} / \binom{n+k}{k}$, for $n = 0, 1, 2, \dots$, we say $C_n^{(k)}$ is the Cesàro mean of order k of the first n + 1 terms of the sequence $s_0^{(0)}, s_1^{(0)}, \dots, s_n^{(0)}, \dots$ and if $\lim_{n\to\infty} C_n^{(k)} = A$ (finite) for some k, then we say that $\sum_{n=0}^{\infty} a_n$ is Cesàro summable of order k to the sum A, or simply summable (C, k) to A.

A remark concerning notation should be made here. By examining the definitions of (C, 1) summability and (C, k) summability it is seen that $s_n = s_n^{(0)}$ for all n.

As an example of (C, k) summability, consider the following series:

$$1 = 2 + 3 - 4 + 5 - 6 + 7 - \dots .$$

Here $s_0 = 1$, $s_1 = -1$, $s_2 = 2$, $s_3 = -2$, ...

$$s_n = \begin{cases} n/2 + 1 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

and $\lim_{n\to\infty} \sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1}$ does not exist. However putting k = 2 in the definition of (C, k) summability gives

$$s_n^{(2)} = s_0^{(1)} + s_1^{(1)} + s_2^{(1)} + \ldots + s_n^{(1)}$$

(7) or

$$s_n^{(2)} = (n+1)\, s_0 + ns_1 + (n-1)\, s_2 + \ldots + 2s_{n-1} + s_n.$$

Now

$$(n+1)\, s_0 + ns_1 = (n+1) + n(-1) = 1$$

$$(n-1)s_2 + (n-2)s_3 = (n-1)2 + (n-2)(-2) = 2$$

etc.

If n is even, say n = 2r, the sum contains an odd number of terms, the last of which is

$$s_n = n/2 + 1 = r + 1,$$

and the next to last two give

$$3s_{n-2} + 2s_{n-1} = 3(n/2) + 2(-n/2) = r.$$

Hence equation (7) now reduces to

$$s_n^{(2)} = 1 + 2 + \ldots + r + (r+1) = \frac{1}{2}(r+1)(r+2). \tag{8}$$

Similarly, if n is odd, say n = 2r + 1, the sum contains an even number of terms ending in

$$2s_{n-1} + s_n = 2((n-1)/2) + 1 - (n+1)/2$$
$$= (n+1)/2 = r + 1$$

Hence relation (8) holds for both even and odd values of n.

$$\binom{n+2}{2} = \frac{1}{2}(n+1)(n+2) = \begin{cases} (r+1)(2r+1) & \text{for n even} \\ \\ (r+1)(2r+3) & \text{for n odd.} \end{cases}$$

Therefore for n even or odd,

$$\lim_{n\to\infty} C_n^{(2)} = \lim_{n\to\infty} \frac{s_n^{(2)}}{\binom{n+2}{2}} = \lim_{n\to\infty} \frac{(r+1)(r+2)}{2(r+1)(2r+1 \text{ or } 3)} = \frac{1}{4}$$

and the series $1 - 2 + 3 - 4 + 5 - 6 + \ldots$ is a doubly indeterminate series summable (C, 2) to the value $\frac{1}{4}$.

Cesàro did not make the first generalization concerning summation by arithmetic means. The first and most obvious generalization was made by Hölder, who defined a sequence of methods called the (H, k) methods. The general (H, k) method is defined as follows:

$$H_n^{(r+1)} = \frac{H_0^{(r)} + H_1^{(r)} + \ldots + H_n^{(r)}}{r+1}, \quad H_n^{(0)} = s_n,$$

where $\sum u_n$ is said to be (H, k) summable to S if $\lim_{n\to\infty} H_n^{(k)} = S$. The (H, 0) method is simply ordinary convergence. The (H, 1) method is the same as the (C, 1) method. Since the (H, k) sum is the (C, 1) sum of (H, k-1), (H, q) summability will always imply (H, q+p) summability for any integer $p \geq 0$. In fact, the (H, k) and (C, k) methods are equivalent for any integer $k > -1$. (For proof see [2], page 103).

The general summability (C, α), for α any real number > -1, can be defined in a fashion similar to that for (C, k), for k a non-negative integer. For such a definition the reader is referred to Orthogonal Functions by Sansone, p. 94. Incidentally, the series $\sum_{n=0}^{\infty}(-1)^n$ is (C, α) summable to $\frac{1}{2}$ for any $\alpha > 0$.

Abel Summability

Consider the two series,

(1) $$\sum_{n=0}^{\infty} a_n$$

(2) and $$\sum_{n=0}^{\infty} a_n r^n .$$

Assume the series given by (2) converges for $0 < r < 1$. (This will always be the case if the terms given in the series (1) are bounded). Let $\sigma(r)$ be the sum of the series (2) and let $\lim_{r \to 1} \sigma(r) = \sigma$. If this is the case, the series (1) is said to be summable by Abel's method to the value σ, or Abel summable to σ.

Abel's method can be used to sum a divergent series. Consider, for example, the series $\sum_{n=0}^{\infty} (-1)^n$, already studied. This series is also summable by Abel's method to the value $\frac{1}{2}$. In this case,

$$\sigma(r) = 1 - r + r^2 - r^3 + \ldots = \frac{1}{1 + r} , \quad \text{for } |r| < 1,$$

and therefore $\lim_{r \to 1} \sigma(r) = \frac{1}{2}$.

The following lemma will be needed to prove that Abel's method is regular.

Lemma 1: Let $\sum_{n=0}^{\infty} a_n$ be a convergent series (with real or complex terms).

Then the series $\sum_{n=0}^{\infty} a_n r^n$ converges for $0 \leq r \leq 1$, and its sum $\sigma(r)$ is continuous on the interval $[0, 1]$. The proof can be found in [8], page 108.

Theorem 1: Abel's method of summation is regular.

Proof: If the series (1) converges to σ, then the above lemma implies that the series (2) converges and its sum $\sigma(r)$ is continuous on the closed interval $0 \le r \le 1$. This means that

$$\lim_{r\to 1} \sigma(r) = \sigma(1) = \sigma,$$

which proves the theorem.

Theorem 2: If $\sum_{n=0}^{\infty} a_n$ is (C, 1) summable to the value σ, then $\varphi(r) = \sum_{n=0}^{\infty} a_n r^n$ exists for $0 \le r < 1$ and the series $\sum_{n=0}^{\infty} a_n$ is summable by Abel's method to σ. (i.e., A series which is (C, 1) summable is Abel summable to the same value).

Proof: Let $\sum_{n=0}^{\infty} a_n$ be summable to the value S by the method of Cesàro, (C, 1), and let

$$\varphi(r) = \sum_{n=0}^{\infty} a_n r^n \qquad (0 \le r < 1).$$

Then

$$\frac{\varphi(r)}{1 - r} = \left(\sum_{n=0}^{\infty} a_n r^n\right)\left(\sum_{n=0}^{\infty} r^n\right) = \sum_{n=0}^{\infty} s_n r^n.$$

$$\frac{\varphi(r)}{(1 - r)^2} = \sum_{n=0}^{\infty} (n + 1)\, \sigma_n r^n$$

(3) $$\varphi(r) = (1 - r)^2 \sum_{n=0}^{\infty} (n + 1)\, \sigma_n r^n$$

As a special case of (3), take $a_0 = 1$, $a_n = 0$ if $n > 0$, then

$$1 = (1 - r)^2 \sum_{n=0}^{\infty} (n + 1) r^n. \tag{4}$$

Therefore,

$$\varphi(r) - S = (1 - r)^2 \sum_{n=0}^{\infty} (n + 1)(\sigma_n - S) r^n$$

Let $\varepsilon > 0$ be given. Then there exists an integer N such that if $n \geq N$, $|\sigma_n - S| < \varepsilon$. Therefore,

$$|\varphi(r) - S| \leq |1 - r|^2 \sum_{n=0}^{N} (n + 1) |\sigma_n - S| r^n + |1 - r|^2 \varepsilon \sum_{n=N+1}^{\infty} (n + 1) r^n.$$

Hence, by (4),

$$|\varphi(r) - S| \leq |1 - r|^2 \sum_{n=0}^{N} (n + 1) |\sigma_n - S| r^n + \varepsilon$$

Now, taking the limit,

$$\lim_{r \to 1} |\varphi(r) - S| < \lim_{r \to 1} \left[|1 - r|^2 \sum_{n=0}^{N} (n + 1) |\sigma_n - S| r^n + \varepsilon \right] = \varepsilon$$

Hence,

$$\lim_{r \to 1} \varphi(r) = S.$$

From this it follows that a series which is summable by Cesàro's (C, 1) method is summable to the same value by Abel's method. The converse of this theorem is not true as can be seen from the following example. Consider the series $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$. This series is not summable by Cesàro's (C, 1) method since $\lim_{n\to\infty} \sigma_n$ does not exist. But $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$ is summable by Abel's method:

$$\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)r^n = \frac{d}{dr}\sum_{n=0}^{\infty}(-1)^n r^n$$

$$= \frac{d}{dr}\left(\frac{1}{1+r}\right) = \frac{-1}{(1+r)^2} = -\frac{1}{4} .$$

Thus this series is summable by Abel's method but not by Cesàro's method.

Fourier Series

Before continuing, some facts concerning Fourier series need to be established. A function f(x) is said to be periodic of period 2π, if it is defined for all real x and if $f(x + 2\pi) = f(x)$ for all x. The transition from functions having period 2π to functions having any period T ($T > 0$) can be effected by a change of scale. In fact, suppose that f(t) has period T. Then a new variable x can be introduced such that f(t), as a function of x, has period 2π. If

(1a) $$t = \frac{Tx}{2\pi}$$

so that

(1b) $$x = \frac{2\pi t}{T},$$

then $x + 2\pi$ corresponds to $t + T$, which means that f, as a function of x, has period 2π.

A trigonometric polynomial of order n can be defined as

(2) $$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos \frac{\pi k x}{t} + b_k \sin \frac{\pi k x}{t})$$

where a_0 is a constant. It follows that $s_n(x)$ is periodic of period $T = 2t$.

Now consider a function f(x) that is periodic with period 2π that has the expansion

(3) $$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where the sign $\sim$ is read "is represented by". The sign $\sim$ is used when the series is formed without knowing in advance whether it converges to the function f(x). The sign $\sim$ can be replaced by the sign = only if the series actually does converge and its sum is f(x). The coefficients a_0, a_k, and b_k need to be determined for k = 1, 2, 3, ... It is assumed that the series (3) and the series to be written soon, can be integrated term-by-term. This is permitted in the case of uniform convergence.

It is also assumed that the function f(x) is integrable. Thus, integrating both sides of (3) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\, dx \sim \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{k=1}^{\infty} (a_k \int_{-\pi}^{\pi} \cos kx\, dx + b_k \int_{-\pi}^{\pi} \sin kx\, dx).$$

Since, for $n \neq 0$,

(4a) $$\int_{-\pi}^{\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_{x=-\pi}^{x=\pi} = 0$$

and

(4b) $$\int_{-\pi}^{\pi} \sin nx\, dx = \left[-\frac{\cos nx}{n}\right]_{x=-\pi}^{x=\pi} = 0,$$

all integrals in the sum vanish, so that

(5) $$\int_{-\pi}^{\pi} f(x)\, dx \sim \pi a_0.$$

Next, multiply both sides of (3) by cos nx, $n \neq 0$, and integrate the result from $-\pi$ to π, as before, obtaining

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx \sim \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{k=1}^{\infty} (a_k \int_{-\pi}^{\pi} \cos kx \cos nx dx$$
$$+ b_k \int_{-\pi}^{\pi} \sin kx \cos nx \, dx).$$

Using (4a) the first integral on the right vanishes. Since the trigonometric functions are pairwise orthogonal, all integrals in the sum also vanish except one. The only one remaining is the coefficient of a_n:

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi.$$

Thus

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx \sim a_n \pi. \tag{6}$$

In a similar manner

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx \sim b_n \pi. \tag{7}$$

From (5), (6), and (7) it follows that it is suitable to let

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad (n = 0, 1, 2, \dots), \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad (n = 1, 2, 3, \dots). \end{aligned} \tag{8}$$

The coefficients a_n and b_n defined in (8) can be calculated and are called the Fourier coefficients of the function f(x), with respect to the trigonometric functions and the trigonometric series with these coefficients is called the

Fourier series of f(x).

As an example of a function expanded in a Fourier series, consider the function $f(x) = x$, $(-\pi < x < \pi)$. Since f(x) is an odd function,

$$a_n = 0 \qquad (n = 0, 1, 2, \dots),$$

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx \, dx = -\frac{2}{\pi n}\left[x \cos nx\right]_{x=0}^{x=\pi} + \frac{2}{\pi n}\int_0^\pi \cos nx \, dx$$

$$= -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1}.$$

Therefore, for $-\pi < x < \pi$,

$$x \sim 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right).$$

Suppose f(x) is of period 2π and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

then the n^{th} partial sum $s_n(x)$ can be expressed as

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

If the values of the Fourier coefficients are substituted,

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos kt \cos kx + \sin kt \sin kx)\right] dt$$

or

$$(9) \qquad s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \cos k(t-x) \right] dt.$$

But, it can be shown that

$$(10) \qquad \frac{1}{2} + \sum_{k=1}^{n} \cos ku = \frac{\sin(n+\frac{1}{2})u}{2 \sin (u/2)} .$$

Substituting (10) into (9) with (t - x) replacing u,

$$s_n(x) = \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(x+u) \frac{\sin(n+\frac{1}{2})u}{2 \sin (u/2)} du.$$

Since $f(x+u)$ and $\dfrac{\sin(n+\frac{1}{2})\,u}{2 \sin (u/2)}$ are periodic in u with period 2π and the interval $[-\pi - x, \pi - x]$ has length 2π, the integral over the given interval is the same as the integral over $[-\pi, \pi]$. Hence

$$(11) \qquad s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin (n+\frac{1}{2})\, u}{2 \sin (u/2)} du.$$

The equation (11) will be used in the next chapter.

Application of Summability Procedures to Fourier Series

The idea of summability is especially applicable in the study of Fourier series, since there exist continuous functions whose Fourier series diverge at some points. First, an integral formula for the arithmetic mean of the partial sums of a Fourier series will be developed. Suppose

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

and

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

For the arithmetic mean of the partial sums,

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + \ldots + s_n(x)}{n+1} ,$$

we get

$$\sigma_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \frac{n-k+1}{n+1} (a_k \cos kx + b_k \sin kx).$$

From (11) of the previous chapter it is known that

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin (n + \frac{1}{2})u}{2 \sin (u/2)} du;$$

therefore

$$\sigma_n(x) = \frac{1}{\pi(n+1)} \int_{-\pi}^{\pi} \frac{f(x+u)}{2 \sin (u/2)} \sum_{k=0}^{n} \sin (k + \frac{1}{2}) u \, du.$$

From the identity,

$$\sum_{k=0}^{n} \sin\left(k + \frac{1}{2}\right)u = \frac{\sum_{k=0}^{n} 2 \sin(u/2) \sin\left(k + \frac{1}{2}\right)u}{2 \sin(u/2)}$$

$$= \frac{\sum_{k=0}^{n} (\cos ku - \cos(k+1)u)}{2 \sin(u/2)} = \frac{1 - \cos nu}{2 \sin(u/2)}$$

$$= \frac{\sin^2(nu/2)}{\sin(u/2)} \quad ,$$

it follows that

$$\text{(1)} \qquad \sigma_n(x) = \frac{1}{\pi(n+1)} \int_{-\pi}^{\pi} f(x+u) \frac{\sin^2(nu/2)}{2 \sin^2(u/2)} \, du \, ,$$

which is the desired formula. The integral given in (1) is sometimes called the Fejér integral expression for the mean $\sigma_n(x)$. As a consequence, if $f(x) = 1$ for all x, then $s_n(x) = 1$ for $n = 1, 2, \ldots$ and therefore $\sigma_n(x) = 1$ for $n = 1, 2, \ldots$. This then implies that

$$\text{(2)} \qquad 1 = \frac{1}{\pi(n+1)} \int_{-\pi}^{\pi} \frac{\sin^2(nu/2)}{2 \sin^2(u/2)} \, du \qquad n = 1, 2, \ldots \, .$$

The function $\dfrac{\sin^2(nu/2)}{2 \sin^2(u/2)}$ is called Fejér's kernel.

Theorem 1: The Fourier series of an absolutely integrable function $f(x)$ (i. e. $\int_a^b |f(x)| dx$ exists) of period 2π is (C, 1) summable to $f(x)$ at every point of continuity and to the value

$$\frac{f(x+0) + f(x-0)}{2}$$

at every point of jump discontinuity.

This theorem is usually called Fejér's Theorem.

Proof: Since at points of continuity, it is also true that

$$f(x) = \frac{f(x+0) + f(x-0)}{2} \quad ,$$

it is sufficient to prove the relation

$$\lim_{n\to\infty} \sigma_n(x) = \frac{f(x+0) + f(x-0)}{2} \quad , \quad \text{for all } x.$$

To show this it is sufficient to prove that

$$(3) \qquad \lim_{n\to\infty} \frac{1}{\pi n} \int_0^{\pi} f(x+u) \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du = \frac{f(x+0)}{2} \; ,$$

and

$$(4) \qquad \lim_{n\to\infty} \frac{1}{\pi n} \int_{-\pi}^{0} f(x+u) \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du = \frac{f(x-0)}{2}$$

Both integral expressions are obtained in the same way, hence only the first will be considered. Since the integrand of (2) is an even function,

$$(5) \qquad \frac{1}{2} = \frac{1}{\pi n} \int_0^{\pi} \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du \; ,$$

so that

$$\frac{f(x+0)}{2} = \frac{1}{\pi n} \int_0^{\pi} f(x+0) \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du \; .$$

Thus from (3), it follows that it remains to prove the formula

$$(6) \qquad \lim_{n\to\infty} \frac{1}{\pi n} \int_0^{\pi} [f(x+u) - f(x+0)] \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du = 0.$$

Let $\varepsilon > 0$ be given; then, since $\lim\limits_{\substack{u \to x \\ u > x}} f(x+u) = f(x+0)$, it follows that

(7) $$|f(x+u) - f(x+0)| < \varepsilon$$

for $0 < u \leq \delta$, if $\delta > 0$ is sufficiently small. Now divide the integral in (6) into two integrals:

(8) $$\begin{cases} I_1 = \dfrac{1}{\pi n} \displaystyle\int_0^{\delta} [f(x+u) - f(x+0)] \dfrac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du \\[2em] I_2 = \dfrac{1}{\pi n} \displaystyle\int_{\delta}^{\pi} [f(x+u) - f(x+0)] \dfrac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du. \end{cases}$$

Then (7) implies that

$$|I_1| < \frac{\varepsilon}{\pi n} \int_0^{\delta} \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du < \frac{\varepsilon}{\pi n} \int_0^{\pi} \frac{\sin^2(nu/2)}{2\sin^2(u/2)} \, du,$$

which implies, from (5), that

(9) $$|I_1| < \frac{\varepsilon}{2} \qquad \text{for any } n.$$

For the other integral,

$$|I_2| \leq \frac{1}{2\pi n \sin^2(\delta/2)} \int_{\delta}^{\pi} |f(x+u) - f(x+0)| \, du,$$

hence

(10) $$|I_2| < \frac{\varepsilon}{2} \qquad \text{for all sufficiently large } n.$$

The relation (6) follows from (8), (9), and (10), which proves the theorem.

A discussion similiar to the one above concerning Cesàro summability can also be presented involving Abel summability. First, calculate the sum of the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\varphi \qquad (0 \le r < 1).$$

For this sum, consider the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n, \quad z = r(\cos\varphi + i\sin\varphi).$$

Since $|z| = r < 1$

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1}{2} + \frac{z}{1-z} = \frac{1+z}{2(1-z)} = \frac{1 + r\cos\varphi + ir\sin\varphi}{2(1 - r\cos\varphi - ir\sin\varphi)}$$

$$= \frac{(1 + r\cos\varphi + ir\sin\varphi)(1 - r\cos\varphi + ir\sin\varphi)}{2[(1 - r\cos\varphi)^2 + r^2\sin^2\varphi]}$$

$$= \frac{1 - r^2 + 2ir\sin\varphi}{2(1 - 2r\cos\varphi + r^2)} .$$

But

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1}{2} + \sum_{n=1}^{\infty} r^n (\cos n\varphi + i\sin n\varphi),$$

therefore

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\varphi = \frac{1}{2} \quad \frac{1 - r^2}{1 - 2r\cos\varphi + r^2} \qquad (0 \le r < 1).$$

The function $\dfrac{1 - r^2}{1 - 2\, r\cos\varphi + r^2}$ of the variables r and φ is called Poisson's kernel.

By applying Abel's method to a function whose Fourier series is known, the following series is formed:

$$f(x, r) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx), \tag{11}$$

where $0 \leq r < 1$.

For convenience of study it is best to represent the function $f(x, r)$ as an integral. Recalling the definitions of a_n and b_n, (11) can be written as

$$f(x, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt + \frac{1}{\pi} \sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} f(t) \cos n(t - x)\, dt. \tag{12}$$

Since the series $\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(t - x)$ converges uniformly in t, it can therefore be integrated term-by-term. Thus (12) can be written as

$$f(x, r) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(t - x)\right] dt,$$

or

$$f(x, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1 - r^2}{1 - 2\, r\cos(t - x) + r^2}\, dt \qquad (0 \leq r < 1). \tag{13}$$

The integral given in (13) is known as Poisson's integral. Note that if $f(x) \equiv 1$, then $a_0/2 = 1$, $a_n = 0$, $b_n = 0$, for $n > 0$, and hence $f(x, r) \equiv 1$, so that (13) becomes

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2 r\cos(t - x) + r^2} \, dt \qquad (0 \le r < 1).$$

Theorem 2: Let f(x) be an absolutely integrable function of period 2π. Then $\lim_{r \to 1} f(x, r) = f(x)$ at every point where f(x) is continuous, and

$$\lim_{r \to 1} f(x, r) = \frac{f(x + 0) + f(x - 0)}{2}$$

at every point where f(x) has a jump discontinuity.

In other words, the Fourier series of f(x) is summable by Abel's method to the value f(x) at every point of continuity of f(x) and to the value $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at every point of jump discontinuity of f(x). The proof of this theorem is essentially identical with the proof of Theorem 1 given earlier in this chapter.

Some other applications will now be stated without proof; the proofs can be found in any standard text on the subject.

Theorem 3: If two continuous functions of period 2π have the same Fourier series, then they coincide. (i.e. The Fourier series of a continuous function determines that function completely).

Theorem 4: If two absolutely integrable functions of period 2π have the same Fourier series, then they coincide on $[0, 2\pi]$ except, at most, finitely many points.

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SUMMABILITY PROCEDURES APPLIED TO FOURIER SERIES

by

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ABSTRACT

This report includes a general discussion on summability of series followed by an introduction to summability of Fourier series. Since the definition of the "sum" of a series is merely a definition, it can be made more general. These more general definitions, called summability procedures, can even be used to find the "sum" of a divergent series. A summability process must satisfy the following conditions in order to keep a close analogy between convergence and summability:

I. If $\sum_{k=0}^{\infty} u_k = S$, then $\sum_{k=1}^{\infty} u_k = S - u_0$, and conversely.

II. If $\sum_{k=0}^{\infty} u_k = S$, $\sum_{k=0}^{\infty} v_k = T$, then $\sum_{k=0}^{\infty} (u_k + v_k) = S + T$.

III. If $\sum_{k=0}^{\infty} u_k = S$, then $\sum_{k=0}^{\infty} \alpha u_k = \alpha S$, for α any constant.

IV. The summability process must be regular. A process is said to "regular" if the process sums every convergent series and when applied to a convergent series gives the sum of the series in the usual sense.

The first method of summation considered was developed by Cesàro. Cesàro's method, or (C, 1) method, is defined as follows: Consider the series $\sum_{n=0}^{\infty} a_n$, and let

$$\left.\begin{aligned} s_n &= a_0 + a_1 + \cdots + a_n, \\ \text{and} \qquad \sigma_n &= \frac{s_0 + s_1 + \cdots + s_n}{n+1} \end{aligned}\right\} \quad (n = 0, 1, 2, \ldots)$$

If $\lim_{n\to\infty} \sigma_n = A$, then $\sum_{n=v}^{\infty} a_n$ is Cesàro summable, or (C, 1) summable, to A.

Also, (C, 1) summability can be generalized further to (C, k) summability for any given positive integer k. Let

$$s_n^{(0)} = a_0 + a_1 + \cdots + a_n,$$

and, in general,

$$s_n^{(k)} = s_0^{(k-1)} + s_1^{(k-1)} + \cdots + s_n^{(k-1)}. \quad (k = 1, 2, \ldots)$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable (C, k) to A if $\lim_{n\to\infty} C_n^{(k)} = A$, where $C_n^{(k)}$, the Cesàro mean of order k, is given by, $C_n^{(k)} = \dfrac{s_n^{(k)}}{\binom{n+k}{k}}$

A second method of summation was developed by Abel. Abel's method is defined as follows. Consider the two series,

$$\sum_{n=0}^{\infty} a_n, \tag{1}$$

and

$$\sum_{n=0}^{\infty} a_n r^k. \tag{2}$$

Assume the series (2) converges for $0 < r < 1$. (This will always be true if the terms of (1) are bounded). Let $\sigma(r)$ be the sum of (2) and let $\lim_{r \to 1} \sigma(r) = \sigma$. If this is the case, the series (1) is said to be summable by Abel's method to σ. It can be shown that a series that is (C, 1) summable is also Abel summable to the same value. The converse is not true.

A function f(x) that is periodic with period 2π with the expansion

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is said to be represented as a Fourier series with a_0, a_n and b_n the Fourier coefficients defined as follows:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx; \; n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx; \; n = 1, 2, \ldots \; .$$

The n^{th} partial sum of the Fourier series can be expressed as

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + u) \frac{\sin (n + \frac{1}{2}) u}{2 \sin (u/2)} du \; .$$

Applying Cesaro's (C, 1) method to a Fourier series, it can be shown that

$$\sigma_n(x) = \frac{1}{\pi(n+1)} \int_{-\pi}^{\pi} f(x + u) \frac{\sin^2 (nu/2)}{2 \sin^2 (u/2)} du.$$

The above integral is usually called Fejér's integral and the function $\dfrac{\sin^2(nu/2)}{2\sin^2(u/2)}$ is called Fejér's kernel.

Theorem 1: The Fourier series of an absolutely integrable function f(x) (i.e. $\int_a^b |f(x)|\,dx$ exists) of period 2π is (C, 1) summable to f(x) at every point of jump discontinuity to the value $\frac{1}{2}[f(x+0) + f(x-0)]$. This theorem is usually called Fejér's theorem.

Similarly applying Abel's method to a function whose Fourier series is known, the following series is formed:

$$f(x, r) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos nx + bn \sin nx).$$

This can be written as

$$f(x, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1 - r^2}{1 - 2r\cos(t - x) + r^2}\, dt \qquad (0 \le r < 1).$$

This integral is known as Poisson's integral and the function $\dfrac{1 - r^2}{1 - 2r\cos\varphi + r^2}$ is called Poisson's kernel.

Theorem 2: The Fourier series of f(x) is summable by Abel's method to the value f(x) at every point of continuity of f(x) and to the value $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point of jump discontinuity of f(x).