FLUID INJECTION THROUGH ONE SIDE OF A LONG VERTICAL CHANNEL BY QUASILINEARIZATION

by

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DEDICATION

The author dedicates this report to his parents, Mr. & Ms. Norbert J. Sidorowicz, without whose assistance and guidance this report would not have been possible.

TABLE OF CONTENTS

NOMENCLATURE

A	Constant
C	Constant
C_p	Specific heat capacity
C_j	Arbitrary constants
$\{f_i\}$	Functional
$f(\eta)$	Normal velocity function
$f'(\eta)$	Tangential velocity function
$f_i(\eta), f_i(1-\xi)$	Perturbation solutions
g	Gravitational constant
g(x)	Integration function
H	Plate height
$h(\eta)$	Axial velocity function
$h_i(\eta), h_i(1-\xi)$	Perturbation solutions
J	Jacobian
Pe	Peclet number
P^*	Relative pressure
Q	Heat transfer rate
Re	Cross flow Reynolds number
t	Independent variable
T(y)	Temperature distribution
$T(\eta)$	Transformed temperature distribution
T_i	Temperature
U	Plate separation
u_i	Velocity components
V	Injection velocity

W	Plate width
$\overline{X}_i$	Unit body force
$\{\dot{X}_i\}$	Derivative of a vector function
X_i	Vector function components
x_k^0	Initial conditions
x_j^f	Final conditions
x	Cartesian coordinate
y	Cartesian coordinate
$y(t)$	Function
$y_i(t)$	Perturbation solutions
z	Cartesian coordinate
α	Thermal diffusity
ε	Perturbation parameter
η	Dimensionless distance
$\theta(\eta)$	Temperature distribution function
κ	Thermal conductivity
ν	Kinematic viscosity
ξ	Transformed dimensionless distance
$\xi(y)$	Temperature distribution function
ρ	Density
τ	Wall shear stress

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Technological advancements in mechanical and chemical engineering in the past two decades have made the subject of diffusion phenomena in a flowing gas stream of concern. More specifically, the cooling of electronic equipment, transpiration cooling processes and the modeling of biomechanical equipment frequently involve flow in channels with porous walls. The solution to problems involving diffusion phenomena requires that the magnitude and direction of the velocity at any point in the flow channel be known.

Wang [1] applied the perturbation technique and a semi-inverse numerical method to obtain solutions for the problem of fluid injection through one side of a long vertical channel. Terrill [2], Berman [3], Sellars [4] and Yuan [5] all employed the perturbation method to obtain solutions for the flow in a channel with porous walls. All of the investigations were concerned with nonlinear, two point, boundary value problems. The perturbation technique is perhaps the simplest approximate method available for achieving solutions to the mentioned boundary value problems. Another approximate method, the quasilinearization technique, is in most cases directly applicable to nonlinear, two point, boundary value problems. Apparently, the solution to flow problems, like those cited above, by quasilinearization has received limited attention.

The quasilinearization technique is basically a numerical integration method. When selecting a numerical integration technique, one should consider that the stability and convergence are the two most important criteria [6], since they determine the computational time involved and the accuracy of the approximate solution. The quasilinearization technique, more commonly known as a modified Newton's method [7], is for most cases stable and will converge quadradically if it converges [8]. Because it satisfies the criteria in most cases, the quasilinearization technique is apparently ideal for achieving solutions to problems of this type. Huang [9] recently demonstrated that the quasilinearization technique is well suited to flow problems involving diffusion phenomena.

It is the aim of this report to investigate the problem studied by Wang [1], by the perturbation and quasilinearization techniques. Furthermore, it will demonstrate the versatility and ease in working with the quasilinearization technique. The second chapter of the report deals with the derivation of the problem's governing equations from the Navier-Stokes equations and an energy equation. Assumptions made regarding the flow's phenomena are listed in that chapter. The third chapter describes and implements each approximate method. The quasilinearization technique is described and implemented first, followed by the perturbation technique. The fourth chapter consists of extensive numerical results for the two methods. Following the fourth chapter is a chapter of conclusions based on the numerical results.

CHAPTER II

PROBLEM DESCRIPTION

The model to be investigated is shown in Figure 1. A fluid having a constant injection velocity, V, is injected through the porous plate, which

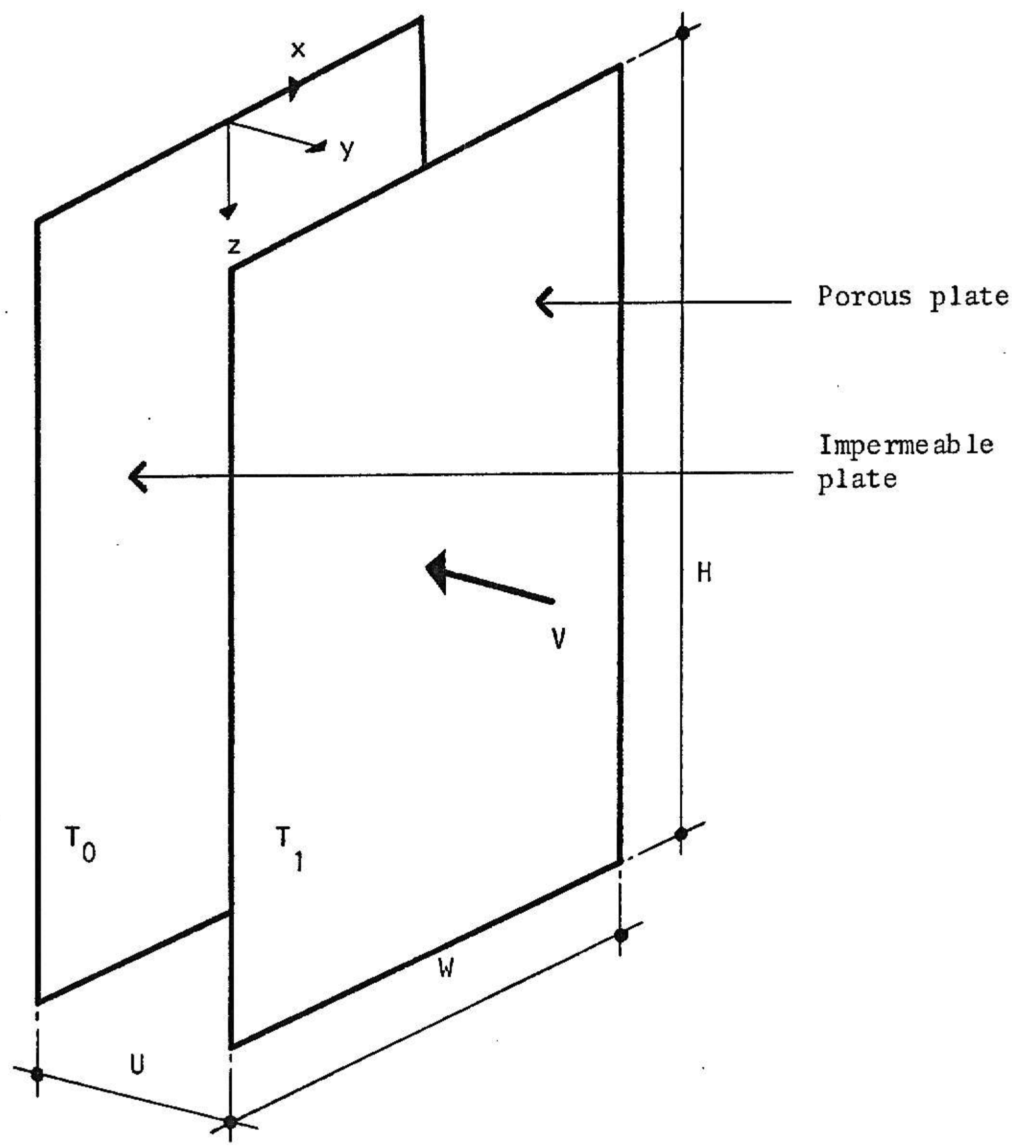


Figure 1: Physical Model

has a temperature T_1, at y = U. The distance between the plates is U. The fluid will strike the impermeable plate, which has a temperature T_0, at y = 0.0. It will flow, due to the action of gravity, out through the opening at the bottom of the plates in the z direction and outwards through the openings at the sides in the x direction. H is the height of the plates and W is their widths, as shown in Figure 1.

In order to reduce the complexity of the problem, the following assumptions are made:

1. The fluid is homogeneous.
2. The fluid is incompressible.
3. Steady state exists.
4. All fluid properties are constant.
5. The dimensions of the plates are such that the edge effects can be ignored, i.e., H>>W>>U.
6. The pressure distribution is independent of z.
7. The velocity in the z direction is independent of z.

By making use of knowledge regarding flow towards a stagnation point and transforming y by $\eta = y/U$, one obtains an expression for the velocity in the y direction:

$$u_y = -Vf(\eta), \tag{2-1}$$

where $f(\eta)$ is an unknown function. The boundary conditions for $f(\eta)$ are:

$$f(0) = 0.0,$$

$$f(1) = 1.0.$$

Making use of assumption 7 and the continuity equation [10],

$$u_{k,k},$$

an expression for the velocity in the x direction is obtained,

$$u_x = Vx/U\ f'(\eta). \tag{2-2}$$

The boundary conditions for $f'(\eta)$ are:

$$f'(0) = 0.0,$$

$$f'(1) = 0.0.$$

Upon applying assumptions 5 and 7 one obtains the following expression for the velocity in the z direction,

$$u_z = C \cdot h(\eta) \tag{2-3}$$

where C is an arbitrary constant and $h(\eta)$ is another unknown function. $h(\eta)$ has the following boundary conditions:

$$h(0) = 0.0,$$

$$h(1) = 0.0.$$

The substitution of equations (2-1) and (2-2) into the Navier-Stokes equations,

$$u_j u_{i,j} = \underline{\overline{X}}_i - \frac{1}{\rho} P^*_{,i} + \nu\nabla^2 u_i,$$

in the x and y directions yields,

$$f'^2 - ff'' = U^2 P^*_{,x}/V^2 x\rho + \nu f'''/UV, \tag{2-4}$$

$$\nu f''/VU + ff' = -UP^*_{,y}/V^2\rho. \tag{2-5}$$

Upon integrating equation (2-5) with respect to y, one obtains the equation for P^*, which is the relative pressure at a point (x,y) with respect to the stagnation pressure [11]:

$$P^* = -\rho/2(2V^2 g(x)/U - 2\nu u_{y,y} + u_y^{\,2}).$$

In the above equation ρ is the density of the fluid, ν is its kinematic viscosity and $g(x)$ is an unknown function.

The equation for P^* is now substituted into equation (2-4) so that,

$$Ug'(x)/x = -1/Re(f'''-Re(f'^2 - ff'')), \tag{2-6}$$

results. Re is the crossflow Reynolds number and is equal to VU/ν. Since equation (2-6) must be valid for all x and y,

$$Ug'(x)/x = -1/Re(f'''-Re(f'^2 - ff'')),$$

$$= A.$$

A is a constant and therefore,

$$g(x) = Ax^2/2U.$$

The relative pressure, P^*, can now be expressed as,

$$P^* = -\rho/2(AV^2x^2/U^2 - 2\nu u_{y,y} + u_y{}^2). \quad (2\text{-}7)$$

From equations (2-4) and (2-7), the first governing differential equation is obtained,

$$f''' - Re(f'^2 + ff'') + Re\cdot A = 0.0. \quad (2\text{-}8)$$

A governing differential equation involving $h(\eta)$ is desired. By substitution of equations (2-1) and (2-3) into the Navier-Stokes equation in the z direction, the second governing differential equation is obtained,

$$h'' + Re\cdot f\cdot h' + U^2g/\nu C = 0.0.$$

Since C was an arbitrary constant, one can let $C = U^2g/\nu$, then the second governing differential equation becomes,

$$h'' + Re\cdot f\cdot h' + 1.0 = 0.0. \quad (2\text{-}9)$$

One must also consider that the flow field has a temperature gradient. If one assumes that T_1, at $y = U$, is larger than T_0, at $y = 0.0$, the temperature distribution can be described by,

$$T(y) = T_0 + (T_1-T_0)\ \xi(y),$$

where $\xi(y)$ is an unknown function. $\xi(y)$ has the boundary conditions:

$$\xi(0) = 0.0,$$

$$\xi(U) = 1.0.$$

The heat will be transferred from the porous plate by convection. The rate at which this heat will be transferred, Q, will be given by,

$$Q = -\kappa(T_1-T_0)\ \xi'(y),$$

where κ is the thermal conductivity. The variable y may be transformed by writing $\eta = y/U$, allowing one to obtain,

$$\theta(\eta) = \xi(y),$$

with the boundary conditions,

$$\theta(0) = 0.0,$$

$$\theta(1) = 1.0.$$

The temperature distribution then becomes,

$$T(\eta) = T_0 + (T_1 - T_0)\ \theta(\eta), \tag{2-10}$$

and the heat transfer rate is now given by,

$$Q = -\kappa(T_1 - T_0)\ \theta'(\eta)/U.$$

Since heat will be transferred by convection, the energy equation is [12],

$$u_y T_{,\eta} = \alpha T_{,\eta\eta}, \tag{2-11}$$

where α is the thermal diffusity and,

$$\alpha = \kappa/C_p\rho.$$

In the above expression C_p is the specific heat capacity at a constant pressure. Upon substituting equations (2-1) and (2-10) into equation (2-11) one obtains,

$$\theta'' + Pe \cdot f \cdot \theta' = 0.0, \tag{2-12}$$

where Pe is the Peclet number and is the product of the Reynolds number and the Prandtl number.

Equation (2-12) is the third governing differential equation to be solved. This fluid injection problem with its three coupled governing differential equations and two point boundary conditions is to be investigated by both the quasilinearization and perturbation techniques in the next chapter.

CHAPTER III

METHODS OF SOLUTION

3.1. Quasilinearization technique

Quasilinearization is known as the generalized Newton-Raphson technique for functional equations [8]. To apply this method, one must consider the M^{th} order, nonlinear, ordinary differential equation,

$$X^{(M)} = f(t;X',X'',\ldots,X^{M-1}).$$

This differential equation can be reduced to a system of M first order differential equations of the form shown in the following expression,

$$\{\dot{X}_i\} = \{f_i(t;X_1,X_2,\ldots,X_M)\} \quad , \quad i = 1,2,\ldots,M. \tag{3-1}$$

In this expression t is the independent variable. The X_i are the dependent variables and,

$$\dot{X}_1 = X_2,$$

$$\dot{X}_2 = X_3,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\dot{X}_M = f(t;X_1,X_2,\ldots,X_M).$$

With the aid of a Taylor series the functional, f_i, can be expanded and linearized. The functional is expanded about the functions $X_{i,0}$ and the constant and linear terms retained. Equation (3-1) is then approximated by,

$$\{\dot{X}_i\} = \{f_i(t;X_{1,0},X_{2,0},\ldots,X_{M,0})\}$$
$$+ J(X_{i,0})(\{X_i\} - \{X_{i,0}\}). \tag{3-2}$$

In equation (3-2), $J(X_{i,0})$ is the Jacobian of the functional, f_i, evaluated at $X_{i,0}$. It is,

$$J(X_{i,0}) = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \ldots\frac{\partial f_1}{\partial X_M} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \ldots\frac{\partial f_2}{\partial X_M} \\ & & \cdot \\ & & \cdot \\ & & \cdot \\ \frac{\partial f_M}{\partial X_1} & \frac{\partial f_M}{\partial X_2} & \ldots\frac{\partial f_M}{\partial X_M} \end{bmatrix}_{X_{i,0}} .$$

Since the values for $X_{i,0}$ are assumed to be known, equation (3-2) is linear and can be solved numerically. Upon solving for X_i in equation (3-2) and calling the solution $X_{i,1}$, the functional is expanded about $X_{i,1}$ as for $X_{i,0}$ and a solution for $X_{i,2}$ can be obtained. Continuing this process leads to a vector recurrence formula shown below,

$$\{\dot{X}_{i,n+1}\} = \{f_i(t;X_{1,n},X_{2,n},\ldots,X_{M,n})\}$$
$$+ J(X_{i,n})(\{X_{i,n+1}\} - \{X_{i,n}\}). \tag{3-3}$$

The subscript following the comma indicates the number of the iteration. This iteration process is of second order [13], which indicates that the error for the n+1 iteration will be approximately the square of the error for the n^{th} iteration. If the iteration process converges, the limit $\lim_{n \to \infty}$, $X_{i,n+1}$ will approach the general solution, X_i, of equation (3-1) quadradically [14]. One can simplify equation (3-3) to obtain a more convenient form as below,

$$\{\dot{X}_{i,n+1}\} = J(X_{i,n})\{X_{i,n+1}\} + \{P_i\}_{X_{i,n}} . \tag{3-4}$$

One may now consider that the M^{th} order differential equation represents a two point boundary value problem. Assuming that there are m final values given, the final conditions are,

$$X_{j,n+1}(t_f) = x_j^f \quad , \quad j = 1,2,\ldots,m. \tag{3-5}$$

Upon letting (i-m) initial values be given, the initial conditions are,

$$X_{k,n+1}(0) = x_k^0 \quad , \quad k = m+1, m+2, \ldots, i, \tag{3-6a}$$

and the m missing initial conditions are given by m arbitrary constants as below,

$$X_{j,n+1}(0) = C_{j,n+1} \quad , \quad j = 1,2,\ldots,m. \tag{3-6b}$$

As was previously mentioned, equation (3-4) is linear, and an approximate solution of equation (3-1) can be obtained by using the principle of superposition and a numerical integration technique, such as a Runge-Kutta algorithm. For the boundary conditions of equations (3-6a) and (3-6b) it is sufficient for one to find one set of particular solutions and m linearly independent homogeneous solutions and treat the problem as if it were an initial value problem.

If the vector, $\{X_{pi,n+1}(t)\}$, is a solution of equation (3-4), i.e.,

$$\{\dot{X}_{pi,n+1}\} = J(X_{i,n})\{X_{pi,n+1}\} + \{P_i\}_{X_{i,n}} ,$$

which satisfies the initial conditions of equation (3-6a) and the m linearly independent vectors, $\{X_{hji,n+1}(t)\}$, are distinct solutions of,

$$\{\dot{X}_{hji,n+1}\} = J(X_{i,n})\{X_{hji,n+1}\} \quad , \quad j = 1,2,\ldots,m,$$

which satisfy the boundary conditions,

$$\sum_{j=1}^{m} C_{j,n+1} \{\dot{X}_{hji,n+1}(0)\} = 0.0 \quad , \quad i > m,$$

an approximate solution can be obtained. Using superposition an approximation of the general solution of equation (3-1) corresponding to the initial conditions of equations (3-6a) and (3-6b) will be,

$$\{X_{gi,n+1}(t)\} = \{X_{pi,n+1}(t)\} + \sum_{j=1}^{m} C_{j,n+1} \{X_{hji,n+1}(t)\},$$

$$0 \leq t \leq t_f.$$

One should notice that in obtaining a solution to equation (3-4) only the initial conditions were used. At the final point, t_f, one can set the first m equations of the approximate solution equal to the final conditions of equation (3-5) as below,

$$X_i^f = X_{pi,n+1}(t_f) + \sum_{j=1}^{m} C_{j,n+1} X_{hji,n+1}(t_f),$$

$$i = 1,2,\ldots,m.$$

Since there are m final conditions, these m equations constitute a system of m algebraic equations in m unknowns and the $C_{j,n+1}$ are uniquely determined.

Once the $C_{j,n+1}$ are determined a solution, $X_{i,n+1}$, is found by superposition of the numerical values of the solution vectors. An improved general solution vector, X_i, may be found by substituting $X_{i,n+1}$ for $X_{i,n}$ in equation (3-4). This procedure can be continued until successive approximate solutions, which differ by a designated error, yield a fairly accurate approximate solution to equation (3-1).

3.1.1. Implementation

To implement this technique for the fluid injection problem defined in Chapter II, the following governing differential equations of the problem are considered:

$$f''' - Re(f'^2 - ff'') + ReA = 0.0, \tag{3-7a}$$

$$h'' + Re \cdot f \cdot h' + 1.0 = 0.0, \tag{3-7b}$$

$$\theta'' + Pe \cdot f \cdot \theta' = 0.0. \tag{3-7c}$$

One can differentiate equation (3-7a) to obtain,

$$f'^{V} - Re(f'f'' - ff''') = 0.0. \tag{3-8}$$

Equations (3-8), (3-7b) and (3-7c) are reduced to one system of first order equations as shown below:

$$\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \\ \dot{X}_7 \\ \dot{X}_8 \end{Bmatrix} = \begin{Bmatrix} f' \\ f'' \\ f''' \\ f'^{V} \\ h' \\ h'' \\ \theta' \\ \theta'' \end{Bmatrix} = \begin{Bmatrix} X_2 \\ X_3 \\ X_4 \\ Re(X_2X_3 - X_1X_4) \\ X_6 \\ -ReX_1X_6 - 1 \\ X_8 \\ -PeX_1X_8 \end{Bmatrix}$$

If one assumes the function values, $X_{i,0}$, and expands the functional about them, using a recurrence process one obtains a form identical to equation (3-4) (see equation (3-9)) after simplification. The governing differential equations have now been reduced, linearized and a vector recurrence formula established. An approximation of the general solution, X_i, can now be found by superposition and numerical integration. Use must now be made of the initial conditions of the problem,

$$
\begin{aligned}
f(0) &= X_1(0) = 0.0, \\
f'(0) &= X_2(0) = 0.0, \\
f''(0) &= X_3(0) = C_{1,n+1}, \\
f'''(0) &= X_4(0) = C_{2,n+1}, \\
h(0) &= X_5(0) = 0.0, \\
h'(0) &= X_6(0) = C_{3,n+1}, \\
\theta(0) &= X_7(0) = 0.0, \\
\theta'(0) &= X_8(0) = C_{4,n+1}.
\end{aligned}
$$

The $C_{j,n+1}$ denote the missing initial conditions. An approximate solution satisfying the initial conditions is,

$$
X_{gi,n+1}(0) = \begin{Bmatrix} \\ \\ \\ \phi \\ \\ \\ \\ \end{Bmatrix} + C_{1,n+1}\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + C_{2,n+1}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + C_{3,n+1}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} + C_{4,n+1}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}
$$

The $\{\phi\}$ vector represents the initial values for the one set of particular solutions which is obtained by integrating equation (3-9) subject to those initial values. The remaining vectors are the initial values for the homogeneous solutions which are found by integrating the homogeneous part of equation (3-9) subject to those initial values. Each solution vector is obtained by integrating from $\eta = 0.0$ to $\eta = 1.0$ with the aid of the program in Appendix I. Making use of the final conditions of the problem at $\eta = 1.0$,

$$f(1) = X_1(1) = 1.0,$$

$$f'(1) = X_2(1) = 0.0,$$

$$h(1) = X_5(1) = 0.0,$$

$$\theta(1) = X_7(1) = 1.0,$$

the $C_{j,n+1}$ are uniquely determined. An approximate solution, $X_{i,n+1}$, may be found and checked against the assumed solution, $X_{i,n}$, to determine if it is within a prescribed error. If the error test fails the process is repeated with the last approximate solution being the assumed solution, i.e., $X_{i,n}$. This iteration process continues until the error test is passed or until a limiting number of iterations have been made.

$$
\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \\ \dot{X}_7 \\ \dot{X}_8 \end{Bmatrix}^{n+1}
=
\begin{bmatrix}
0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-ReX_4 & ReX_3 & ReX_2 & -ReX_1 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
-ReX_6 & 0.0 & 0.0 & 0.0 & 0.0 & -ReX_1 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
-PeX_8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -PeX_1
\end{bmatrix}^{n}
\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{Bmatrix}^{n+1}
+
\begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ Re(X_1X_4-X_2X_3) \\ 0.0 \\ Re(X_1X_6)-1.0 \\ 0.0 \\ PeX_1X_8 \end{Bmatrix}^{n}
$$

Equation (3-9)

3.2. Perturbation technique

The perturbation technique, an expansion of a solution as power series in a parameter, is one of the simplest and most useful approximation techniques [15]. In fluid mechanics the perturbation parameter is usually a dimensionless quantity such as the Reynolds number, Mach number, etc.. To apply this method the following differential equation is considered,

$$\varepsilon y''(t) + y'(t) = h(t), \tag{3-10}$$

subject to the boundary conditions,

$$y(0) = 0.0,$$

$$y'(1) = 1.0.$$

In equation (3-10) ε is the perturbation parameter and $h(t)$ is a known function. The general solution of equation (3-10) can be expressed as a power series such as,

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots \quad . \tag{3-11}$$

In equation (3-11) the $y_i(t)$, $i = 0,1,2,\ldots$ are unknown polynomials to be determined. Substituting equation (3-11) into equation (3-10) and equating coefficients of ε yields an infinite set of equations which can be solved recursively to yield the unknown polynomials. The equations to be solved are,

$$y_0'(t) = h(t),$$

$$\varepsilon(y_0''(t) + y_1'(t)) = 0.0,$$

$$\varepsilon^2(Y_1''(t) + Y_2'(t)) = 0.0,$$

$$\varepsilon^3(y_2''(t) + y_3'(t)) = 0.0,$$

with the boundary conditions,

$$y_0(0) = 0.0, \qquad y_0'(1) = 1.0,$$

$$y_1(0) = 0.0, \qquad y_1'(1) = 0.0,$$

$$y_2(0) = 0.0, \qquad y_2'(1) = 0.0,$$

$$\vdots \qquad\qquad \vdots$$

If the perturbation parameter, ε, is small the set of equations and equation (3-11) can truncated after $o(\varepsilon^2)$. A fairly accurate approximate solution to equation (3-10), satisfying the given boundary conditions can then be found and is given by,

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t).$$

3.2.1. Implementation

For the fluid injection problem defined in Chapter II the perturbation theory is applicable to the first two governing differential equations (equations (3-8) and (2-9)). In these two equations,

$$f'^{V} - Re(f'f'' - ff''') = 0.0, \tag{3-8}$$

$$h'' + Re\cdot f\cdot h' + 1.0 = 0.0, \tag{2-9}$$

the perturbation parameter will be the cross flow Reynolds number. For convenience the perturbation parameter is assumed to be small in magnitude, hence, approximate solutions to equations (3-8) and (2-9) will be truncated after $o(Re^2)$.

Equation (3-8) is investigated under the assumption that,

$$f(\eta) = f_0(\eta) + Re\cdot f_1(\eta) + Re^2 f_2(\eta), \tag{3-12}$$

and that the boundary conditions are:

$$f_0(0)=f_1(0)=f_2(0)=f_0'(0)=f_1'(0)=f_2'(0)=0.0,$$

$$f_1(1)=f_2(1)=f_0'(1)=f_1'(1)=f_2'(1)=0.0,$$

$$f_0(1)=1.0.$$

Upon substituting equation (3-12) into equation (3-8), equating coefficients of Re and neglecting coefficients higher than o(Re^2) one obtains a set of three differential equations. The differential equations which can be solved recursively are,

$$f_0'^{V} = 0.0, \tag{3-13a}$$

$$Re(f_1'^{V} - f_0'f_0'' + f_0f_0''') = 0.0, \tag{3-13b}$$

$$Re^2(f_2'^{V} - f_1'f_0'' - f_0'f_1'' + f_1f_0''' + f_1'''f_0) = 0.0. \tag{3-13c}$$

f_0 is determined from equation (3-13a). Upon making the transformation $\eta = 1 - \xi$, $d\eta = -d\xi$, one obtains:

$$f_0(1-\xi) = 2\xi^3 - 3\xi^2 + 1. \tag{3-14}$$

Equation (3-14) is substituted into equation (3-13b). Upon integrating and solving for the integration constants one can determine f_1,

$$f_1(1-\xi) = -2/35\xi^7 + 1/5\xi^6 - 3/10\xi^5 + 1/2\xi^4$$
$$-43/70\xi^3 + 19/70\xi^2. \tag{3-15}$$

Substituting equations (3-14) and (3-15) into equation (3-13c), integrating and solving for the integration constants yields for f_2,

$$f_2(1-\xi) = -4/5775\xi^{11} + 2/525\xi^{10} - 1/210\xi^9$$
$$- 11/560\xi^8 + 191/2450\xi^7 - 68/525\xi^6$$
$$+27/175\xi^5 - 43/280\xi^4 + 32189/323400\xi^3$$
$$- 17719/646800\xi^2.$$

The approximate solution of equation (3-8) is then given by,

$$\begin{aligned} f(1-\xi) = {} & 2\xi^3 - 3\xi^2 + 1 \\ & + Re(-2/35\xi^7 + 1/5\xi^6 - 3/10\xi^5 + 1/2\xi^4 \\ & \quad -43/70\xi^3 + 19/70\xi^2) \\ & + Re^2(-4/5775\xi^{11} + 2/525\xi^{10} - 1/210\xi^9 \\ & \quad - 11/560\xi^8 + 191/2450\xi^7 - 68/525\xi^6 \\ & \quad + 27/175\xi^5 - 43/280\xi^4 + 32189/323400\xi^3 \\ & \quad - 17719/646800\xi^2). \end{aligned}$$

Differentiating f one obtains f',

$$\begin{aligned} f'(1-\xi) = {} & -6\xi^2 + 6\xi \\ & + Re(14/35\xi^6 - 6/5\xi^5 + 3/2\xi^4 -2\xi^3 \\ & \quad + 129/70\xi^2 - 38/70\xi) \\ & + Re^2(44/5775\xi^{10} - 20/525\xi^9 + 9/210\xi^8 \\ & \quad + 88/560\xi^7 - 1337/2450\xi^6 + 408/525\xi^5 \\ & \quad - 135/175\xi^4 + 172/280\xi^3 \\ & \quad - 96567/323400\xi^2 + 35438/646800\xi) \end{aligned}$$

Equation (2-9) is investigated under the assumption that,

$$h(\eta) = h_0(\eta) + Re\cdot h_1(\eta) + Re^2 h_2(\eta),$$

and that the boundary conditions are:

$$h_0(0) = h_1(0) = h_2(0) = 0.0,$$
$$h_0(1) = h_1(1) = h_2(1) = 0.0.$$

After equating coefficients of Re and neglecting coefficients of $o(Re^3)$ and greater one finds:

$$h_0'' + 1.0 = 0.0,$$
$$Re(f_0 h_0' + h_1'') = 0.0,$$
$$Re^2(h_2'' + f_1 h_0' + f_0 h_1') = 0.0.$$

Upon transforming η and following the same procedure used to determine f, h is found and is as follows:

$$
\begin{aligned}
h(1-\xi) = {} & 1/2\xi - 1/2\xi^2 \\
& + Re(-1/15\xi^6 + 1/5\xi^5 - 1/8\xi^4 - 1/6\xi^3 \\
& \quad + 1/4\xi^2 - 11/120\xi) \\
& + Re^2(-13/1575\xi^{10} + 13/315\xi^9 - 9/140\xi^8 \\
& \quad - 11/840\xi^7 + 611/4200\xi^6 - 29/210\xi^5 \\
& \quad - 5/672\xi^4 + 1/12\xi^3 - 11/240\xi^2 + 349/50400\xi).
\end{aligned}
$$

The above solutions for f and h are identical to those used by Wang [1]. With the aid of the program in Appendix II three cross flow Reynolds numbers will be selected to investigate the range of Re for which the solutions, f, f' and h appear to be valid. An approximate solution for θ can be obtained by integrating the following finite difference equation:

$$\frac{d\theta}{dt} = \frac{\theta(i+1) - 2\theta(i) + \theta(i-1)}{h^2} + Pe\cdot f(i)\ \frac{\theta(i+1) - \theta(i-1)}{2h}\ .$$

The above equation is integrated until steady state exists, i.e., $\frac{d\theta}{dt} = 0.0$. The computer program used is given in Appendix III. f(i) is the perturbation solution for f at the i^{th} point and h is the interval. An approximation for θ' can be obtained by investigating the following equation:

$$\theta' = \frac{\theta(i+1) - \theta(i-1)}{2h}\ .$$

In the following chapter the numerical results for the methods in this chapter and the quasilinearization technique are presented.

CHAPTER IV

NUMERICAL RESULTS

Numerical examples are given for the physical model described in Chapter II. Given below are the properties of the fluid, the model dimensions and computational tolerances and requirements needed for the problem to be investigated by quasilinearization:

Fluid

1. Air is the fluid.
2. The Prandtl number is constant and equals 0.7.
3. $\nu = 0.000197$ ft.^{2}/sec..
4. $\rho = 0.0675$ lb$_m$/ft.^{3}.
5. Re has five distinct values: Re = 0.0
   1.0
   5.0
   10.0
   25.0

Model dimensions

1. U = 0.5 inches.

Computational requirements

1. The designated error equals 0.0001.
2. The interval size, h, equals 0.025.
3. The assumed functions, $X_{i,0}$, are constant and equal 0.05.

Equation (3-9) is investigated by making use of the above requirements and the aid of the program in Appendix I. Results for f, f', h, θ and θ' are shown in dimensionless form in Figures 2 through 6 respectively.

In Figure 2 it is evident that the normal velocity function, f, changes relatively very little over the range of Re. It changes approximately 0.15 for a specific change of $\Delta\eta = 0.1$ in the direction toward the impermeable plate for a range of η between 0.3 and 0.6 and Re in the range from 0.0 to 25.0. In Figure 3 one may observe that the maximum value of f' shifts towards the impermeable plate and increases by 0.07 over the range of Re. Inspection of Figure 3 indicates to one that the flow in the x direction is laminar for all values of Re.

From Figure 4, it is apparent that the flow is not laminar for Re $\geq$ 10.0. This is evidenced by the inflection points in the curves for Re = 10.0 and 25.0. The axial velocity, which is just the constant C times this function, decreases and shifts towards the impermeable plate as Re is increased from its minimum to its maximum value.

Figure 5 depicts the temperature distribution function, θ, for the given values of Re. One should observe that as Re is increased, the temperature approaches a constant, approximately, for increasing distances near the porous plate. The plots for θ' (Figure 6) depict how the heat transfer rate behaves for various Re.

As Re is increased, less heat is being transferred initially from the porous plate by convection for an increasing interval of η. For Re = 0.0, θ and θ' behave as a motionless medium in the y direction. An increase in Re to a value much greater than 25.0 would result in an almost constant temperature distribution with heat transfer occurring only near the impermeable plate.

Figures 7 and 8 depict the behavior of the velocity in the x direction for x between 0.0 and 1.0' and for Re = 1.0 and 25.0 respectively. Figures 9 and 10 show how the relative pressure behaves for the same values of Re. A two dimensional representation was chosen since the deviation in the y direction was negligible from the plotting standpoint. In Figures 11 and 12 it may be seen that the wall shear stress in the x and z direction at $\eta = 1.0$ decrease very little for substantial increases in Re. At $\eta = 0.0$ the shear stress in the x direction is increasing for increasing Re whereas that in the z direction is behaving like τ_z at $\eta = 1.0$. Figure 13 shows the number of iterations (NIT) required for convergence for the given values of Re.

For the perturbation technique, three Reynolds numbers (0.0, 10.0 and 25.0) were selected and solutions for f, f' and h are found with aid of the program in Appendix II. Figures 14, 15 and 16 show plots for f, f' and h for the selected Re for both the quasilinearization and perturbation techniques. The solid lines show the perturbation solutions while the dashed lines depict the quasilinearization solutions. It may be noticed that for Re $\leq$ 10.0, f and f' by the perturbation method appear to be valid for all η.

Figures 17 and 18 show the solutions for θ and θ' obtained by the combined perturbation and finite difference method. The results show that the solutions are very close to those found by the quasilinearization technique. The reason for the closeness is that the function θ is somewhat insensitive to the function f.

Conclusions based on the material in this chapter will be presented in the next chapter as well as a discussion of the results.

THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.

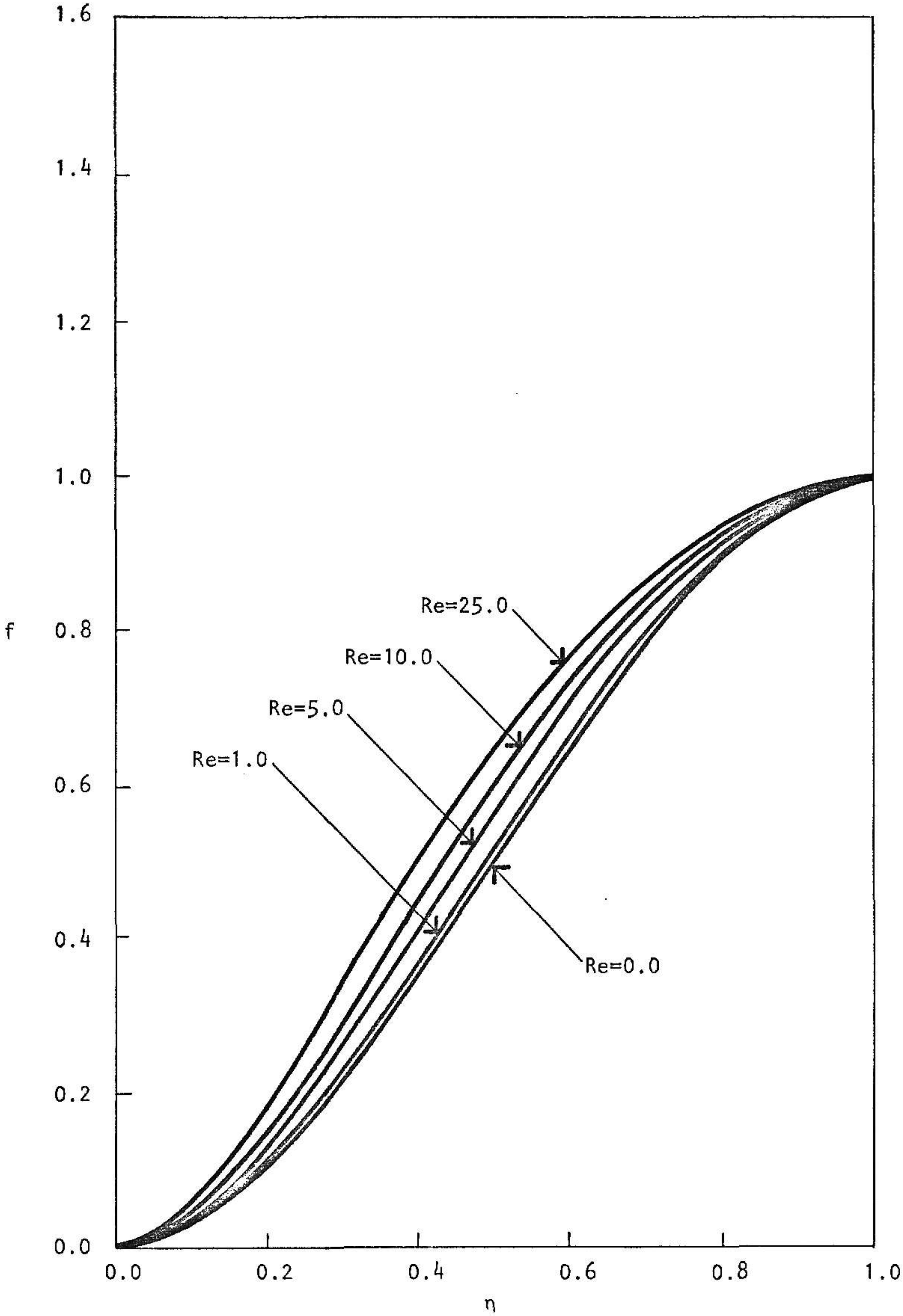


Figure 2: Normal velocity function versus eta by quasilinearization,Re=0,1,5,10,25.

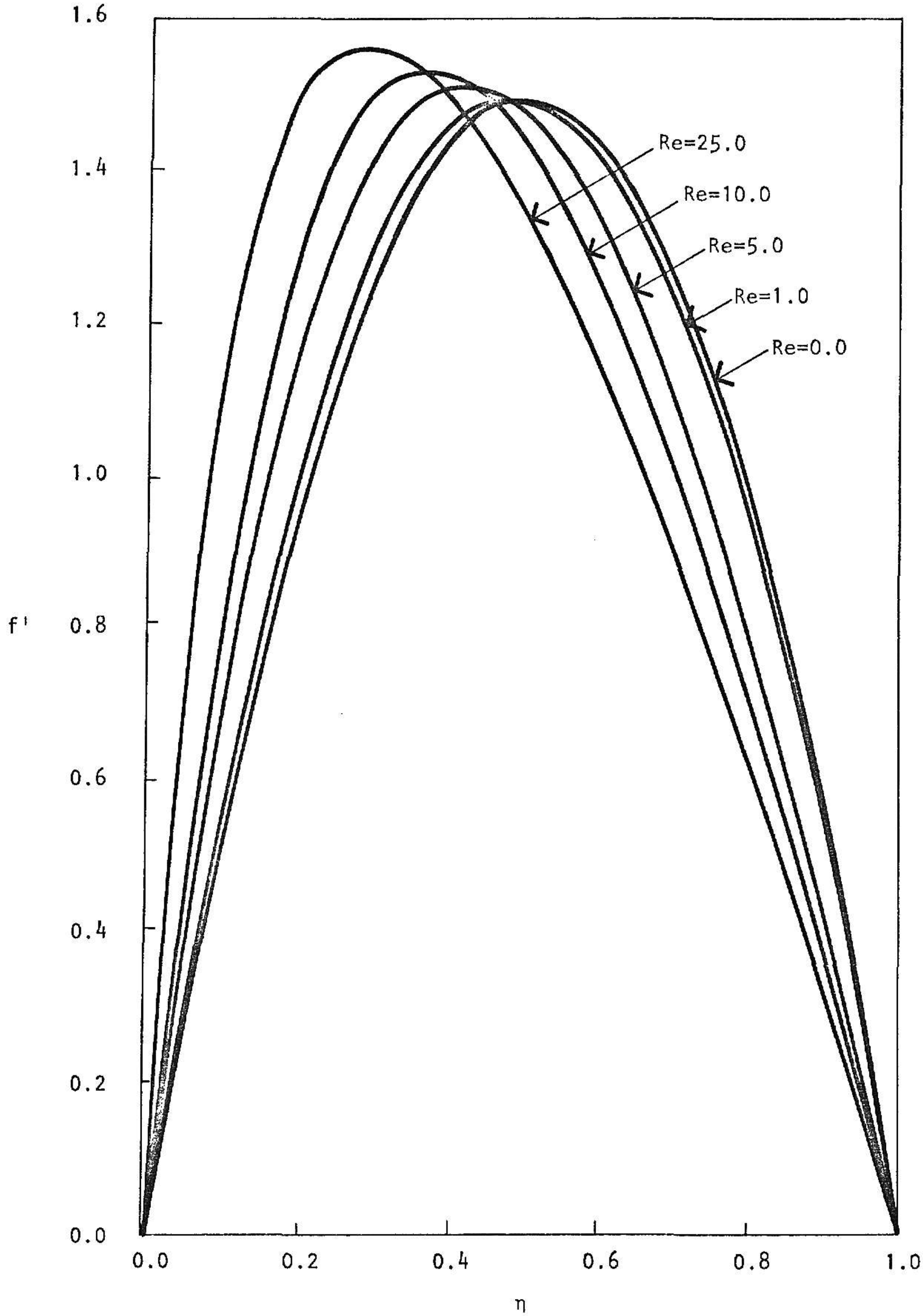


Figure 3: Tangential velocity function versus eta
by quasilinearization, Re=0,1,5,10,25.

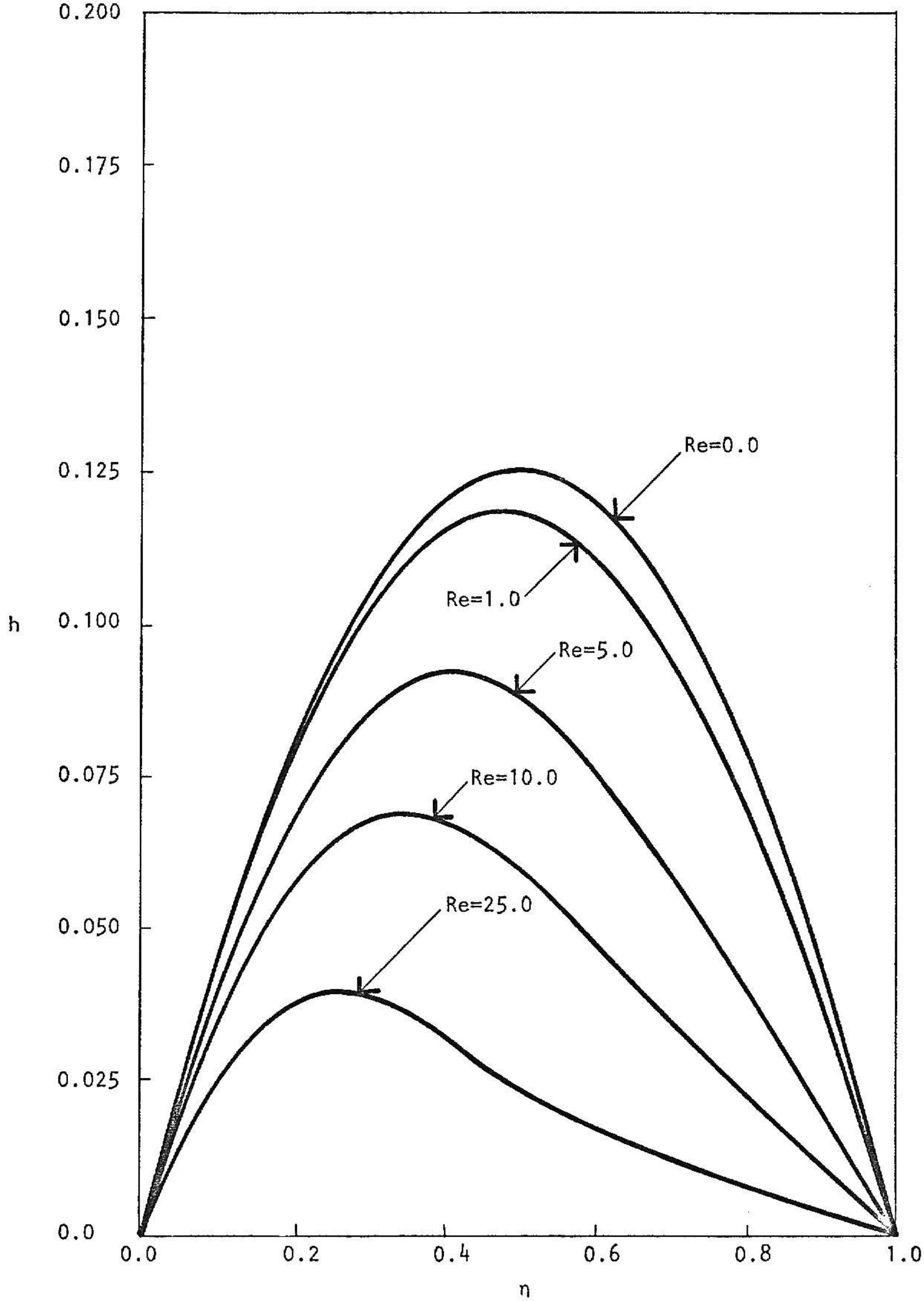


Figure 4: Axial velocity function versus eta by quasilinearization, Re=0,1,5,10,25.

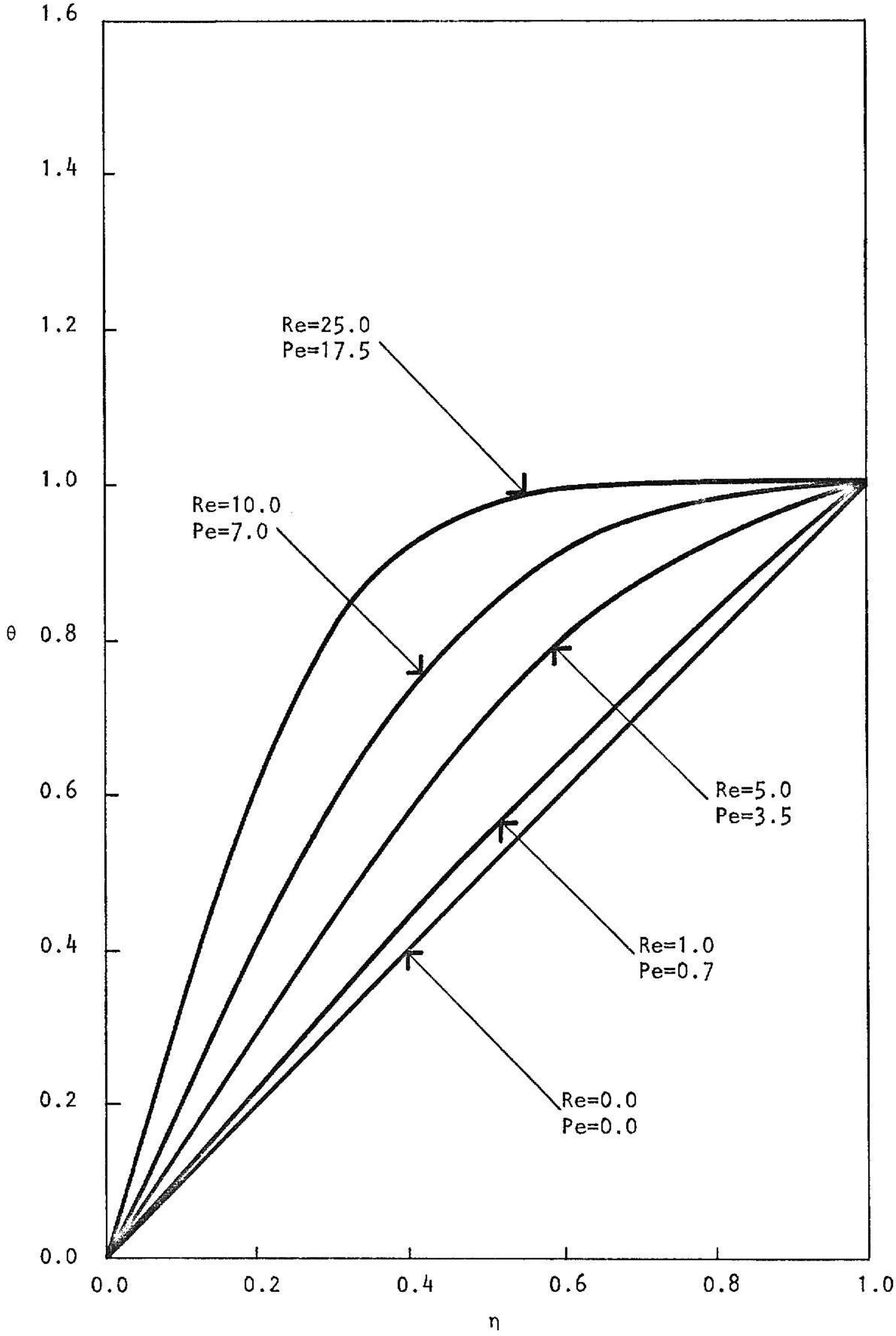


Figure 5: Temperature distribution function versus eta by quasilinearization,Re=0,1,5,10,25.

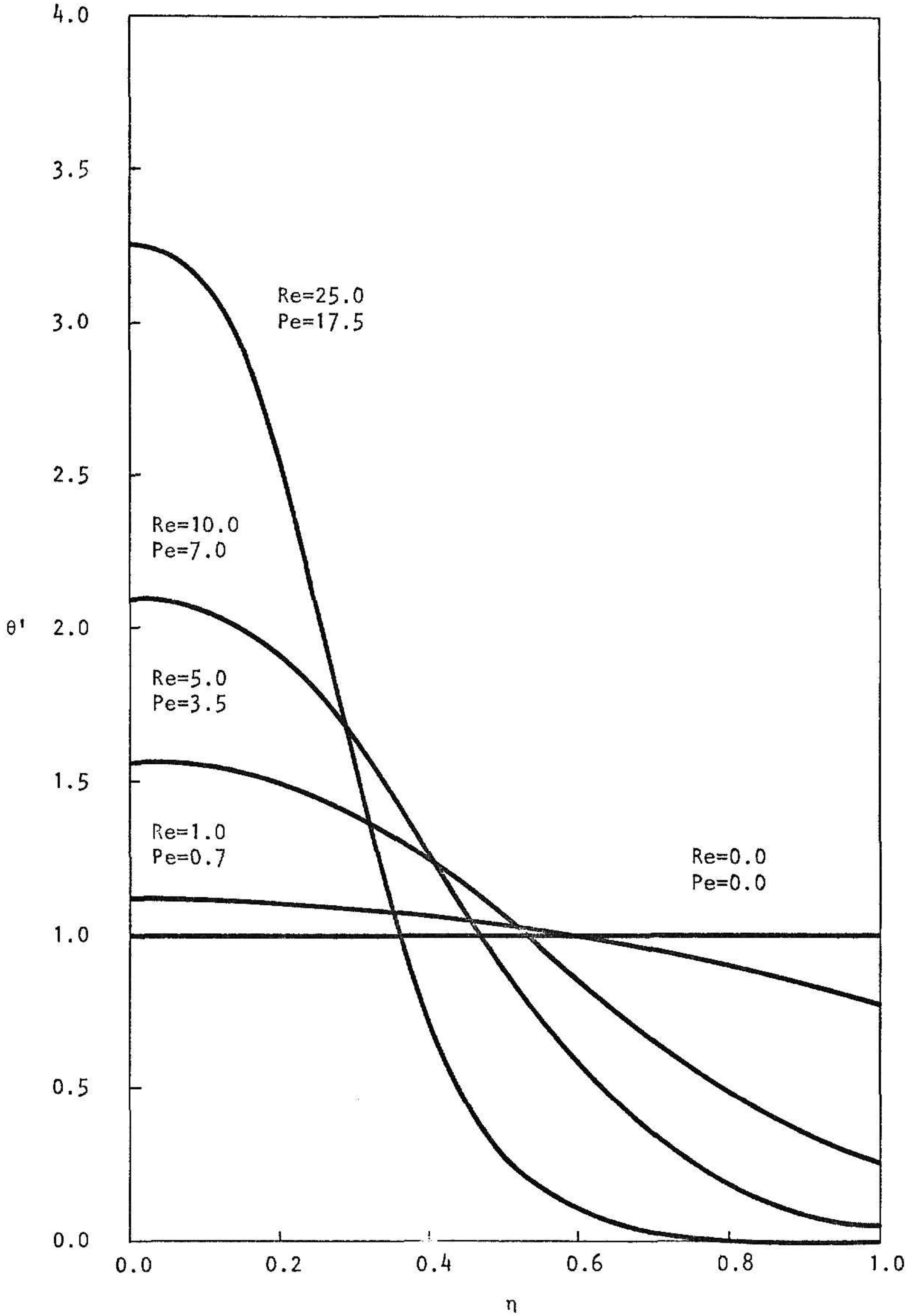
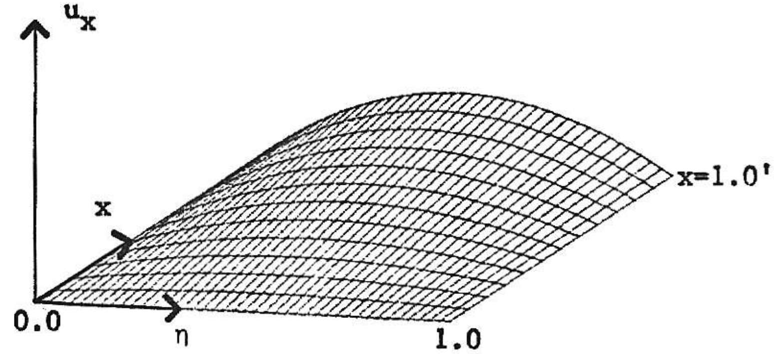


Figure 6: Heat transfer rate function versus eta by quasilinearization, Re=0,1,5,10,25.



VELØCITY IN THE X DIRECTIØN RE=1.0

ALPHA =	0.0	XMIN=	0.0	XMAX=	0.300E 01
BETA =	10.00	YMIN=	0.0	YMAX=	0.600E 01
GAMMA =	75.00	ZMIN=	0.0	ZMAX=	0.514E 00

Figure 7: Velocity distribution in the x direction by quasilinearization, Re=1.0.

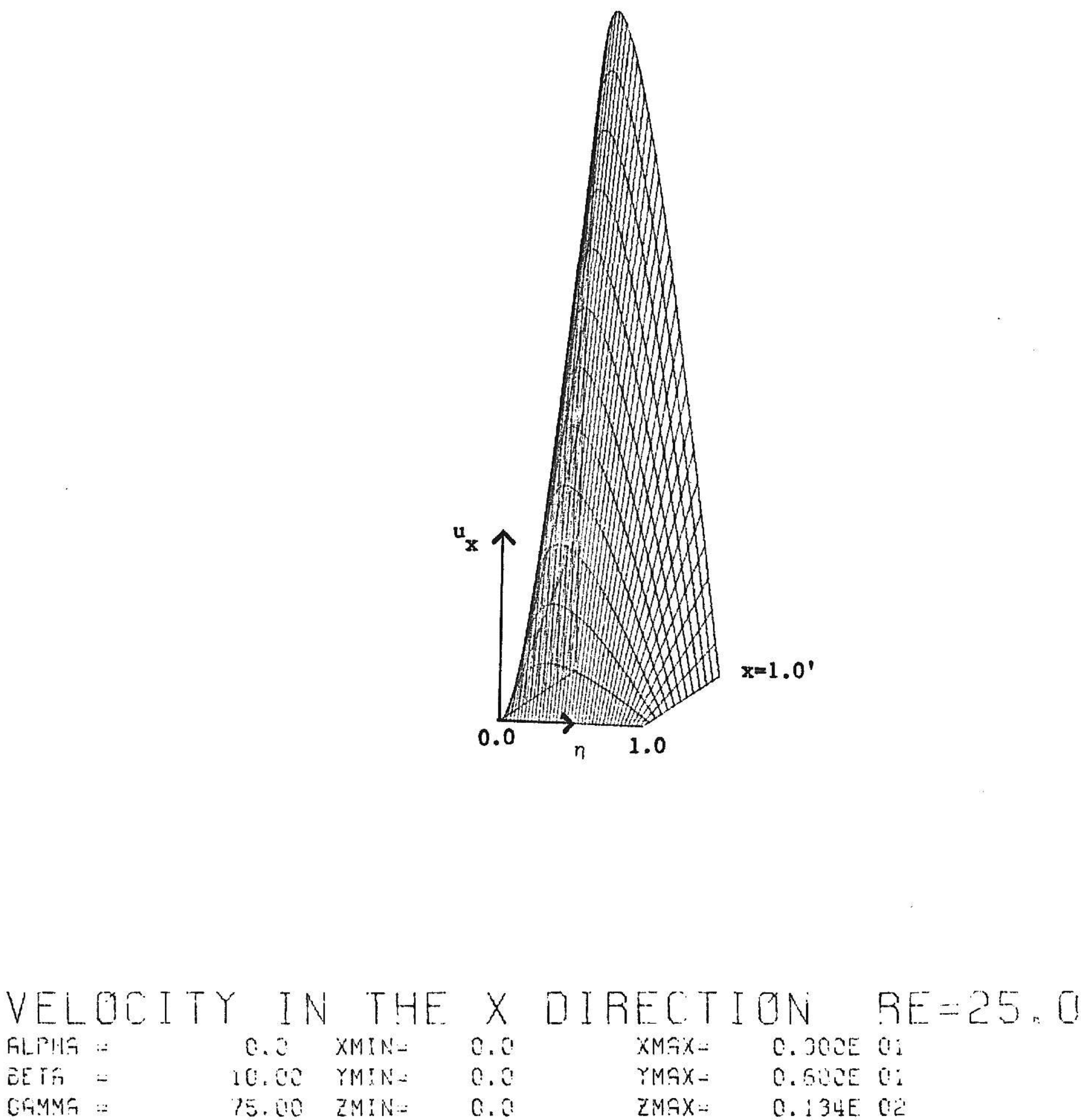


Figure 8: Velocity distribution in the x direction by quasilinearization, Re=25.0

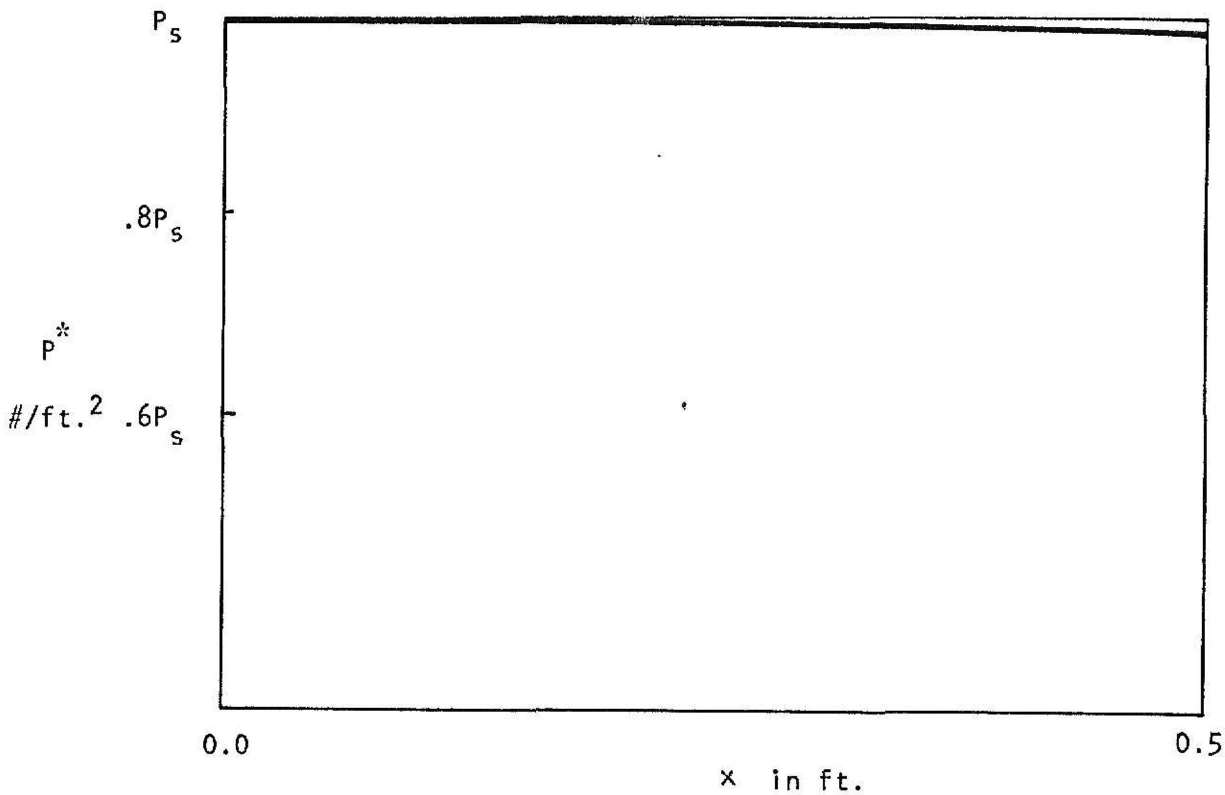


Figure 9: Relative pressure at $\eta = 0.5$ versus x by quasilinearization,Re=1.

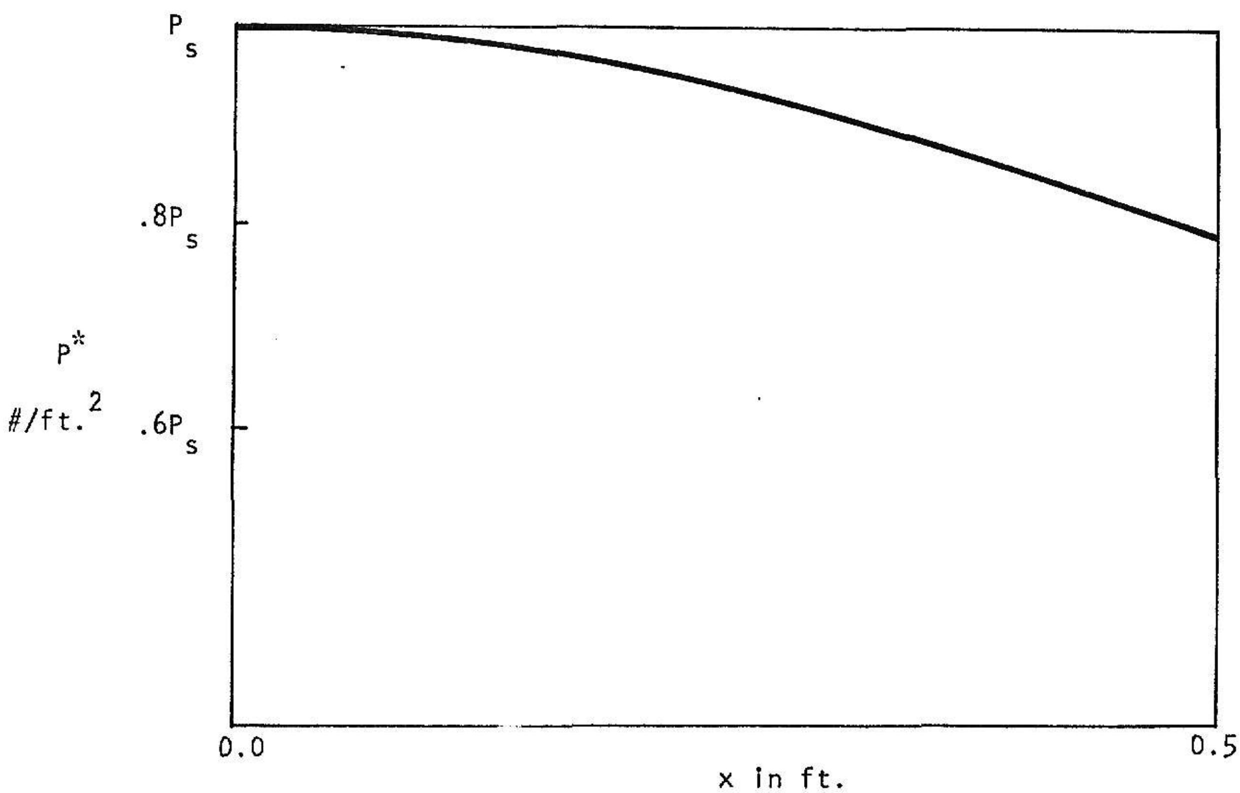


Figure 10: Relative pressure at $\eta = 0.5$ versus x by quasilinearization,Re=25.

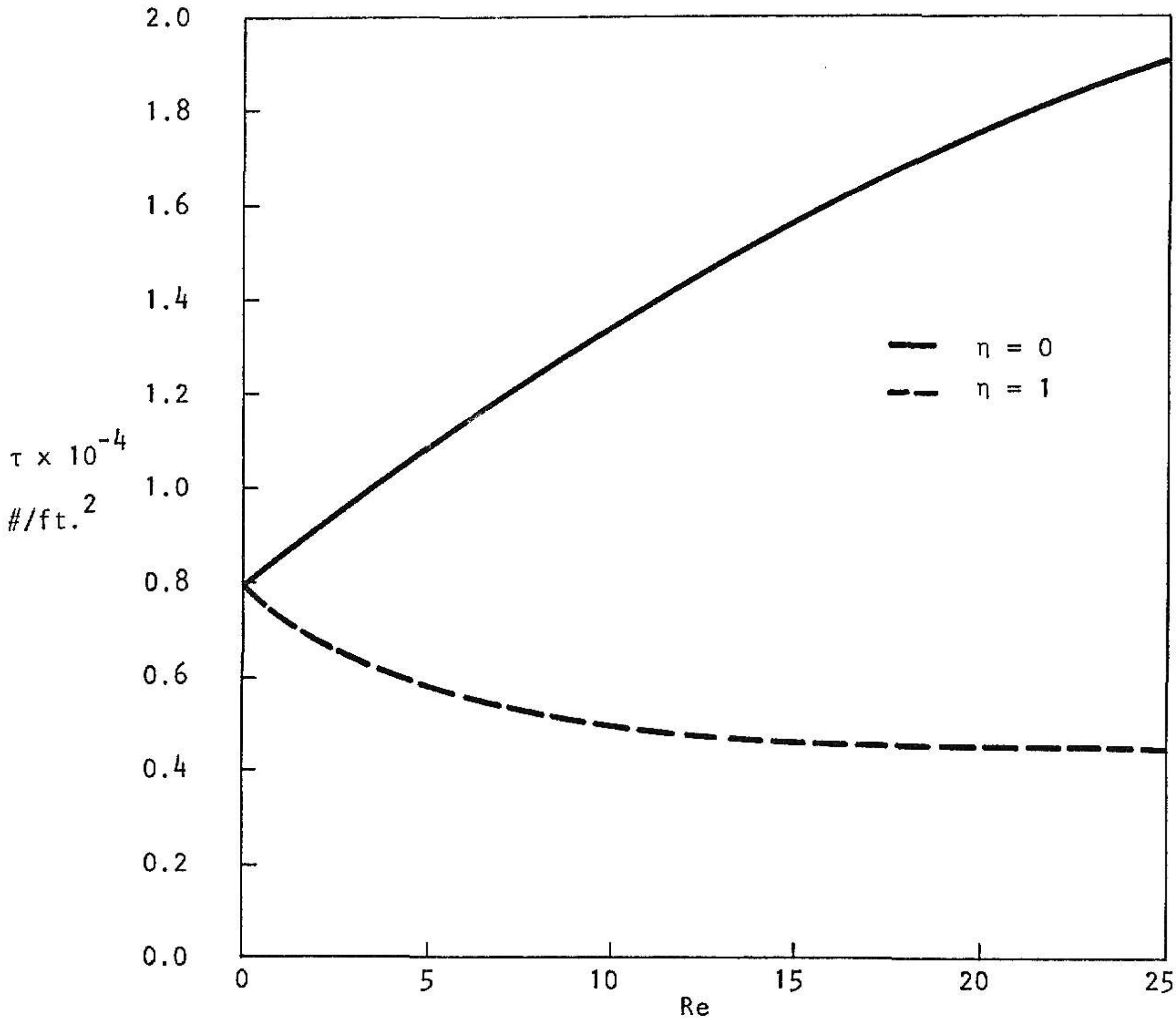


Figure 11: Wall shear stress in the x direction versus Re.

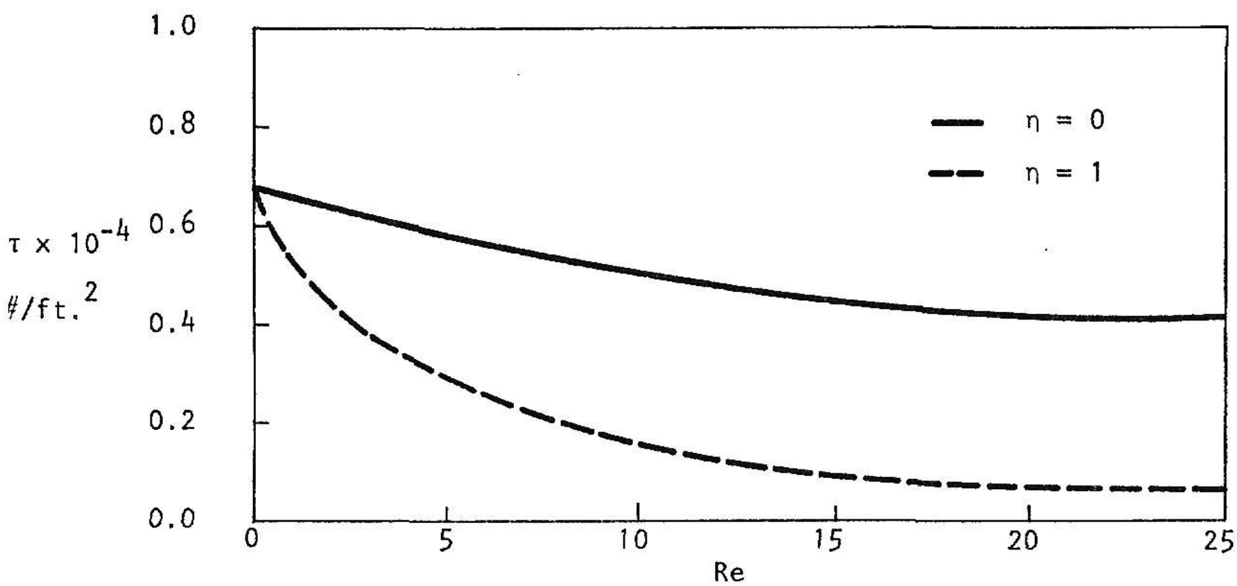


Figure 12: Wall shear stress in the z direction versus Re.

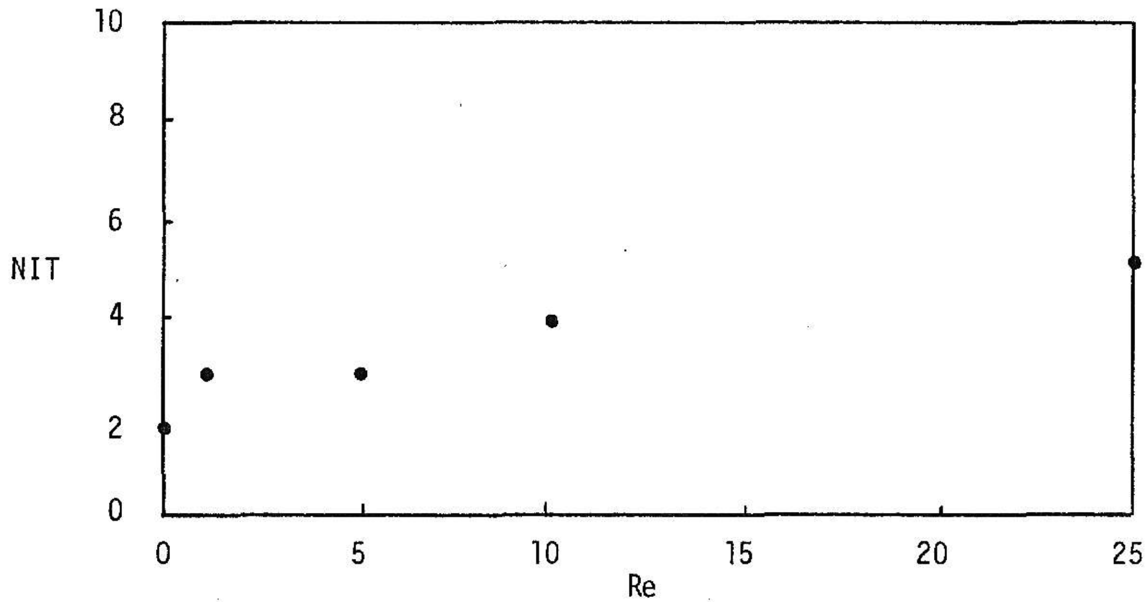


Figure 13: Number of iterations required for convergence versus Re.

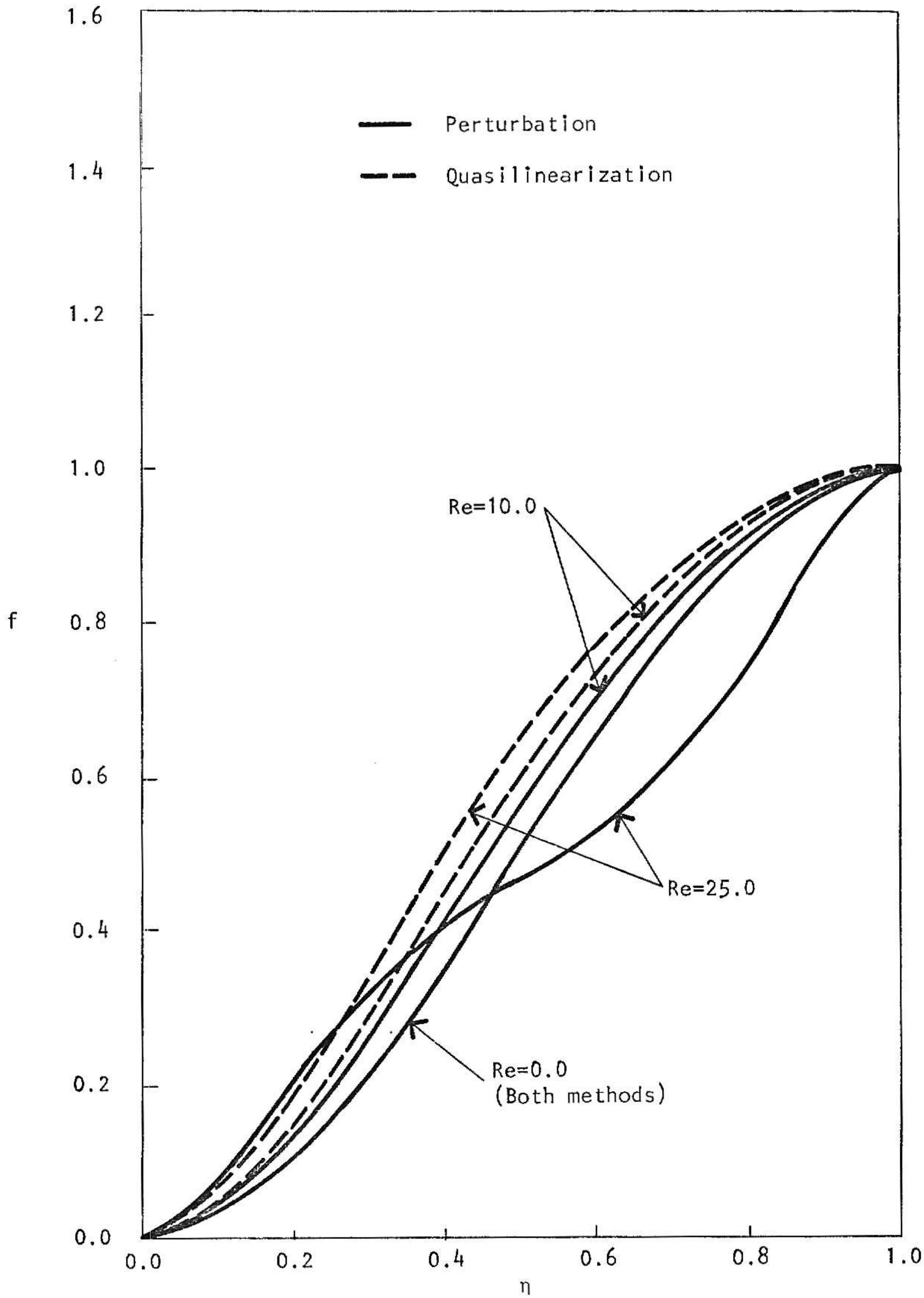


Figure 14: Normal velocity function versus eta
by perturbation,Re=0,10,25.

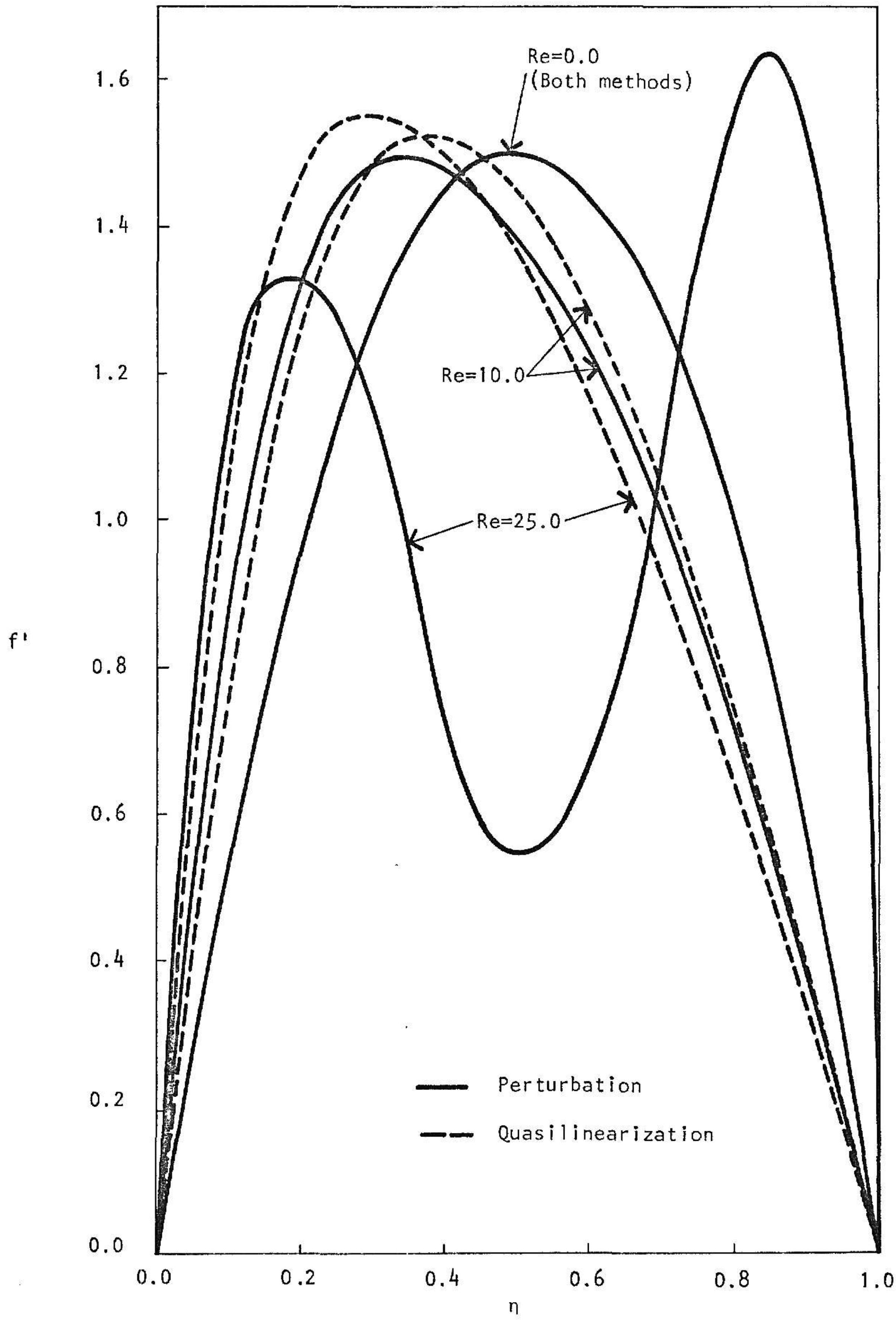


Figure 15: Tangential velocity function versus eta by perturbation, Re=0,10,25.

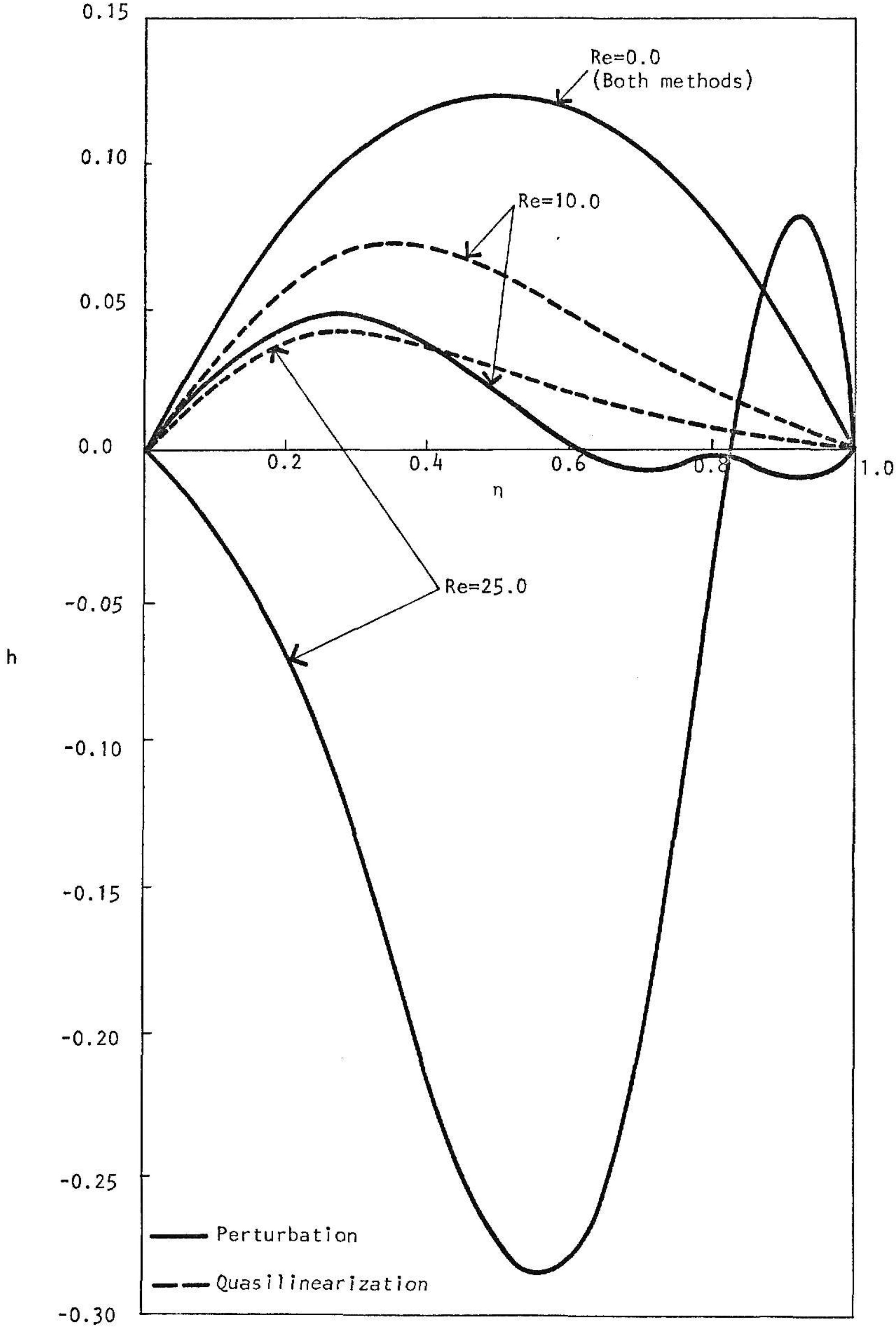


Figure 16: Axial velocity function versus eta
by perturbation,Re=0,10,25.

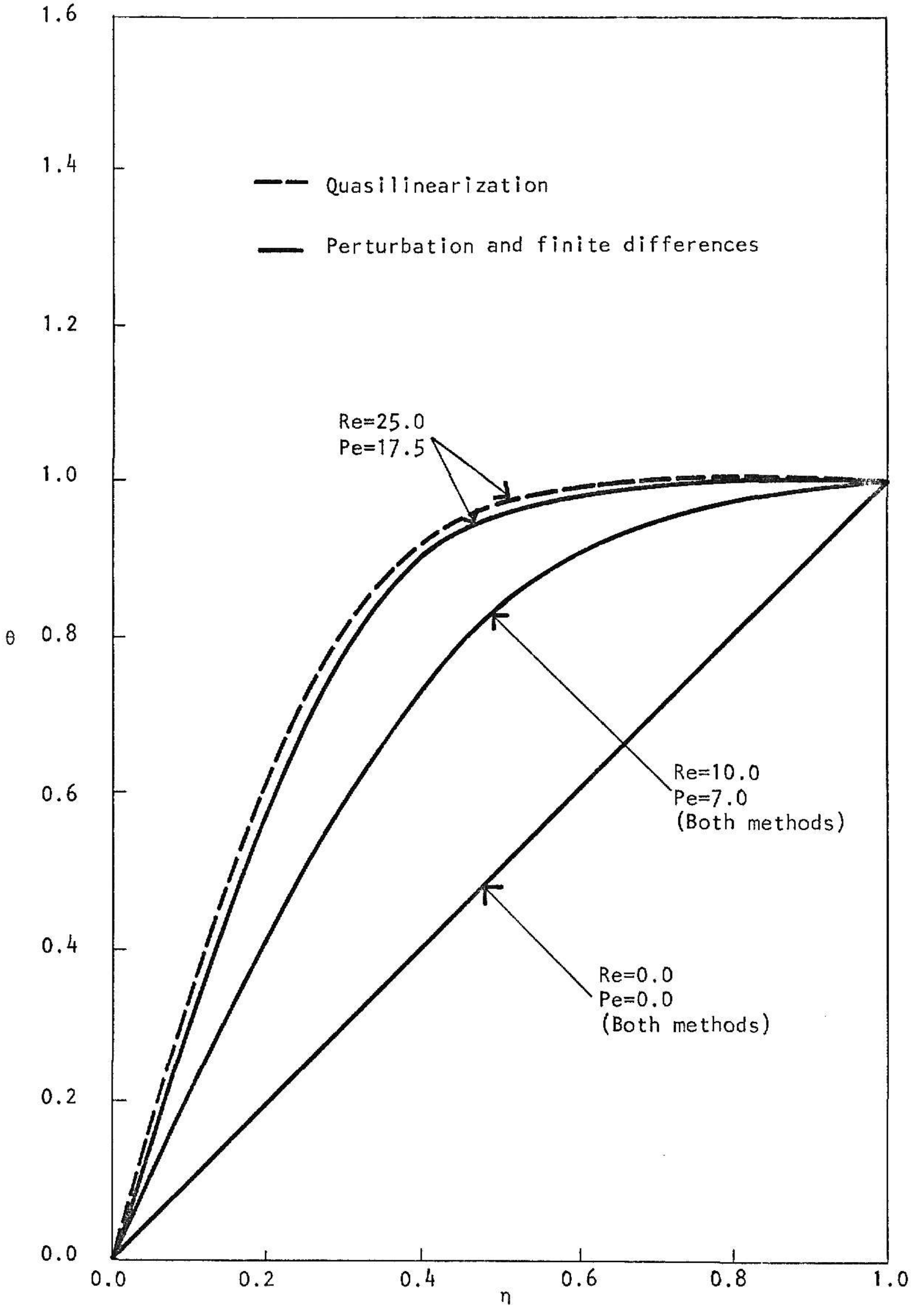


Figure 17: Temperature distribution function versus eta by perturbation and finite differences, Re=0,10,25.

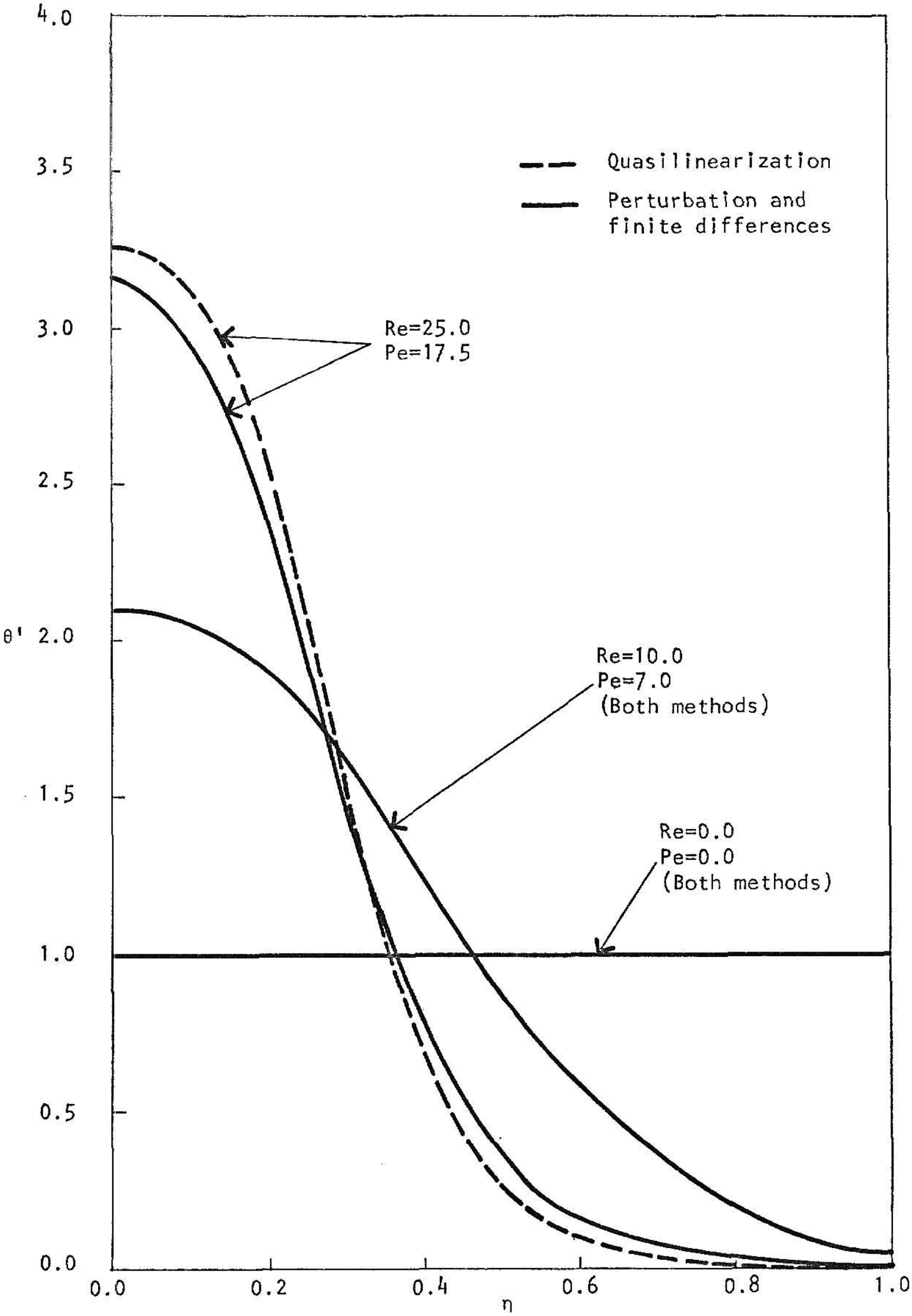


Figure 18: Heat transfer rate function versus eta
by perturbation and finite differences
Re=0,10,25.

CHAPTER V

DISCUSSION AND CONCLUSIONS

The physical model and its solutions by the quasilinearization and perturbation methods for various cross flow Reynolds numbers have been studied. A comparison of the results for the quasilinearization technique and the perturbation method for three selected Reynolds numbers was made. It is evident, by inspection of the computer programs in the Appendices, that the quasilinearization method requires considerably more computational time than the perturbation technique.

Inspecting the curves of Figures 14, 15 and 16 one can see that, although the quasilinearization technique requires approximately six times the computational time, the results obtained by it are far superior to those obtained by the perturbation technique. Furthermore, the quasilinearization method allows one to readily obtain solutions for the temperature distribution and heat transfer rate functions which the singular perturbation technique does not. For larger cross flow Reynolds numbers, the perturbation method requires that one solve a nonlinear differential equation with two point boundary conditions to obtain a solution for f. Because of the apparent difficulty involved in obtaining an approximate solution by the perturbation method, one would be more inclined to solve the governing differential equation for f.

The quasilinearization technique allows one to obtain accurate, approximate solutions for moderately large cross flow Reynolds numbers without the need of solving the governing differential equation. Furthermore, results obtained by quasilinearization are good to a designated error.

From the inspection of Figure 13 it is evident that the quasilinearization technique converges fairly rapidly. For each cross flow Reynolds number, except the initial one, the preceding solution is used as the assumed function values. This process aids in reducing the number of iterations required for convergence.

The problem presented in this report is ideal for being solved by the quasilinearization technique. The solutions are far superior to those obtained by the perturbation method. The type of problems mentioned in the introduction have not been investigated thoroughly by the quasilinearization technique.

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BIBLIOGRAPHY

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APPENDIX I

```
      IMPLICIT REAL*8(A-H,O-Z)
C
C
C     FLUID INJECTION THROUGH ONE SIDE OF A LONG VERTICAL
C     CHANNEL BY QUASIILINEARIZATION
C
C
C             REFERENCES
C
C
C                  1.  CHIEN,SHUN-FAN,"LAMINAR FLOW THROUGH AN
C                      ANNULUS WITH PORUS WALLS",MASTERS REPORT,
C                      KSU,1972
C
C
C                  2.  WANG,CHANG-Y;,SKALAK,FRANCIS,'FLUID INJECTION
C                      THROUGH ONE SIDE OF A LONG VERTICAL CHANNEL',
C                      AICHE JOURNAL,VOL 20,#3,5-74,PAGE 603
C
C
      DIMENSION R(5),YH3(8),YH4(8),YP(8),Y( 50,8),A( 50,8),DY(8),
     1YH3S( 50,8),YH4S( 50,8),YPS( 50,8),ER( 50,8),Q(8),C(4,5),
     1YH6(8),YH8(8),YH6S( 50,8),YH8S( 50,8),P(5),T( 50),X1(43),
     1S( 50,8)
  100 FORMAT(3I3)
  101 FORMAT(F15.8)
  102 FORMAT(8X,'THE SECOND DERIVATIVE OF F AT ZERO EQUALS   ',
     1E16.8,/,
     18X,'THE SECOND DERIVATIVE OF F AT ONE EQUALS    ',E16.8,/,
     18X,'THE THIRD DERIVATIVE OF F AT ZERO EQUALS    ',E16.8,/,
     18X,'THE FIRST DERIVATIVE OF H AT ZERO EQUALS    ',E16.8,/,
     18X,'THE FIRST DERIVATIVE OF H AT ONE EQUALS     ',E16.8,//)
  103 FORMAT(8X,F8.4,5(2X,F8.4))
  104 FORMAT('1',7X,'PRESSURE DISTRIBUTION IN THE POSITIVE ',
     1'X DIRECTION FOR VARIOUS ETA',/,
     18X,'THE INJECTION VELOCITY IS    ',E16.8,'   FT PER SEC',//)
  105 FORMAT('1',7X,'REYNOLDS NUMBER EQUALS                 ',E16.8,/,
     18X,'PECLET NUMBER EQUALS                ',E16.8,/,
     18X,'THE INTERVAL SIZE EQUALS            ',E16.8,/,
     18X,'THE NUMBER OF ITERATIONS EQUALS  ',I2,//)
  106 FORMAT('1',2X,I4,' ITERATIONS HAVE BEEN PERFORMED WITHOUT
     1THE SOLUTION CONVERGING WITHIN ERRMIN',//)
  107 FORMAT(11X,'ETA',8X,'F',8X,'FDOT',7X,'H',9X,'T',8X,'TDOT')
  108 FORMAT(5F10.5)
  109 FORMAT(8X,E16.8,4X,E16.8,4X,E16.8)
  110 FORMAT(12X,'ETA EQUALS',10X,'X EQUALS',12X,'P* EQUALS')
  111 FORMAT(3F25.8)
C
C
      READ(5,100) NEG,LL,NORM
      READ(5,111) VISCNU,U,RHO
      READ(5,108) R
      READ(5,108) P
C
C     COMPUTATION OF THE INTERVAL SIZE H
C
      H=1.0/(LL-1)
C
```

```
C
C     ASSUMED VALUES OF Y ARE READ IN HERE
C
C
      READ(5,101) AINIT
      DO 202 J=1,LL
      DO 202 I=1,NEQ
  202 Y(J,I)=AINIT
C
C
C     COMPUTATION OF THE SOLUTION FOR VARIOUS REYNOLDS NUMBERS
C
C
      DO 702 NORN=1,NORM
      RE=R(NORN)
      PE=P(NORN)
      V=RE*VISCNU/U
      INT=LL-1
C
C     NIT IS THE NUMBER OF ITERATIONS
C
      NIT=0
  200 NIT=NIT+1
      IF(NIT.GT.15) GO TO 701
      DO 201 I=1,LL
      DO 201 J=1,NEQ
  201 S(I,J)=Y(I,J)
C
C
C               CONSTRUCTION OF THE PARTICULAR AND HOMOGENEOUS
C               SOLUTIONS.  THE PARTICULAR SOLUTION SATISFIES
C               NEQ-M INITIAL CONDITIONS.  THE HOMOGENEOUS
C               SOLUTIONS ARE FORCED TO SATISFY THE REMAINING
C               FINAL CONDITIONS.
C
C
                DO 300 I=1,NEQ
                YH3(I)=0.0
                YH4(I)=0.0
                YH6(I)=0.0
                YH8(I)=0.0
  300           YP(I)=0.0
                YH3(3)=1.0
                YH4(4)=1.0
                YH6(6)=1.0
                YH8(8)=1.0
C
C
C
C
C               RKG INTEGRATION OF YH3
C
C                  YH3 STORED AND Q(I) SET EQUAL TO ZERO
C
                   DO 301 I=1,NEQ
                   Q(I)=0.0
  301              YH3S(1,I)=YH3(I)
C
```

```
C              INITIALIZE X ( STARTING VALUE )
C
               X=0.0
C
      DO 303 JJ=1,INT
      DO 302 J=1,NEQ
  302 A(JJ,J)=Y(JJ,J)
      CALL RKG(H,X,YH3,DY,Q,A,JJ,RE,1,NEQ,PE)
      DO 303 N=1,NEQ
  303 YH3S(JJ+1,N)=YH3(N)
C
C
C              END OF INTEGRATION FOR YH3
C
C
C              RKG INTEGRATION OF YH4
C
C                 YH4 STORED AND Q(I) SET EQUAL TO ZERO
C
                  DO 304 I=1,NEQ
                  Q(I)=0.0
  304             YH4S(1,I)=YH4(I)
C
C              INITIALIZE X ( STARTING VALUE )
C
               X=0.0
C
      DO 306 JJ=1,INT
      DO 305 J=1,NEQ
  305 A(JJ,J)=Y(JJ,J)
      CALL RKG(H,X,YH4,DY,Q,A,JJ,RE,1,NEQ,PE)
      DO 306 N=1,NEQ
  306 YH4S(JJ+1,N)=YH4(N)
C
C
C              END OF INTEGRATION FOR YH4
C
C
C
C              RKG INTEGRATION OF YH6
C
C                 YH6 STORED AND Q(I) SET EQUAL TO ZERO
C
                  DO 310 I=1,NEQ
                  Q(I)=0.0
  310             YH6S(1,I)=YH6(I)
C
C              INITIALIZE X ( STARTING VALUE )
C
               X=0.0
C
      DO 312 JJ=1,INT
      DO 311 J=1,NEQ
  311 A(JJ,J)=Y(JJ,J)
      CALL RKG(H,X,YH6,DY,Q,A,JJ,RE,1,NEQ,PE)
      DO 312 N=1,NEQ
  312 YH6S(JJ+1,N)=YH6(N)
C
```

```
C
C               END OF INTEGRATION FOR YH6
C
C               RKG INTEGRATION OF YH8
C
C                  YH8 STORED AND Q(I) SET EQUAL TO ZERO
C
                   DO 313 I=1,NEQ
                   Q(I)=0.0
  313              YH8S(1,I)=YH8(I)
C
C               INITIALIZE X ( STARTING VALUE )
C
                X=0.0
C
      DO 315 JJ=1,INT
      DO 314 J=1,NEQ
  314 A(JJ,J)=Y(JJ,J)
      CALL RKG(H,X,YH8,DY,Q,A,JJ,RE,1,NEQ,PE)
      DO 315 N=1,NEQ
  315 YH8S(JJ+1,N)=YH8(N)
C
C
C               END OF INTEGRATION FOR YH8
C
C
C
C
C
C               RKG INTEGRATION OF THE PARTICULAR SOLUTION
C
C                  YP STORED AND Q(I) SET EQUAL TO ZERO
C
                   DO 307 I=1,NEQ
                   Q(I)=0.0
  307              YPS(1,I)=YP(I)
C
C               INITIALIZE X ( STARTING VALUE )
C
                X=0.0
C
      DO 309 JJ=1,INT
      DO 308 J=1,NEQ
  308 A(JJ,J)=Y(JJ,J)
      CALL RKG(H,X,YP,DY,Q,A,JJ,RE,2,NEQ,PE)
      DO 309 N=1,NEQ
  309 YPS(JJ+1,N)=YP(N)
C
C               END OF INTEGRATION FOR YP
C
C
C            SOLVING FOR THE CONSTANTS IN THE HOMOGENEOUS SOLUTIONS
C
C
      C(1,1)=YH3S(LL,1)
      C(1,2)=YH4S(LL,1)
      C(1,3)=YH6S(LL,1)
      C(1,4)=YH8S(LL,1)
```

```
      C(1,5)=1.0-YPS(LL,1)
      C(2,1)=YH3S(LL,2)
      C(2,2)=YH4S(LL,2)
      C(2,3)=YH6S(LL,2)
      C(2,4)=YH8S(LL,2)
      C(2,5)=-YPS(LL,2)
      C(3,1)=YH3S(LL,5)
      C(3,2)=YH4S(LL,5)
      C(3,3)=YH6S(LL,5)
      C(3,4)=YH8S(LL,5)
      C(3,5)=-YPS(LL,5)
      C(4,1)=YH3S(LL,7)
      C(4,2)=YH4S(LL,7)
      C(4,3)=YH6S(LL,7)
      C(4,4)=YH8S(LL,7)
      C(4,5)=1.0-YPS(LL,7)
C
      CALL GAUSSJ(C)
C
      BC1=C(1,5)
      BC2=C(2,5)
      BC3=C(3,5)
      BC4=C(4,5)
C
C
C     COMPUTATION OF THE SOLUTION
C
C
      DO 400 I=1,LL
      DO 400 J=1,NEQ
  400 Y(I,J)=YPS(I,J)+BC1*YH3S(I,J)+BC2*YH4S(I,J)+BC3*YH6S(I,J)+
     1BC4*YH8S(I,J)
C
C
C             ERROR TEST
C
C
              ERRMIN=0.0001
              DO 401 I=1,LL
              DO 401 J=1,NEQ
              ER(I,J)=DABS(Y(I,J)-S(I,J))
              IF(ER(I,J).GT.ERRMIN) GO TO 200
  401         CONTINUE
C
C
C     OUTPUT
C
      WRITE(6,105) RE,PE,H,NIT
      WRITE(6,102) Y(1,3),Y(LL,3),Y(1,4),Y(1,6),Y(LL,6)
      WRITE(6,107)
      X=-H
      DO 699 J=1,LL
      X=X+H
  699 WRITE(6,103) X,Y(J,1),Y(J,2),Y(J,5),Y(J,7),Y(J,8)
      IF(NORN.EQ.1) GO TO 702
      WRITE(6,104)V
      J=-3
  801 J=J+4
```

```
      WRITE(6,110)
      ETA=H*(J-1)
      X=-0.1
  800 X=X+0.1
      FLPRES=(-RHO/2.0)*(((-Y(1,4)/RE)*(V/U)**2.0)*(X)**2.0
     1+(V**2.0)*(Y(J,1)**2.0)
     1+2.0*VISCNU*V*Y(J,2)/U)
      WRITE(6,109) ETA,X,FLPRES
      IF(X.LT.0.48) GO TO 800
      IF(J.LT.LL) GO TO 801
      GO TO 702
  701 NIT=NIT-1
      WRITE(6,106) NIT
  702 CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE RKG(H,X,Y,DY,Q,D,K,RE,NP,NEQ,PE)
      IMPLICIT REAL*8(A-H,O-Z)
C
C
C     THE AUTHOR WISHES TO THANK DR. HUGH WALKER FOR PROVIDING
C     THIS SUBROUTINE.  SPRING SEMESTER 1975, ENGINEERING
C     ANALYSIS I.
C
C
      DIMENSION Y(NEQ),DY(NEQ),Q(NEQ),A(2),D( 50,NEQ)
      A(1)=0.29289321881345
      A(2)=1.70710678118654
      H2=0.5*H
      CALL DERIV(X,Y,DY,D,K,RE,NP,NEQ,PE)
      DO 100 I=1,NEQ
      B=H2*DY(I)-Q(I)
      Y(I)=Y(I)+B
  100 Q(I)=Q(I)+3.0*B-H2*DY(I)
      X=X+H2
      DO 101 J=1,2
      CALL DERIV(X,Y,DY,D,K,RE,NP,NEQ,PE)
      DO 101 I=1,NEQ
      B=A(J)*(H*DY(I)-Q(I))
      Y(I)=Y(I)+B
  101 Q(I)=Q(I)+3.0*B-A(J)*H*DY(I)
      X=X+H2
      CALL DERIV(X,Y,DY,D,K,RE,NP,NEQ,PE)
      DO 102 I=1,NEQ
      B=(H*DY(I)-2.0*Q(I))/6.0
      Y(I)=Y(I)+B
  102 Q(I)=Q(I)+3.0*B-H2*DY(I)
      RETURN
      END
```

```
      SUBROUTINE DERIV(X,Y,DY,D,K,RE,NP,NEQ,PE)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(NEQ),DY(NEQ),D( 50,NEQ)
      DO 100 I=1,3
  100 DY(I)=Y(I+1)
      DY(5)=Y(6)
      DY(7)=Y(8)
      DY( 4 )=RE*(-Y(1)*D(K,4)+Y(2)*D(K,3)+Y(3)*D(K,2)-Y(4)*D(K,1)
      DY(6)=RE*(-Y(1)*D(K,6)-Y(6)*D(K,1))
      DY(8)=PE*(-Y(1)*D(K,8)-Y(8)*D(K,1))
      IF(NP.EQ.2) GO TO 101
      GO TO 102
  101 DY( 4 )=DY( 4 )+RE*(D(K,1)*D(K,4)-D(K,2)*D(K,3))
      DY(6)=DY(6)+RE*(D(K,1)*D(K,6))-1.0
      DY(8)=DY(8)+PE*D(K,1)*D(K,8)
  102 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE GAUSSJ(A)
      IMPLICIT REAL*8(A-H,O-Z)
C
C
C     THE AUTHOR WISHES TO THANK DR. HUGH WALKER FOR PROVIDING
C     THE BASIC FORM OF THIS SUBROUTINE.  SPRING SEMESTER 1975,
C     ENGINEERING ANALYSIS I.
C
C
      DIMENSION A(4,5)
      N=4
      N1=N+1
      DO 200 J=1,N
      DIV=A(J,J)
      S=1.0/DIV
      DO 201 K=J,N1
  201 A(J,K)=A(J,K)*S
      DO 202 I=1,N
      IF(I-J) 203,202,203
  203 AIJ=-A(I,J)
      DO 204 K=J,N1
  204 A(I,K)=A(I,K)+AIJ*A(J,K)
  202 CONTINUE
  200 CONTINUE
      RETURN
      END
```

```
REYNOLDS NUMBER EQUALS                0.0
PECLET NUMBER EQUALS                  0.0
THE INTERVAL SIZE EQUALS              0.24999999D-01
THE NUMBER OF ITERATIONS EQUALS       2

THE SECOND DERIVATIVE OF F AT ZERO EQUALS        0.60000017D 01
THE SECOND DERIVATIVE OF F AT ONE EQUALS        -0.60000017D 01
THE THIRD DERIVATIVE OF F AT ZERO EQUALS        -0.12000005D 02
THE FIRST DERIVATIVE OF H AT ZERO EQUALS         0.49999993D 00
THE FIRST DERIVATIVE OF H AT ONE EQUALS         -0.49999993D 00
```

ETA	F	FDOT	H	T	TDOT
0.0	0.0	0.0	0.0	0.0	1.0000
0.0250	0.0018	0.1463	0.0122	0.0250	1.0000
0.0500	0.0073	0.2850	0.0237	0.0500	1.0000
0.0750	0.0160	0.4163	0.0347	0.0750	1.0000
0.1000	0.0280	0.5400	0.0450	0.1000	1.0000
0.1250	0.0430	0.6563	0.0547	0.1250	1.0000
0.1500	0.0608	0.7650	0.0637	0.1500	1.0000
0.1750	0.0812	0.8663	0.0722	0.1750	1.0000
0.2000	0.1040	0.9600	0.0800	0.2000	1.0000
0.2250	0.1291	1.0463	0.0872	0.2250	1.0000
0.2500	0.1563	1.1250	0.0937	0.2500	1.0000
0.2750	0.1853	1.1963	0.0997	0.2750	1.0000
0.3000	0.2160	1.2600	0.1050	0.3000	1.0000
0.3250	0.2482	1.3163	0.1097	0.3250	1.0000
0.3500	0.2818	1.3650	0.1137	0.3500	1.0000
0.3750	0.3164	1.4063	0.1172	0.3750	1.0000
0.4000	0.3520	1.4400	0.1200	0.4000	1.0000
0.4250	0.3883	1.4663	0.1222	0.4250	1.0000
0.4500	0.4253	1.4850	0.1237	0.4500	1.0000
0.4750	0.4625	1.4963	0.1247	0.4750	1.0000
0.5000	0.5000	1.5000	0.1250	0.5000	1.0000
0.5250	0.5375	1.4963	0.1247	0.5250	1.0000
0.5500	0.5748	1.4850	0.1237	0.5500	1.0000
0.5750	0.6117	1.4663	0.1222	0.5750	1.0000
0.6000	0.6480	1.4400	0.1200	0.6000	1.0000
0.6250	0.6836	1.4063	0.1172	0.6250	1.0000
0.6500	0.7183	1.3650	0.1137	0.6500	1.0000
0.6750	0.7518	1.3163	0.1097	0.6750	1.0000
0.7000	0.7840	1.2600	0.1050	0.7000	1.0000
0.7250	0.8147	1.1963	0.0997	0.7250	1.0000
0.7500	0.8438	1.1250	0.0937	0.7500	1.0000
0.7750	0.8709	1.0463	0.0872	0.7750	1.0000
0.8000	0.8960	0.9600	0.0800	0.8000	1.0000
0.8250	0.9188	0.8663	0.0722	0.8250	1.0000
0.8500	0.9393	0.7650	0.0637	0.8500	1.0000
0.8750	0.9570	0.6563	0.0547	0.8750	1.0000
0.9000	0.9720	0.5400	0.0450	0.9000	1.0000
0.9250	0.9840	0.4163	0.0347	0.9250	1.0000
0.9500	0.9928	0.2850	0.0237	0.9500	1.0000
0.9750	0.9982	0.1463	0.0122	0.9750	1.0000
1.0000	1.0000	0.0000	-0.0000	1.0000	1.0000

REYNOLDS NUMBER EQUALS 0.10000000D 01
PECLET NUMBER EQUALS 0.70000000D 00
THE INTERVAL SIZE EQUALS 0.24999999D-01
THE NUMBER OF ITERATIONS EQUALS 3

THE SECOND DERIVATIVE OF F AT ZERO EQUALS 0.64546901D 01
THE SECOND DERIVATIVE OF F AT ONE EQUALS -0.55089772D 01
THE THIRD DERIVATIVE OF F AT ZERO EQUALS -0.14370047D 02
THE FIRST DERIVATIVE OF H AT ZERO EQUALS 0.49100100D 00
THE FIRST DERIVATIVE OF H AT ONE EQUALS -0.41536146D 00

ETA	F	FDOT	H	T	TDOT
0.0	0.0	0.0	0.0	0.0	1.1075
0.0250	0.0020	0.1569	0.0120	0.0277	1.1075
0.0500	0.0078	0.3048	0.0233	0.0554	1.1074
0.0750	0.0171	0.4437	0.0340	0.0831	1.1071
0.1000	0.0299	0.5738	0.0441	0.1107	1.1067
0.1250	0.0458	0.6950	0.0535	0.1384	1.1060
0.1500	0.0646	0.8073	0.0623	0.1660	1.1049
0.1750	0.0861	0.9109	0.0705	0.1936	1.1034
0.2000	0.1100	1.0059	0.0781	0.2212	1.1016
0.2250	0.1363	1.0922	0.0849	0.2487	1.0992
0.2500	0.1646	1.1701	0.0912	0.2761	1.0963
0.2750	0.1947	1.2395	0.0968	0.3035	1.0929
0.3000	0.2265	1.3005	0.1017	0.3308	1.0888
0.3250	0.2597	1.3534	0.1060	0.3579	1.0842
0.3500	0.2941	1.3980	0.1096	0.3850	1.0790
0.3750	0.3295	1.4347	0.1126	0.4119	1.0731
0.4000	0.3657	1.4633	0.1150	0.4386	1.0666
0.4250	0.4026	1.4841	0.1167	0.4652	1.0595
0.4500	0.4399	1.4971	0.1177	0.4916	1.0517
0.4750	0.4774	1.5025	0.1181	0.5178	1.0433
0.5000	0.5149	1.5003	0.1179	0.5438	1.0342
0.5250	0.5523	1.4905	0.1171	0.5695	1.0246
0.5500	0.5894	1.4734	0.1157	0.5950	1.0144
0.5750	0.6260	1.4489	0.1137	0.6202	1.0037
0.6000	0.6618	1.4172	0.1110	0.6452	0.9925
0.6250	0.6968	1.3784	0.1078	0.6698	0.9807
0.6500	0.7307	1.3326	0.1041	0.6942	0.9685
0.6750	0.7633	1.2797	0.0998	0.7183	0.9560
0.7000	0.7946	1.2200	0.0949	0.7420	0.9430
0.7250	0.8243	1.1535	0.0895	0.7654	0.9297
0.7500	0.8522	1.0803	0.0837	0.7885	0.9162
0.7750	0.8782	1.0004	0.0773	0.8112	0.9024
0.8000	0.9022	0.9141	0.0704	0.8336	0.8885
0.8250	0.9239	0.8213	0.0631	0.8556	0.8744
0.8500	0.9432	0.7221	0.0553	0.8773	0.8602
0.8750	0.9599	0.6167	0.0471	0.8987	0.8460
0.9000	0.9740	0.5052	0.0385	0.9196	0.8318
0.9250	0.9851	0.3877	0.0295	0.9402	0.8177
0.9500	0.9933	0.2642	0.0200	0.9605	0.8036
0.9750	0.9983	0.1349	0.0102	0.9804	0.7897
1.0000	1.0000	0.0000	-0.0000	1.0000	0.7760

PRESSURE DISTRIBUTION IN THE POSITIVE X DIRECTION FOR VARIOUS ETA
THE INJECTION VELOCITY IS 0.47472008D-02 FT PER SEC

ETA EQUALS	X EQUALS	P* EQUALS
0.0	0.0	0.0
0.0	0.99999964D-01	-0.62954891D-04
0.0	0.19999993D 00	-0.25181957D-03
0.0	0.29999989D 00	-0.56659402D-03
0.0	0.39999986D 00	-0.10072783D-02
0.0	0.49999982D 00	-0.15738723D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D-01	0.0	-0.87349839D-06
0.99999994D-01	0.99999964D-01	-0.63828390D-04
0.99999994D-01	0.19999993D 00	-0.25269306D-03
0.99999994D-01	0.29999989D 00	-0.56746752D-03
0.99999994D-01	0.39999986D 00	-0.10081518D-02
0.99999994D-01	0.49999982D 00	-0.15747458D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.19999999D 00	0.0	-0.15393225D-05
0.19999999D 00	0.99999964D-01	-0.64494214D-04
0.19999999D 00	0.19999993D 00	-0.25335889D-03
0.19999999D 00	0.29999989D 00	-0.56813335D-03
0.19999999D 00	0.39999986D 00	-0.10088176D-02
0.19999999D 00	0.49999982D 00	-0.15754116D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.29999998D 00	0.0	-0.20173566D-05
0.29999998D 00	0.99999964D-01	-0.64972248D-04
0.29999998D 00	0.19999993D 00	-0.25383692D-03
0.29999998D 00	0.29999989D 00	-0.56861138D-03
0.29999998D 00	0.39999986D 00	-0.10092956D-02
0.29999998D 00	0.49999982D 00	-0.15758896D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.39999998D 00	0.0	-0.23277191D-05
0.39999998D 00	0.99999964D-01	-0.65282611D-04
0.39999998D 00	0.19999993D 00	-0.25414728D-03
0.39999998D 00	0.29999989D 00	-0.56892174D-03
0.39999998D 00	0.39999986D 00	-0.10096060D-02
0.39999998D 00	0.49999982D 00	-0.15762000D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.49999997D 00	0.0	-0.24838294D-05
0.49999997D 00	0.99999964D-01	-0.65438721D-04
0.49999997D 00	0.19999993D 00	-0.25430340D-03
0.49999997D 00	0.29999989D 00	-0.56907785D-03
0.49999997D 00	0.39999986D 00	-0.10097621D-02
0.49999997D 00	0.49999982D 00	-0.15763561D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.59999996D 00	0.0	-0.24889953D-05
0.59999996D 00	0.99999964D-01	-0.65443887D-04
0.59999996D 00	0.19999993D 00	-0.25430856D-03
0.59999996D 00	0.29999989D 00	-0.56908302D-03
0.59999996D 00	0.39999986D 00	-0.10097673D-02
0.59999996D 00	0.49999982D 00	-0.15763613D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.69999996D 00	0.0	-0.23360372D-05
0.69999996D 00	0.99999964D-01	-0.65290929D-04
0.69999996D 00	0.19999993D 00	-0.25415560D-03
0.69999996D 00	0.29999989D 00	-0.56893006D-03
0.69999996D 00	0.39999986D 00	-0.10096143D-02
0.69999996D 00	0.49999982D 00	-0.15762083D-02

ETA EQUALS	X EQUALS	P* EQUALS
0.79999995D 00	0.0	-0.20094948D-05
0.79999995D 00	0.99999964D-01	-0.64964386D-04
0.79999995D 00	0.19999993D 00	-0.25382906D-03
0.79999995D 00	0.29999989D 00	-0.56860352D-03
0.79999995D 00	0.39999986D 00	-0.10092878D-02
0.79999995D 00	0.49999982D 00	-0.15758818D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.89999995D 00	0.0	-0.14900190D-05
0.89999995D 00	0.99999964D-01	-0.64444910D-04
0.89999995D 00	0.19999993D 00	-0.25330958D-03
0.89999995D 00	0.29999989D 00	-0.56808404D-03
0.89999995D 00	0.39999986D 00	-0.10087683D-02
0.89999995D 00	0.49999982D 00	-0.15753623D-02
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D 00	0.0	-0.76058713D-06
0.99999994D 00	0.99999964D-01	-0.63715479D-04
0.99999994D 00	0.19999993D 00	-0.25258015D-03
0.99999994D 00	0.29999989D 00	-0.56735461D-03
0.99999994D 00	0.39999986D 00	-0.10080389D-02
0.99999994D 00	0.49999982D 00	-0.15746329D-02

REYNOLDS NUMBER EQUALS 0.50000000D 01
PECLET NUMBER EQUALS 0.35000000D 01
THE INTERVAL SIZE EQUALS 0.24999999D-01
THE NUMBER OF ITERATIONS EQUALS 4

THE SECOND DERIVATIVE OF F AT ZERO EQUALS 0.81788023D 01
THE SECOND DERIVATIVE OF F AT ONE EQUALS -0.43086597D 01
THE THIRD DERIVATIVE OF F AT ZERO EQUALS -0.24630688D 02
THE FIRST DERIVATIVE OF H AT ZERO EQUALS 0.44615801D 00
THE FIRST DERIVATIVE OF H AT ONE EQUALS -0.20208820D 00

ETA	F	FDOT	H	T	TDOT
0.0	0.0	0.0	0.0	0.0	1.5590
0.0250	0.0025	0.1968	0.0108	0.0390	1.5589
0.0500	0.0097	0.3782	0.0211	0.0779	1.5581
0.0750	0.0213	0.5445	0.0306	0.1169	1.5561
0.1000	0.0368	0.6960	0.0396	0.1557	1.5521
0.1250	0.0560	0.8329	0.0478	0.1945	1.5459
0.1500	0.0783	0.9557	0.0554	0.2330	1.5369
0.1750	0.1036	1.0648	0.0623	0.2713	1.5247
0.2000	0.1315	1.1608	0.0684	0.3092	1.5091
0.2250	0.1616	1.2442	0.0739	0.3467	1.4899
0.2500	0.1936	1.3154	0.0786	0.3837	1.4669
0.2750	0.2272	1.3751	0.0826	0.4200	1.4402
0.3000	0.2622	1.4238	0.0859	0.4556	1.4096
0.3250	0.2983	1.4621	0.0884	0.4905	1.3754
0.3500	0.3353	1.4904	0.0903	0.5244	1.3378
0.3750	0.3728	1.5093	0.0914	0.5573	1.2970
0.4000	0.4107	1.5193	0.0919	0.5892	1.2532
0.4250	0.4487	1.5208	0.0917	0.6200	1.2069
0.4500	0.4866	1.5144	0.0910	0.6495	1.1585
0.4750	0.5243	1.5004	0.0897	0.6779	1.1083
0.5000	0.5616	1.4792	0.0878	0.7049	1.0568
0.5250	0.5982	1.4512	0.0855	0.7307	1.0045
0.5500	0.6341	1.4168	0.0828	0.7552	0.9517
0.5750	0.6690	1.3762	0.0796	0.7783	0.8989
0.6000	0.7029	1.3299	0.0761	0.8001	0.8465
0.6250	0.7355	1.2780	0.0723	0.8206	0.7948
0.6500	0.7667	1.2209	0.0683	0.8399	0.7442
0.6750	0.7965	1.1588	0.0640	0.8579	0.6949
0.7000	0.8246	1.0920	0.0595	0.8746	0.6473
0.7250	0.8510	1.0206	0.0549	0.8902	0.6015
0.7500	0.8756	0.9450	0.0502	0.9047	0.5577
0.7750	0.8983	0.8653	0.0453	0.9181	0.5160
0.8000	0.9188	0.7817	0.0404	0.9305	0.4766
0.8250	0.9373	0.6946	0.0354	0.9420	0.4394
0.8500	0.9535	0.6039	0.0304	0.9525	0.4045
0.8750	0.9675	0.5101	0.0253	0.9622	0.3718
0.9000	0.9790	0.4132	0.0203	0.9711	0.3415
0.9250	0.9881	0.3136	0.0152	0.9793	0.3133
0.9500	0.9947	0.2113	0.0101	0.9868	0.2873
0.9750	0.9987	0.1067	0.0051	0.9937	0.2633
1.0000	1.0000	-0.0000	-0.0000	1.0000	0.2412

PRESSURE DISTRIBUTION IN THE POSITIVE X DIRECTION FOR VARIOUS ETA
THE INJECTION VELOCITY IS 0.23736004D-01 FT PER SEC

ETA EQUALS	X EQUALS	P* EQUALS
0.0	0.0	0.0
0.0	0.99999964D-01	-0.53953278D-03
0.0	0.19999993D 00	-0.21581311D-02
0.0	0.29999989D 00	-0.48557950D-02
0.0	0.39999986D 00	-0.86325245D-02
0.0	0.49999982D 00	-0.13488320D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D-01	0.0	-0.53193620D-05
0.99999994D-01	0.99999964D-01	-0.54485214D-03
0.99999994D-01	0.19999993D 00	-0.21634505D-02
0.99999994D-01	0.29999989D 00	-0.48611144D-02
0.99999994D-01	0.39999986D 00	-0.86378438D-02
0.99999994D-01	0.49999982D 00	-0.13493639D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.19999999D 00	0.0	-0.91576346D-05
0.19999999D 00	0.99999964D-01	-0.54869041D-03
0.19999999D 00	0.19999993D 00	-0.21672888D-02
0.19999999D 00	0.29999989D 00	-0.48649527D-02
0.19999999D 00	0.39999986D 00	-0.86416821D-02
0.19999999D 00	0.49999982D 00	-0.13497477D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.29999998D 00	0.0	-0.12137032D-04
0.29999998D 00	0.99999964D-01	-0.55166981D-03
0.29999998D 00	0.19999993D 00	-0.21702682D-02
0.29999998D 00	0.29999989D 00	-0.48679321D-02
0.29999998D 00	0.39999986D 00	-0.86446615D-02
0.29999998D 00	0.49999982D 00	-0.13500457D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.39999998D 00	0.0	-0.14762155D-04
0.39999998D 00	0.99999964D-01	-0.55429494D-03
0.39999998D 00	0.19999993D 00	-0.21728933D-02
0.39999998D 00	0.29999989D 00	-0.48705572D-02
0.39999998D 00	0.39999986D 00	-0.86472866D-02
0.39999998D 00	0.49999982D 00	-0.13503082D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.49999997D 00	0.0	-0.17247445D-04
0.49999997D 00	0.99999964D-01	-0.55678023D-03
0.49999997D 00	0.19999993D 00	-0.21753786D-02
0.49999997D 00	0.29999989D 00	-0.48730425D-02
0.49999997D 00	0.39999986D 00	-0.86497719D-02
0.49999997D 00	0.49999982D 00	-0.13505567D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.59999996D 00	0.0	-0.19508410D-04
0.59999996D 00	0.99999964D-01	-0.55904119D-03
0.59999996D 00	0.19999993D 00	-0.21776395D-02
0.59999996D 00	0.29999989D 00	-0.48753034D-02
0.59999996D 00	0.39999986D 00	-0.86520329D-02
0.59999996D 00	0.49999982D 00	-0.13507828D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.69999996D 00	0.0	-0.21235299D-04
0.69999996D 00	0.99999964D-01	-0.56076808D-03
0.69999996D 00	0.19999993D 00	-0.21793664D-02
0.69999996D 00	0.29999989D 00	-0.48770303D-02
0.69999996D 00	0.39999986D 00	-0.86537598D-02
0.69999996D 00	0.49999982D 00	-0.13509555D-01

ETA EQUALS	X EQUALS	P* EQUALS
0.79999995D 00	0.0	-0.21999548D-04
0.79999995D 00	0.99999964D-01	-0.56153233D-03
0.79999995D 00	0.19999993D 00	-0.21801307D-02
0.79999995D 00	0.29999989D 00	-0.48777946D-02
0.79999995D 00	0.39999986D 00	-0.86545240D-02
0.79999995D 00	0.49999982D 00	-0.13510319D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.89999995D 00	0.0	-0.21368282D-04
0.89999995D 00	0.99999964D-01	-0.56090106D-03
0.89999995D 00	0.19999993D 00	-0.21794994D-02
0.89999995D 00	0.29999989D 00	-0.48771633D-02
0.89999995D 00	0.39999986D 00	-0.86538928D-02
0.89999995D 00	0.49999982D 00	-0.13509688D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D 00	0.0	-0.19014678D-04
0.99999994D 00	0.99999964D-01	-0.55854746D-03
0.99999994D 00	0.19999993D 00	-0.21771458D-02
0.99999994D 00	0.29999989D 00	-0.48748097D-02
0.99999994D 00	0.39999986D 00	-0.86515392D-02
0.99999994D 00	0.49999982D 00	-0.13507334D-01

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REYNOLDS NUMBER EQUALS                  0.10000000D 02
PECLET NUMBER EQUALS                    0.70000000D 01
THE INTERVAL SIZE EQUALS                0.24999999D-01
THE NUMBER OF ITERATIONS EQUALS         4

THE SECOND DERIVATIVE OF F AT ZERO EQUALS       0.10052270D 02
THE SECOND DERIVATIVE OF F AT ONE EQUALS       -0.36826020D 01
THE THIRD DERIVATIVE OF F AT ZERO EQUALS       -0.38268227D 02
THE FIRST DERIVATIVE OF H AT ZERO EQUALS        0.39377817D 00
THE FIRST DERIVATIVE OF H AT ONE EQUALS        -0.10370014D 00
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ETA	F	FDOT	H	T	TDOT
0.0	0.0	0.0	0.0	0.0	2.0844
0.0250	0.0030	0.2394	0.0095	0.0521	2.0841
0.0500	0.0118	0.4550	0.0184	0.1042	2.0815
0.0750	0.0256	0.6475	0.0267	0.1562	2.0749
0.1000	0.0440	0.8176	0.0342	0.2079	2.0624
0.1250	0.0663	0.9662	0.0411	0.2592	2.0427
0.1500	0.0921	1.0946	0.0472	0.3099	2.0146
0.1750	0.1209	1.2038	0.0526	0.3599	1.9774
0.2000	0.1521	1.2952	0.0572	0.4087	1.9308
0.2250	0.1855	1.3701	0.0610	0.4563	1.8745
0.2500	0.2205	1.4298	0.0640	0.5024	1.8090
0.2750	0.2569	1.4755	0.0663	0.5467	1.7348
0.3000	0.2942	1.5084	0.0678	0.5891	1.6530
0.3250	0.3322	1.5298	0.0686	0.6293	1.5647
0.3500	0.3706	1.5407	0.0687	0.6673	1.4712
0.3750	0.4091	1.5420	0.0683	0.7028	1.3739
0.4000	0.4476	1.5347	0.0672	0.7359	1.2745
0.4250	0.4858	1.5195	0.0657	0.7666	1.1743
0.4500	0.5235	1.4973	0.0638	0.7947	1.0748
0.4750	0.5606	1.4686	0.0615	0.8203	0.9773
0.5000	0.5969	1.4340	0.0590	0.8436	0.8830
0.5250	0.6323	1.3940	0.0562	0.8645	0.7927
0.5500	0.6666	1.3491	0.0532	0.8832	0.7074
0.5750	0.6997	1.2996	0.0501	0.8999	0.6275
0.6000	0.7315	1.2459	0.0470	0.9147	0.5535
0.6250	0.7620	1.1883	0.0438	0.9276	0.4856
0.6500	0.7909	1.1271	0.0406	0.9390	0.4238
0.6750	0.8183	1.0624	0.0373	0.9489	0.3680
0.7000	0.8440	0.9946	0.0342	0.9575	0.3181
0.7250	0.8680	0.9237	0.0310	0.9648	0.2738
0.7500	0.8902	0.8501	0.0279	0.9712	0.2347
0.7750	0.9105	0.7739	0.0249	0.9766	0.2004
0.8000	0.9288	0.6953	0.0219	0.9812	0.1706
0.8250	0.9452	0.6144	0.0190	0.9852	0.1448
0.8500	0.9595	0.5314	0.0162	0.9885	0.1225
0.8750	0.9718	0.4465	0.0134	0.9913	0.1035
0.9000	0.9818	0.3599	0.0106	0.9937	0.0872
0.9250	0.9897	0.2718	0.0079	0.9957	0.0734
0.9500	0.9954	0.1823	0.0052	0.9974	0.0617
0.9750	0.9989	0.0916	0.0026	0.9988	0.0518
1.0000	1.0000	0.0	-0.0000	1.0000	0.0435

PRESSURE DISTRIBUTION IN THE POSITIVE X DIRECTION FOR VARIOUS ETA
THE INJECTION VELOCITY IS 0.47472008D-01 FT PER SEC

ETA EQUALS	X EQUALS	P* EQUALS
0.0	0.0	0.0
0.0	0.99999964D-01	-0.16765234D-02
0.0	0.19999993D 00	-0.67060938D-02
0.0	0.29999989D 00	-0.15088711D-01
0.0	0.39999986D 00	-0.26824375D-01
0.0	0.49999982D 00	-0.41913086D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D-01	0.0	-0.12584024D-04
0.99999994D-01	0.99999964D-01	-0.16891075D-02
0.99999994D-01	0.19999993D 00	-0.67186778D-02
0.99999994D-01	0.29999989D 00	-0.15101295D-01
0.99999994D-01	0.39999986D 00	-0.26836959D-01
0.99999994D-01	0.49999982D 00	-0.41925670D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.19999999D 00	0.0	-0.21462659D-04
0.19999999D 00	0.99999964D-01	-0.16979861D-02
0.19999999D 00	0.19999993D 00	-0.67275565D-02
0.19999999D 00	0.29999989D 00	-0.15110174D-01
0.19999999D 00	0.39999986D 00	-0.26845838D-01
0.19999999D 00	0.49999982D 00	-0.41934549D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.29999998D 00	0.0	-0.29528514D-04
0.29999998D 00	0.99999964D-01	-0.17060520D-02
0.29999998D 00	0.19999993D 00	-0.67356223D-02
0.29999998D 00	0.29999989D 00	-0.15118240D-01
0.29999998D 00	0.39999986D 00	-0.26853904D-01
0.29999998D 00	0.49999982D 00	-0.41942615D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.39999998D 00	0.0	-0.38584070D-04
0.39999998D 00	0.99999964D-01	-0.17151075D-02
0.39999998D 00	0.19999993D 00	-0.67446779D-02
0.39999998D 00	0.29999989D 00	-0.15127295D-01
0.39999998D 00	0.39999986D 00	-0.26862959D-01
0.39999998D 00	0.49999982D 00	-0.41951670D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.49999997D 00	0.0	-0.48913308D-04
0.49999997D 00	0.99999964D-01	-0.17254368D-02
0.49999997D 00	0.19999993D 00	-0.67550071D-02
0.49999997D 00	0.29999989D 00	-0.15137624D-01
0.49999997D 00	0.39999986D 00	-0.26873288D-01
0.49999997D 00	0.49999982D 00	-0.41962000D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.59999996D 00	0.0	-0.59653225D-04
0.59999996D 00	0.99999964D-01	-0.17361767D-02
0.59999996D 00	0.19999993D 00	-0.67657470D-02
0.59999996D 00	0.29999989D 00	-0.15148364D-01
0.59999996D 00	0.39999986D 00	-0.26884028D-01
0.59999996D 00	0.49999982D 00	-0.41972739D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.69999996D 00	0.0	-0.69308275D-04
0.69999996D 00	0.99999964D-01	-0.17458317D-02
0.69999996D 00	0.19999993D 00	-0.67754021D-02
0.69999996D 00	0.29999989D 00	-0.15158019D-01
0.69999996D 00	0.39999986D 00	-0.26893683D-01
0.69999996D 00	0.49999982D 00	-0.41982394D-01

ETA EQUALS	X EQUALS	P* EQUALS
0.79999995D 00	0.0	-0.76194679D-04
0.79999995D 00	0.99999964D-01	-0.17527181D-02
0.79999995D 00	0.19999993D 00	-0.67822885D-02
0.79999995D 00	0.29999989D 00	-0.15164906D-01
0.79999995D 00	0.39999986D 00	-0.26900570D-01
0.79999995D 00	0.49999982D 00	-0.41989281D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.89999995D 00	0.0	-0.78797552D-04
0.89999995D 00	0.99999964D-01	-0.17553210D-02
0.89999995D 00	0.19999993D 00	-0.67848913D-02
0.89999995D 00	0.29999989D 00	-0.15167509D-01
0.89999995D 00	0.39999986D 00	-0.26903173D-01
0.89999995D 00	0.49999982D 00	-0.41991884D-01
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D 00	0.0	-0.76058713D-04
0.99999994D 00	0.99999964D-01	-0.17525822D-02
0.99999994D 00	0.19999993D 00	-0.67821525D-02
0.99999994D 00	0.29999989D 00	-0.15164770D-01
0.99999994D 00	0.39999986D 00	-0.26900434D-01
0.99999994D 00	0.49999982D 00	-0.41989145D-01

REYNOLDS NUMBER EQUALS 0.25000000D 02
PECLET NUMBER EQUALS 0.17500000D 02
THE INTERVAL SIZE EQUALS 0.24999999D-01
THE NUMBER OF ITERATIONS EQUALS 5

THE SECOND DERIVATIVE OF F AT ZERO EQUALS 0.14308808D 02
THE SECOND DERIVATIVE OF F AT ONE EQUALS -0.31357885D 01
THE THIRD DERIVATIVE OF F AT ZERO EQUALS -0.79812586D 02
THE FIRST DERIVATIVE OF H AT ZERO EQUALS 0.30611645D 00
THE FIRST DERIVATIVE OF H AT ONE EQUALS -0.40209516D-01

ETA	F	FDOT	H	T	TDOT
0.0	0.0	0.0	0.0	0.0	3.2424
0.0250	0.0043	0.3329	0.0073	0.0810	3.2403
0.0500	0.0162	0.6169	0.0140	0.1619	3.2266
0.0750	0.0347	0.8547	0.0200	0.2422	3.1915
0.1000	0.0586	1.0495	0.0253	0.3213	3.1273
0.1250	0.0869	1.2056	0.0297	0.3983	3.0294
0.1500	0.1186	1.3274	0.0333	0.4724	2.8960
0.1750	0.1530	1.4193	0.0361	0.5428	2.7284
0.2000	0.1894	1.4857	0.0379	0.6086	2.5307
0.2250	0.2271	1.5307	0.0390	0.6692	2.3093
0.2500	0.2657	1.5578	0.0394	0.7240	2.0720
0.2750	0.3049	1.5703	0.0391	0.7727	1.8275
0.3000	0.3442	1.5708	0.0383	0.8154	1.5843
0.3250	0.3833	1.5615	0.0371	0.8520	1.3499
0.3500	0.4222	1.5443	0.0356	0.8830	1.1306
0.3750	0.4605	1.5205	0.0339	0.9087	0.9310
0.4000	0.4981	1.4912	0.0320	0.9298	0.7539
0.4250	0.5350	1.4573	0.0302	0.9466	0.6006
0.4500	0.5710	1.4193	0.0283	0.9600	0.4709
0.4750	0.6059	1.3778	0.0265	0.9704	0.3635
0.5000	0.6398	1.3330	0.0247	0.9783	0.2763
0.5250	0.6726	1.2853	0.0230	0.9843	0.2071
0.5500	0.7041	1.2348	0.0214	0.9888	0.1530
0.5750	0.7343	1.1817	0.0198	0.9921	0.1115
0.6000	0.7631	1.1261	0.0183	0.9945	0.0802
0.6250	0.7906	1.0682	0.0169	0.9962	0.0570
0.6500	0.8165	1.0080	0.0156	0.9974	0.0401
0.6750	0.8410	0.9457	0.0143	0.9982	0.0278
0.7000	0.8638	0.8815	0.0130	0.9988	0.0191
0.7250	0.8850	0.8153	0.0118	0.9992	0.0130
0.7500	0.9046	0.7474	0.0106	0.9995	0.0088
0.7750	0.9224	0.6779	0.0095	0.9997	0.0059
0.8000	0.9384	0.6068	0.0083	0.9998	0.0039
0.8250	0.9527	0.5344	0.0072	0.9999	0.0026
0.8500	0.9651	0.4607	0.0062	0.9999	0.0017
0.8750	0.9757	0.3858	0.0051	0.9999	0.0011
0.9000	0.9844	0.3100	0.0041	1.0000	0.0007
0.9250	0.9912	0.2334	0.0030	1.0000	0.0005
0.9500	0.9961	0.1561	0.0020	1.0000	0.0003
0.9750	0.9990	0.0782	0.0010	1.0000	0.0002
1.0000	1.0000	-0.0000	-0.0000	1.0000	0.0001

PRESSURE DISTRIBUTION IN THE POSITIVE X DIRECTION FOR VARIOUS ETA
THE INJECTION VELOCITY IS 0.11868002D 00 FT PER SEC

ETA EQUALS	X EQUALS	P* EQUALS
0.0	0.0	0.0
0.0	0.99999964D-01	-0.87414339D-02
0.0	0.19999993D 00	-0.34965736D-01
0.0	0.29999989D 00	-0.78672905D-01
0.0	0.39999986D 00	-0.13986294D 00
0.0	0.49999982D 00	-0.21853585D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D-01	0.0	-0.41545907D-04
0.99999994D-01	0.99999964D-01	-0.87829798D-02
0.99999994D-01	0.19999993D 00	-0.35007281D-01
0.99999994D-01	0.29999989D 00	-0.78714451D-01
0.99999994D-01	0.39999986D 00	-0.13990449D 00
0.99999994D-01	0.49999982D 00	-0.21857739D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.19999999D 00	0.0	-0.73546534D-04
0.19999999D 00	0.99999964D-01	-0.88149804D-02
0.19999999D 00	0.19999993D 00	-0.35039282D-01
0.19999999D 00	0.29999989D 00	-0.78746452D-01
0.19999999D 00	0.39999986D 00	-0.13993649D 00
0.19999999D 00	0.49999982D 00	-0.21860939D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.29999998D 00	0.0	-0.11604128D-03
0.29999998D 00	0.99999964D-01	-0.88574752D-02
0.29999998D 00	0.19999993D 00	-0.35081777D-01
0.29999998D 00	0.29999989D 00	-0.78788946D-01
0.29999998D 00	0.39999986D 00	-0.13997898D 00
0.29999998D 00	0.49999982D 00	-0.21865189D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.39999998D 00	0.0	-0.17467109D-03
0.39999998D 00	0.99999964D-01	-0.89161050D-02
0.39999998D 00	0.19999993D 00	-0.35140407D-01
0.39999998D 00	0.29999989D 00	-0.78847576D-01
0.39999998D 00	0.39999986D 00	-0.14003761D 00
0.39999998D 00	0.49999982D 00	-0.21871052D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.49999997D 00	0.0	-0.24530525D-03
0.49999997D 00	0.99999964D-01	-0.89867391D-02
0.49999997D 00	0.19999993D 00	-0.35211041D-01
0.49999997D 00	0.29999989D 00	-0.78918210D-01
0.49999997D 00	0.39999986D 00	-0.14010825D 00
0.49999997D 00	0.49999982D 00	-0.21878115D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.59999996D 00	0.0	-0.31967153D-03
0.59999996D 00	0.99999964D-01	-0.90611054D-02
0.59999996D 00	0.19999993D 00	-0.35285407D-01
0.59999996D 00	0.29999989D 00	-0.78992577D-01
0.59999996D 00	0.39999986D 00	-0.14018261D 00
0.59999996D 00	0.49999982D 00	-0.21885552D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.69999996D 00	0.0	-0.38821936D-03
0.69999996D 00	0.99999964D-01	-0.91296532D-02
0.69999996D 00	0.19999993D 00	-0.35353955D-01
0.69999996D 00	0.29999989D 00	-0.79061124D-01
0.69999996D 00	0.39999986D 00	-0.14025116D 00
0.69999996D 00	0.49999982D 00	-0.21892407D 00

ETA EQUALS	X EQUALS	P* EQUALS
0.79999995D 00	0.0	-0.44171306D-03
0.79999995D 00	0.99999964D-01	-0.91831470D-02
0.79999995D 00	0.19999993D 00	-0.35407449D-01
0.79999995D 00	0.29999989D 00	-0.79114618D-01
0.79999995D 00	0.39999986D 00	-0.14030466D 00
0.79999995D 00	0.49999982D 00	-0.21897756D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.89999995D 00	0.0	-0.47246599D-03
0.89999995D 00	0.99999964D-01	-0.92138999D-02
0.89999995D 00	0.19999993D 00	-0.35438202D-01
0.89999995D 00	0.29999989D 00	-0.79145371D-01
0.89999995D 00	0.39999986D 00	-0.14033541D 00
0.89999995D 00	0.49999982D 00	-0.21900831D 00
ETA EQUALS	X EQUALS	P* EQUALS
0.99999994D 00	0.0	-0.47536696D-03
0.99999994D 00	0.99999964D-01	-0.92168008D-02
0.99999994D 00	0.19999993D 00	-0.35441103D-01
0.99999994D 00	0.29999989D 00	-0.79148272D-01
0.99999994D 00	0.39999986D 00	-0.14033831D 00
0.99999994D 00	0.49999982D 00	-0.21901121D 00

APPENDIX II

```
      IMPLICIT REAL*8(A-H,O-Z)
C
C
C     COMPUTATION OF F, FDOT AND H BY THE PERTURBATION METHOD
C
C
      DIMENSION F(41),FDOT(41),H(41),XO(41)
  100 FORMAT(F10.5)
  101 FORMAT('1',10X,'REYNOLDS NUMBER EQUALS',2X,E16.8,///,
     115X,'PSI',16X,'F',17X,'FDOT',15X,'H',//)
  102 FORMAT( 9X,E16.8,2X,E16.8,2X,E16.8,2X,E16.8)
      J=0
      T=0.025
  399 Z=-T
      I=0
      READ(5,100) R
      WRITE(6,101) R
      J=J+1
  400 Z=Z+T
C
C     THE INDEPENDENT VARIABLE ETA IS TRANSFORMED HERE
C
      X=1.0-Z
      IF(X.LT.0.00001) X=0.0
C
      I=I+1
C
C     COMPUTATION OF THE NORMAL VELOCITY FUNCTION
C
      F(I)=2.*X**3-3.*X**2+1.0
     1+R*((-2./35.)*X**7+(1./5.)*X**6-(3./10.)*X**5+0.5*X**4-(4
     170.)*X**3+(19./70.)*X**2)+(R**2)*((-4./5775.)*X**11+(2./
     1525.)*X**10-(1./210.)*X**9-(11./560.)*X**8+(191./2450.)*X
     1-(68./525.)*X**6+(27./175.)*X**5-(43./280.)*X**4
     1+(32189./323400.)*X**3-(17719./646800.)*X**2)
C
C     COMPUTATION OF THE TANGENTIAL VELOCTIY FUNCTION
C
      FDOT(I)=-6.*X**2+6.*X+R*((14./35.)*X**6-(6./5.)*X**5+1.5*
     12.*X**3+(129./70.)*X**2-(38./70.)*X)+
     1(R**2)*((44./5775.)*X**10-(20./525.)*X**9+(9./210.)*X**8+
     1(88./560.)*X**7-(1337./2450.)*X**6+(408./525.)*X**5-
     1(135./175.)*X**4+(172./280.)*X**3-(96567./323400.)*X**2+
     1(35438./646800.)*X)
C
C     COMPUTATION OF THE AXIAL VELOCITY FUNCTION
C
      H(I)=.5*X-.5*X**2+
     1R*((-1./15.)*X**6+.2*X**5-(1./8.)*X**4-(1./6.)*X**3+.25*X
     1(11./120.)*X)+
     1(R**2)*((-13./1575.)*X**10+(13./315.)*X**9-(9./140.)*X**8
     1(11./840.)*X**7+(611./4200.)*X**6-(29./210.)*X**5-(5./672
     1**4+(1./12.)*X**3-(11./240.)*X**2+(349./50400.)*X)
C
C
      XO(I)=X
C
      WRITE(6,102) XO(I),F(I),FDOT(I),H(I)
```

```
IF(Z.LT.0.98) GO TO 400
IF(J.LT.3) GO TO 399
STOP
END
```

REYNOLDS NUMBER EQUALS 0.0

XI	F	FDOT	H
0.10000000D 01	0.0	0.0	0.0
0.97500000D 00	0.18437498D-02	0.14624999D 00	0.12187499D-01
0.95000000D 00	0.72499992D-02	0.28499998D 00	0.23749999D-01
0.92500000D 00	0.16031248D-01	0.41624998D 00	0.34687498D-01
0.90000001D 00	0.27999997D-01	0.53999997D 00	0.44999998D-01
0.87500001D 00	0.42968745D-01	0.65624997D 00	0.54687497D-01
0.85000001D 00	0.60749993D-01	0.76499996D 00	0.63749997D-01
0.82500001D 00	0.81156241D-01	0.86624996D 00	0.72187497D-01
0.80000001D 00	0.10399999D 00	0.95999996D 00	0.79999996D-01
0.77500001D 00	0.12909374D 00	0.10462500D 01	0.87187496D-01
0.75000001D 00	0.15624998D 00	0.11250000D 01	0.93749996D-01
0.72500002D 00	0.18528123D 00	0.11962500D 01	0.99687496D-01
0.70000002D 00	0.21599998D 00	0.12600000D 01	0.10500000D 00
0.67500002D 00	0.24821872D 00	0.13162500D 01	0.10968750D 00
0.65000002D 00	0.28174997D 00	0.13650000D 01	0.11375000D 00
0.62500002D 00	0.31640622D 00	0.14062500D 01	0.11718750D 00
0.60000002D 00	0.35199997D 00	0.14400000D 01	0.12000000D 00
0.57500003D 00	0.38834371D 00	0.14662500D 01	0.12218750D 00
0.55000003D 00	0.42524996D 00	0.14850000D 01	0.12375000D 00
0.52500003D 00	0.46253121D 00	0.14962500D 01	0.12468750D 00
0.50000003D 00	0.49999996D 00	0.15000000D 01	0.12500000D 00
0.47500003D 00	0.53746870D 00	0.14962500D 01	0.12468750D 00
0.45000003D 00	0.57474995D 00	0.14850000D 01	0.12375000D 00
0.42500003D 00	0.61165620D 00	0.14662500D 01	0.12218750D 00
0.40000004D 00	0.64799995D 00	0.14400000D 01	0.12000000D 00
0.37500004D 00	0.68359370D 00	0.14062501D 01	0.11718750D 00
0.35000004D 00	0.71824995D 00	0.13650001D 01	0.11375001D 00
0.32500004D 00	0.75178120D 00	0.13162501D 01	0.10968751D 00
0.30000004D 00	0.78399995D 00	0.12600001D 01	0.10500001D 00
0.27500004D 00	0.81471870D 00	0.11962501D 01	0.99687510D-01
0.25000004D 00	0.84374995D 00	0.11250001D 01	0.93750011D-01
0.22500005D 00	0.87090620D 00	0.10462502D 01	0.87187513D-01
0.20000005D 00	0.89599995D 00	0.96000017D 00	0.80000014D-01
0.17500005D 00	0.91884371D 00	0.86625019D 00	0.72187516D-01
0.15000005D 00	0.93924996D 00	0.76500021D 00	0.63750018D-01
0.12500005D 00	0.95703122D 00	0.65625023D 00	0.54687520D-01
0.10000005D 00	0.97199997D 00	0.54000026D 00	0.45000021D-01
0.75000055D-01	0.98396873D 00	0.41625028D 00	0.34687523D-01
0.50000057D-01	0.99274998D 00	0.28500031D 00	0.23750025D-01
0.25000058D-01	0.99815624D 00	0.14625033D 00	0.12187528D-01
0.0	0.10000000D 01	0.0	0.0

XI	F	FDOT	H
0.10000000D 01	0.72177500D-06	-0.49918890D-05	0.20861626D-05
0.97500000D 00	0.31249293D-02	0.24573236D 00	0.80714404D-02
0.95000000D 00	0.12077343D-01	0.46630340D 00	0.15506095D-01
0.92500000D 00	0.26234091D-01	0.66217813D 00	0.22283667D-01
0.90000001D 00	0.44986373D-01	0.83408489D 00	0.28371897D-01
0.87500001D 00	0.67746372D-01	0.98296662D 00	0.33732722D-01
0.85000001D 00	0.93952096D-01	0.11099402D 01	0.38326147D-01
0.82500001D 00	0.12307126D 00	0.12162590D 01	0.42113905D-01
0.80000001D 00	0.15460425D 00	0.13032782D 01	0.45062880D-01
0.77500001D 00	0.18808628D 00	0.13724231D 01	0.47148257D-01
0.75000001D 00	0.22308876D 00	0.14251605D 01	0.48356349D-01
0.72500002D 00	0.25922003D 00	0.14629725D 01	0.48687044D-01
0.70000002D 00	0.29612542D 00	0.14873334D 01	0.48155810D-01
0.67500002D 00	0.33348679D 00	0.14996888D 01	0.46795206D-01
0.65000002D 00	0.37102161D 00	0.15014381D 01	0.44655839D-01
0.62500002D 00	0.40848159D 00	0.14939181D 01	0.41806713D-01
0.60000002D 00	0.44565092D 00	0.14783901D 01	0.38334938D-01
0.57500003D 00	0.48234429D 00	0.14560288D 01	0.34344757D-01
0.55000003D 00	0.51840455D 00	0.14279130D 01	0.29955888D-01
0.52500003D 00	0.55370025D 00	0.13950178D 01	0.25301151D-01
0.50000003D 00	0.58812296D 00	0.13582095D 01	0.20523420D-01
0.47500003D 00	0.62158450D 00	0.13182406D 01	0.15771905D-01
0.45000003D 00	0.65401406D 00	0.12757467D 01	0.11197814D-01
0.42500003D 00	0.68535520D 00	0.12312434D 01	0.69494599D-02
0.40000004D 00	0.71556288D 00	0.11851247D 01	0.31668846D-02
0.37500004D 00	0.74460033D 00	0.11376605D 01	-0.23899792D-04
0.35000004D 00	0.77243598D 00	0.10889944D 01	-0.25169430D-02
0.32500004D 00	0.79904023D 00	0.10391403D 01	-0.42325308D-02
0.30000004D 00	0.82438218D 00	0.98797919D 00	-0.51233593D-02
0.27500004D 00	0.84842622D 00	0.93525243D 00	-0.51805501D-02
0.25000004D 00	0.87112845D 00	0.88055482D 00	-0.44393367D-02
0.22500005D 00	0.89243292D 00	0.82332420D 00	-0.29842413D-02
0.20000005D 00	0.91226752D 00	0.76282839D 00	-0.95355314D-03
0.17500005D 00	0.93053955D 00	0.69814853D 00	0.14570871D-02
0.15000005D 00	0.94713077D 00	0.62815843D 00	0.39921941D-02
0.12500005D 00	0.96189197D 00	0.55149927D 00	0.63352354D-02
0.10000005D 00	0.97463674D 00	0.46654928D 00	0.81089139D-02
0.75000055D-01	0.98513444D 00	0.37138758D 00	0.88771245D-02
0.50000057D-01	0.99310227D 00	0.26375191D 00	0.81487628D-02
0.25000058D-01	0.99819610D 00	0.14098944D 00	0.53835500D-02
0.0	0.10000000D 01	0.0	0.0

REYNOLDS NUMBER EQUALS 0.25000000D 02

XI	F	FDOT	H
0.10000000D 01	0.45110937D-05	-0.11082739D-04	0.10803342D-04
0.97500000D 00	0.47199798D-02	0.36656192D 00	-0.57257434D-02
0.95000000D 00	0.17797158D-01	0.66916677D 00	-0.12124442D-01
0.92500000D 00	0.37661731D-01	0.91001402D 00	-0.19269919D-01
0.90000001D 00	0.62809974D-01	0.10925011D 01	-0.27281222D-01
0.87500001D 00	0.91835556D-01	0.12209853D 01	-0.36294034D-01
0.85000001D 00	0.12345086D 00	0.13005732D 01	-0.46443734D-01
0.82500001D 00	0.15650329D 00	0.13369270D 01	-0.57849340D-01
0.80000001D 00	0.18998684D 00	0.13360866D 01	-0.70598477D-01
0.77500001D 00	0.22304956D 00	0.13043072D 01	-0.84733628D-01
0.75000001D 00	0.25499701D 00	0.12479125D 01	-0.10023996D 00
0.72500002D 00	0.28529229D 00	0.11731624D 01	-0.11703508D 00
0.70000002D 00	0.31355300D 00	0.10861353D 01	-0.13496111D 00
0.67500002D 00	0.33954529D 00	0.99262422D 00	-0.15377945D 00
0.65000002D 00	0.36317565D 00	0.89804617D 00	-0.17316849D 00
0.62500002D 00	0.38448045D 00	0.80736419D 00	-0.19272476D 00
0.60000002D 00	0.40361376D 00	0.72502112D 00	-0.21196756D 00
0.57500003D 00	0.42083366D 00	0.65488419D 00	-0.23034752D 00
0.55000003D 00	0.43648720D 00	0.60019951D 00	-0.24725890D 00
0.52500003D 00	0.45099443D 00	0.56355500D 00	-0.26205595D 00
0.50000003D 00	0.46483157D 00	0.54685067D 00	-0.27407309D 00
0.47500003D 00	0.47851348D 00	0.55127451D 00	-0.28264880D 00
0.45000003D 00	0.49257560D 00	0.57728246D 00	-0.28715293D 00
0.42500003D 00	0.50755544D 00	0.62458054D 00	-0.28701707D 00
0.40000004D 00	0.52397360D 00	0.69210721D 00	-0.28176752D 00
0.37500004D 00	0.54231435D 00	0.77801366D 00	-0.27106015D 00
0.35000004D 00	0.56300566D 00	0.87963987D 00	-0.25471657D 00
0.32500004D 00	0.58639853D 00	0.99348381D 00	-0.23276074D 00
0.30000004D 00	0.61274549D 00	0.11151613D 01	-0.20545515D 00
0.27500004D 00	0.64217791D 00	0.12393534D 01	-0.17333557D 00
0.25000004D 00	0.67468181D 00	0.13597391D 01	-0.13724326D 00
0.22500005D 00	0.71007179D 00	0.14689094D 01	-0.98353619D-01
0.20000005D 00	0.74796252D 00	0.15582603D 01	-0.58199977D-01
0.17500005D 00	0.78773732D 00	0.16178611D 01	-0.18691409D-01
0.15000005D 00	0.82851301D 00	0.16362952D 01	0.17876756D-01
0.12500005D 00	0.86910045D 00	0.16004686D 01	0.48820904D-01
0.10000005D 00	0.90795990D 00	0.14953851D 01	0.71076838D-01
0.75000055D-01	0.94315023D 00	0.13038821D 01	0.81218375D-01
0.50000057D-01	0.97227101D 00	0.10063252D 01	0.75487968D-01
0.25000058D-01	0.99239646D 00	0.58025766D 00	0.49840354D-01
0.0	0.10000000D 01	0.0	0.0

APPENDIX III

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION YO(41),YODOT(41)
      COMMON /GEAR9/ HUSED,NQUSED,NSTEP,NFE,NJE
  100 FORMAT('1',/////,19X,'TOUT EQUALS           ',E16.8)
  101 FORMAT(19X,E16.8,19X,E16.8)
  102 FORMAT(19X,'STEP SIZE USED        ',E16.8,/,
     119X,'ORDER USED          ',I4,/,
     119X,'NUMBER OF STEPS         ',I4,/,
     119X,'NUMBER OF FUNCTION EVALUATIONS      ',I4,/
     119X,'NUMBER OF JACOBIAN EVALUATIONS      ',I4,/)
  103 FORMAT(//,25X,'T',31X,'TDOT',//)
      ML=1
      MU=1
      N=41
      TO=0.0
      HO=0.0000001
      TOUT=0.0
      EPS=10.0
      KLEZ=-4
      MF=22
      INDEX=1
      DO 200 I=1,41
  200 YO(I)=(I-1.0)/40.0
  201 KLEZ=KLEZ+1
      TOUT=10.0**(KLEZ)
      CALL DRIVEB(N,TO,HO,YO,TOUT,EPS,MF,INDEX,ML,MU)
      WRITE(6,100) TOUT
      WRITE(6,102) HUSED,NQUSED,NSTEP,NFE,NJE
      WRITE(6,103)
      DO 301 I=2,40
  301 YODOT(I)=(YO(I+1)-YO(I-1))/0.05
      YODOT(1)=2.0*YODOT(2)-YODOT(3)
      YODOT(41)=2.0*YODOT(40)-YODOT(39)
      DO 300 I=1,41
  300 WRITE(6,101) YO(I),YODOT(I)
      IF(INDEX.LT.0) GO TO 1000
      TO=TOUT
      IF(KLEZ.LT.3) GO TO 201
 1000 CONTINUE
      STOP
      END
```

```
SUBROUTINE PDB(N,T,Y,PD,H0,ML,MU)
RETURN
END
```

```
      SUBROUTINE DIFFUN(N,T,Y,YDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION F(41)
      DIMENSION Y(N),YDOT(N)
      R=10.0
      P=7.0
      XDH=(1.0/40.0)**2
      YDOT(1)=0.0
      YDOT(41)=0.0
      DO 100 I=2,40
      X=1.0-((I-1.0)/40.0)
      F(I)=2.*X**3-3.*X**2+1.0
     1+R*((-2./35.)*X**7+(1./5.)*X**6-(3./10.)*X**5+0.5*X**4-(43.
     170.)*X**3+(19./70.)*X**2)+(R**2)*((-4./5775.)*X**11+(2./
     1525.)*X**10-(1./210.)*X**9-(11./560.)*X**8+(191./2450.)*X**
     1-(68./525.)*X**6+(27./175.)*X**5-(43./280.)*X**4
     1+(32189./323400.)*X**3-(17719./646800.)*X**2)
      YDOT(I)=(Y(I-1)-2.0*Y(I)+Y(I+1))/XDH
     1+ P *F(I)*((Y(I+1)-Y(I-1))/0.05)
  100 CONTINUE
      RETURN
      END
```

```
TOUT EQUALS            0.99999993D-03
STEP SIZE USED           0.99999966D-03
ORDER USED             1
NUMBER OF STEPS             4
NUMBER OF FUNCTION EVALUATIONS        11
NUMBER OF JACOBIAN EVALUATIONS         2
```

T	TDOT
-0.19057743D-19	0.10016207D 01
0.25054099D-01	0.10025662D 01
0.50128305D-01	0.10035116D 01
0.75229678D-01	0.10046200D 01
0.10035930D 00	0.10057180D 01
0.12551558D 00	0.10067280D 01
0.15069570D 00	0.10076181D 01
0.17589648D 00	0.10083788D 01
0.20111464D 00	0.10090098D 01
0.22634697D 00	0.10095166D 01
0.25159046D 00	0.10099071D 01
0.27684232D 00	0.10101906D 01
0.30209999D 00	0.10103776D 01
0.32736119D 00	0.10104774D 01
0.35262386D 00	0.10105000D 01
0.37788619D 00	0.10104550D 01
0.40314660D 00	0.10103513D 01
0.42840376D 00	0.10101981D 01
0.45365651D 00	0.10100025D 01
0.47890388D 00	0.10097716D 01
0.50414509D 00	0.10095117D 01
0.52937946D 00	0.10092280D 01
0.55460648D 00	0.10089256D 01
0.57982574D 00	0.10086073D 01
0.60503684D 00	0.10082755D 01
0.63023951D 00	0.10079312D 01
0.65543340D 00	0.10075736D 01
0.68061819D 00	0.10072000D 01
0.70579340D 00	0.10068020D 01
0.73095829D 00	0.10063648D 01
0.75611164D 00	0.10058591D 01
0.78125124D 00	0.10052289D 01
0.80637308D 00	0.10043657D 01
0.83146952D 00	0.10030560D 01
0.85652587D 00	0.10008810D 01
0.88151357D 00	0.99701332D 00
0.90637654D 00	0.98981372D 00
0.93100425D 00	0.97602941D 00
0.95517800D 00	0.94919375D 00
0.97846394D 00	0.89644000D 00
0.10000000D 01	0.84368624D 00

TOUT EQUALS 0.99999979D-02
STEP SIZE USED 0.99999966D-02
ORDER USED 1
NUMBER OF STEPS 7
NUMBER OF FUNCTION EVALUATIONS 17
NUMBER OF JACOBIAN EVALUATIONS 3

T	TDOT
-0.13782039D-16	0.10486478D 01
0.26272633D-01	0.10522707D 01
0.52613531D-01	0.10558935D 01
0.79067307D-01	0.10609313D 01
0.10566009D 00	0.10667293D 01
0.13240377D 00	0.10727848D 01
0.15929933D 00	0.10787164D 01
0.18633958D 00	0.10842396D 01
0.21351131D 00	0.10891478D 01
0.24079697D 00	0.10932968D 01
0.26817615D 00	0.10965930D 01
0.29562662D 00	0.10989828D 01
0.32312528D 00	0.11004442D 01
0.35064883D 00	0.11009791D 01
0.37817424D 00	0.11006058D 01
0.40567912D 00	0.10993529D 01
0.43314188D 00	0.10972536D 01
0.46054179D 00	0.10943402D 01
0.48785888D 00	0.10906393D 01
0.51507376D 00	0.10861681D 01
0.54216728D 00	0.10809295D 01
0.56912023D 00	0.10749097D 01
0.59591277D 00	0.10680740D 01
0.62252393D 00	0.10603639D 01
0.64893096D 00	0.10516935D 01
0.67510860D 00	0.10419461D 01
0.70102826D 00	0.10309691D 01
0.72665705D 00	0.10185692D 01
0.75195672D 00	0.10045050D 01
0.77688230D 00	0.98847754D 00
0.80138059D 00	0.97011820D 00
0.82538820D 00	0.94897188D 00
0.84882918D 00	0.92447583D 00
0.87161199D 00	0.89593388D 00
0.89362587D 00	0.86248966D 00
0.91473647D 00	0.82310820D 00
0.93478128D 00	0.77658852D 00
0.95356590D 00	0.72165173D 00
0.97086386D 00	0.65718143D 00
0.98642497D 00	0.58272276D 00
0.10000000D 01	0.50826409D 00

```
TOUT EQUALS              0.99999964D-01
STEP SIZE USED             0.99999966D-01
ORDER USED               1
NUMBER OF STEPS             10
NUMBER OF FUNCTION EVALUATIONS         23
NUMBER OF JACOBIAN EVALUATIONS          4
```

T	TDOT
0.79373620D-14	0.15575803D 01
0.38992739D-01	0.15611744D 01
0.78058713D-01	0.15647684D 01
0.11723116D 00	0.15689681D 01
0.15650711D 00	0.15724021D 01
0.19585126D 00	0.15738565D 01
0.23519993D 00	0.15722846D 01
0.27446548D 00	0.15668197D 01
0.31354091D 00	0.15567853D 01
0.35230474D 00	0.15417022D 01
0.39062602D 00	0.15212894D 01
0.42836920D 00	0.14954574D 01
0.46539888D 00	0.14642953D 01
0.50158397D 00	0.14280512D 01
0.53680144D 00	0.13871070D 01
0.57093931D 00	0.13419502D 01
0.60389894D 00	0.12931434D 01
0.63559648D 00	0.12412949D 01
0.66596368D 00	0.11870304D 01
0.69494800D 00	0.11309680D 01
0.72251208D 00	0.10736978D 01
0.74863289D 00	0.10157661D 01
0.77330038D 00	0.95766374D 00
0.79651607D 00	0.89982034D 00
0.81829140D 00	0.84260228D 00
0.83864618D 00	0.78631406D 00
0.85760710D 00	0.73120269D 00
0.87520631D 00	0.67746474D 00
0.89148033D 00	0.62525406D 00
0.90646901D 00	0.57469050D 00
0.92021486D 00	0.52586911D 00
0.93276247D 00	0.47886815D 00
0.94415826D 00	0.43375679D 00
0.95445031D 00	0.39060144D 00
0.96368833D 00	0.34947008D 00
0.97192381D 00	0.31043511D 00
0.97921009D 00	0.27357393D 00
0.98560251D 00	0.23896725D 00
0.99115845D 00	0.20669513D 00
0.99593726D 00	0.17683104D 00
0.10000000D 01	0.14696694D 00

```
TOUT EQUALS            0.10000000D 01
STEP SIZE USED           0.99999966D 00
ORDER USED             1
NUMBER OF STEPS            13
NUMBER OF FUNCTION EVALUATIONS        29
NUMBER OF JACOBIAN EVALUATIONS         5
```

T	TDOT
0.37672329D-13	0.20646217D 01
0.51566071D-01	0.20621786D 01
0.10310892D 00	0.20597356D 01
0.15455284D 00	0.20533378D 01
0.20577581D 00	0.20412708D 01
0.25661638D 00	0.20221119D 01
0.30688140D 00	0.19947606D 01
0.35635440D 00	0.19584642D 01
0.40480460D 00	0.19128334D 01
0.45199607D 00	0.18578440D 01
0.49769680D 00	0.17938235D 01
0.54168724D 00	0.17214224D 01
0.58376791D 00	0.16415708D 01
0.62376577D 00	0.15554240D 01
0.66153910D 00	0.14643007D 01
0.69698080D 00	0.13696181D 01
0.73002001D 00	0.12728270D 01
0.76062215D 00	0.11753520D 01
0.78878760D 00	0.10785399D 01
0.81454914D 00	0.98361679D 00
0.83796844D 00	0.89165695D 00
0.85913199D 00	0.80356284D 00
0.87814658D 00	0.72005520D 00
0.89513474D 00	0.64167323D 00
0.91023024D 00	0.56878219D 00
0.92357385D 00	0.50158700D 00
0.93530959D 00	0.44015039D 00
0.94558137D 00	0.38441322D 00
0.95453025D 00	0.33421599D 00
0.96229217D 00	0.28932005D 00
0.96899625D 00	0.24942773D 00
0.97476355D 00	0.21420059D 00
0.97970628D 00	0.18327544D 00
0.98392733D 00	0.15627797D 00
0.98752018D 00	0.13283398D 00
0.99056902D 00	0.11257815D 00
0.99314908D 00	0.95160767D-01
0.99532706D 00	0.80252521D-01
0.99716171D 00	0.67547664D-01
0.99870445D 00	0.56765792D-01
0.10000000D 01	0.45983921D-01

```
TOUT EQUALS          0.10000000D 02
STEP SIZE USED         0.99999966D 01
ORDER USED           1
NUMBER OF STEPS          16
NUMBER OF FUNCTION EVALUATIONS        35
NUMBER OF JACOBIAN EVALUATIONS         6
```

T	TDOT
0.42315122D-13	0.208304110D 01
0.52021140D-01	0.208027690D 01
0.10401384D 00	0.207751270D 01
0.15589677D 00	0.207056440D 01
0.20754205D 00	0.205771180D 01
0.25878235D 00	0.203753400D 01
0.30941875D 00	0.200894090D 01
0.35922939D 00	0.197119860D 01
0.40797867D 00	0.192394460D 01
0.45542662D 00	0.186718860D 01
0.50133810D 00	0.180129770D 01
0.54549149D 00	0.172696550D 01
0.58768637D 00	0.164516720D 01
0.62774985D 00	0.155710320D 01
0.66554152D 00	0.146413480D 01
0.70095658D 00	0.136771760D 01
0.73392740D 00	0.126933560D 01
0.76442336D 00	0.117044020D 01
0.79244941D 00	0.107239750D 01
0.81804324D 00	0.976445760D 00
0.84127169D 00	0.883664330D 00
0.86222645D 00	0.794954010D 00
0.88101939D 00	0.711028260D 00
0.89777786D 00	0.632413990D 00
0.91264009D 00	0.559460370D 00
0.92575088D 00	0.492353630D 00
0.93725777D 00	0.431135990D 00
0.94730767D 00	0.375727070D 00
0.95604412D 00	0.325946210D 00
0.96360498D 00	0.281534320D 00
0.97012083D 00	0.242174590D 00
0.97571371D 00	0.207511100D 00
0.98049639D 00	0.177165130D 00
0.98457197D 00	0.150748930D 00
0.98803383D 00	0.127876990D 00
0.99096582D 00	0.108174760D 00
0.99344257D 00	0.912853130D-01
0.99553008D 00	0.768739720D-01
0.99728627D 00	0.646312740D-01
0.99876165D 00	0.542745930D-01
0.10000000D 01	0.439179110D-01

```
TOUT EQUALS          0.10000000D 03
STEP SIZE USED         0.99999966D 02
ORDER USED           1
NUMBER OF STEPS          19
NUMBER OF FUNCTION EVALUATIONS        41
NUMBER OF JACOBIAN EVALUATIONS         7
```

T	TDOT
0.42318193D-13	0.20830435D 01
0.52021197D-01	0.20802792D 01
0.10401395D 00	0.20775150D 01
0.15589694D 00	0.20705666D 01
0.20754228D 00	0.20577138D 01
0.25878262D 00	0.20375359D 01
0.30941907D 00	0.20089427D 01
0.35922975D 00	0.19712002D 01
0.40797907D 00	0.19239460D 01
0.45542705D 00	0.18671898D 01
0.50133855D 00	0.18012986D 01
0.54549197D 00	0.17269661D 01
0.58768685D 00	0.16451676D 01
0.62775034D 00	0.15571033D 01
0.66554202D 00	0.14641347D 01
0.70095707D 00	0.13677173D 01
0.73392788D 00	0.12693352D 01
0.76442383D 00	0.11704396D 01
0.79244985D 00	0.10723967D 01
0.81804366D 00	0.97644483D 00
0.84127209D 00	0.88366331D 00
0.86222682D 00	0.79495292D 00
0.88101974D 00	0.71102712D 00
0.89777818D 00	0.63241283D 00
0.91264038D 00	0.55945922D 00
0.92575114D 00	0.49235249D 00
0.93725800D 00	0.43113488D 00
0.94730788D 00	0.37572602D 00
0.95604430D 00	0.32594521D 00
0.96360514D 00	0.28153338D 00
0.97012097D 00	0.24217373D 00
0.97571382D 00	0.20751030D 00
0.98049648D 00	0.17716440D 00
0.98457204D 00	0.15074828D 00
0.98803389D 00	0.12787640D 00
0.99096586D 00	0.10817424D 00
0.99344261D 00	0.91284860D-01
0.99553011D 00	0.76873578D-01
0.99728629D 00	0.64630935D-01
0.99876165D 00	0.54274303D-01
0.10000000D 01	0.43917671D-01

```
TOUT EQUALS           0.10000000D 04
STEP SIZE USED          0.99999966D 03
ORDER USED            1
NUMBER OF STEPS            22
NUMBER OF FUNCTION EVALUATIONS        47
NUMBER OF JACOBIAN EVALUATIONS         8
```

T	TDOT
0.42318193D-13	0.20830435D 01
0.52021197D-01	0.20802792D 01
0.10401395D 00	0.20775150D 01
0.15589694D 00	0.20705666D 01
0.20754228D 00	0.20577138D 01
0.25878262D 00	0.20375359D 01
0.30941907D 00	0.20089427D 01
0.35922975D 00	0.19712002D 01
0.40797907D 00	0.19239460D 01
0.45542705D 00	0.18671898D 01
0.50133855D 00	0.18012986D 01
0.54549197D 00	0.17269661D 01
0.58768685D 00	0.16451676D 01
0.62775034D 00	0.15571033D 01
0.66554202D 00	0.14641347D 01
0.70095707D 00	0.13677173D 01
0.73392788D 00	0.12693352D 01
0.76442383D 00	0.11704396D 01
0.79244985D 00	0.10723967D 01
0.81804366D 00	0.97644483D 00
0.84127209D 00	0.88366331D 00
0.86222682D 00	0.79495292D 00
0.88101974D 00	0.71102712D 00
0.89777818D 00	0.63241283D 00
0.91264038D 00	0.55945922D 00
0.92575114D 00	0.49235249D 00
0.93725800D 00	0.43113488D 00
0.94730788D 00	0.37572602D 00
0.95604430D 00	0.32594521D 00
0.96360514D 00	0.28153338D 00
0.97012097D 00	0.24217373D 00
0.97571382D 00	0.20751030D 00
0.98049648D 00	0.17716440D 00
0.98457204D 00	0.15074828D 00
0.98803389D 00	0.12787640D 00
0.99096586D 00	0.10817424D 00
0.99344261D 00	0.91284860D-01
0.99553011D 00	0.76873578D-01
0.99728629D 00	0.64630935D-01
0.99876165D 00	0.54274303D-01
0.10000000D 01	0.43917671D-01

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION YO(41),YODOT(41)
      COMMON /GEAR9/ HUSED,NQUSED,NSTEP,NFE,NJE
  100 FORMAT('1',/////,19X,'TOUT EQUALS       ',E16.8)
  101 FORMAT(19X,E16.8,19X,E16.8)
  102 FORMAT(19X,'STEP SIZE USED      ',E16.8,/,
     119X,'ORDER USED       ',I4,/,
     119X,'NUMBER OF STEPS       ',I4,/,
     119X,'NUMBER OF FUNCTION EVALUATIONS     ',I4,/
     119X,'NUMBER OF JACOBIAN EVALUATIONS     ',I4,/)
  103 FORMAT(//,25X,'T',31X,'TDOT',//)
      ML=1
      MU=1
      N=41
      T0=0.0
      H0=0.0000001
      TOUT=0.0
      EPS=10.0
      KLEZ=-4
      MF=22
      INDEX=1
      DO 200 I=1,41
  200 YO(I)=(I-1.0)/40.0
  201 KLEZ=KLEZ+1
      TOUT=10.0**(KLEZ)
      CALL DRIVEB(N,T0,H0,YO,TOUT,EPS,MF,INDEX,ML,MU)
      WRITE(6,100) TOUT
      WRITE(6,102) HUSED,NQUSED,NSTEP,NFE,NJE
      WRITE(6,103)
      DO 301 I=2,40
  301 YODOT(I)=(YO(I+1)-YO(I-1))/0.05
      YODOT(1)=2.0*YODOT(2)-YODOT(3)
      YODOT(41)=2.0*YODOT(40)-YODOT(39)
      DO 300 I=1,41
  300 WRITE(6,101) YO(I),YODOT(I)
      IF(INDEX.LT.0) GO TO 1000
      T0=TOUT
      IF(KLEZ.LT.3) GO TO 201
 1000 CONTINUE
      STOP
      END
```

```
SUBROUTINE PDB(N,T,Y,PD,HO,ML,MU)
RETURN
END
```

```
      SUBROUTINE DIFFUN(N,T,Y,YDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION F(41)
      DIMENSION Y(N),YDOT(N)
      R=25.0
      P=17.5
      XDH=(1.0/40.0)**2
      YDOT(1)=0.0
      YDOT(41)=0.0
      DO 100 I=2,40
      X=1.0-((I-1.0)/40.0)
      F(I)=2.*X**3-3.*X**2+1.0
     1+R*((-2./35.)*X**7+(1./5.)*X**6-(3./10.)*X**5+0.5*X**4-(43.
     170.)*X**3+(19./70.)*X**2)+(R**2)*((-4./5775.)*X**11+(2./
     1525.)*X**10-(1./210.)*X**9-(11./560.)*X**8+(191./2450.)*X*
     1-(68./525.)*X**6+(27./175.)*X**5-(43./280.)*X**4
     1+(32189./323400.)*X**3-(17719./646800.)*X**2)
      YDOT(I)=(Y(I-1)-2.0*Y(I)+Y(I+1))/XDH
     1+ P *F(I)*((Y(I+1)-Y(I-1))/0.05)
  100 CONTINUE
      RETURN
      END
```

```
TOUT EQUALS          0.99999993D-03
STEP SIZE USED         0.99999966D-03
ORDER USED           1
NUMBER OF STEPS          4
NUMBER OF FUNCTION EVALUATIONS       11
NUMBER OF JACOBIAN EVALUATIONS        2
```

T	TDOT
-0.95361118D-19	0.10059720D 01
0.25192075D-01	0.10090422D 01
0.50452107D-01	0.10121124D 01
0.75797693D-01	0.10154292D 01
0.10122356D 00	0.10183383D 01
0.12671460D 00	0.10205718D 01
0.15225215D 00	0.10220539D 01
0.17781729D 00	0.10228065D 01
0.20339248D 00	0.10229021D 01
0.22896240D 00	0.10224402D 01
0.25451448D 00	0.10215329D 01
0.28003904D 00	0.10202962D 01
0.30552929D 00	0.10188446D 01
0.33098127D 00	0.10172857D 01
0.35639357D 00	0.10157188D 01
0.38176721D 00	0.10142333D 01
0.40710523D 00	0.10129069D 01
0.43241255D 00	0.10118058D 01
0.45769552D 00	0.10109830D 01
0.48296169D 00	0.10104797D 01
0.50821950D 00	0.10103245D 01
0.53347792D 00	0.10105334D 01
0.55874617D 00	0.10111100D 01
0.58403341D 00	0.10120427D 01
0.60934830D 00	0.10133055D 01
0.63469868D 00	0.10148554D 01
0.66009107D 00	0.10166286D 01
0.68553011D 00	0.10185364D 01
0.71101789D 00	0.10204549D 01
0.73655286D 00	0.10222131D 01
0.76212854D 00	0.10235691D 01
0.78773131D 00	0.10241699D 01
0.81333703D 00	0.10234834D 01
0.83890547D 00	0.10206760D 01
0.86437083D 00	0.10144025D 01
0.88962559D 00	0.10024325D 01
0.91449245D 00	0.98098968D 00
0.93867508D 00	0.94357195D 00
0.96167104D 00	0.87881650D 00
0.98261590D 00	0.76657918D 00
0.10000000D 01	0.65434187D 00

```
TOUT EQUALS            0.99999979D-02
STEP SIZE USED           0.99999966D-02
ORDER USED             1
NUMBER OF STEPS             7
NUMBER OF FUNCTION EVALUATIONS        17
NUMBER OF JACOBIAN EVALUATIONS         3
```

T	TDOT
-0.32643718D-18	0.11542665D 01
0.29007111D-01	0.11642853D 01
0.58214262D-01	0.11743041D 01
0.87722314D-01	0.11868282D 01
0.11755567D 00	0.11991918D 01
0.14768190D 00	0.12094732D 01
0.17802932D 00	0.12164186D 01
0.20850283D 00	0.12193699D 01
0.23899781D 00	0.12181804D 01
0.26941184D 00	0.12131153D 01
0.29965358D 00	0.12047397D 01
0.32964883D 00	0.11938036D 01
0.35934375D 00	0.11811352D 01
0.38870558D 00	0.11675506D 01
0.41772128D 00	0.11537847D 01
0.44639481D 00	0.11404449D 01
0.47474352D 00	0.11279843D 01
0.50279402D 00	0.11166901D 01
0.53057802D 00	0.11066827D 01
0.55812816D 00	0.10979208D 01
0.58547406D 00	0.10902083D 01
0.61263857D 00	0.10832023D 01
0.63963417D 00	0.10764209D 01
0.66645961D 00	0.10692515D 01
0.69309674D 00	0.10609623D 01
0.71950772D 00	0.10507176D 01
0.74563262D 00	0.10375997D 01
0.77138770D 00	0.10206395D 01
0.79666459D 00	0.99885515D 00
0.82133046D 00	0.97129705D 00
0.84522944D 00	0.93709689D 00
0.86818530D 00	0.89551496D 00
0.89000519D 00	0.84598201D 00
0.91048440D 00	0.78813438D 00
0.92941190D 00	0.72184835D 00
0.94657681D 00	0.64729094D 00
0.96177645D 00	0.56501867D 00
0.97482774D 00	0.47616859D 00
0.98558488D 00	0.38278688D 00
0.99396709D 00	0.28830251D 00
0.10000000D 01	0.19381814D 00

```
TOUT EQUALS          0.99999964D-01
STEP SIZE USED         0.99999966D-01
ORDER USED           1
NUMBER OF STEPS          10
NUMBER OF FUNCTION EVALUATIONS        23
NUMBER OF JACOBIAN EVALUATIONS         4
```

T	TDOT
-0.43084681D-14	0.23493536D 01
0.58730421D-01	0.23513106D 01
0.11756552D 00	0.23532677D 01
0.17639380D 00	0.23472687D 01
0.23492895D 00	0.23272553D 01
0.29275656D 00	0.22891311D 01
0.34938550D 00	0.22308911D 01
0.40430110D 00	0.21525861D 01
0.45701480D 00	0.20560873D 01
0.50710546D 00	0.19446728D 01
0.55424843D 00	0.18225025D 01
0.59823058D 00	0.16940691D 01
0.63895189D 00	0.15637064D 01
0.67641590D 00	0.14352125D 01
0.71071251D 00	0.13116162D 01
0.74199670D 00	0.11950805D 01
0.77046653D 00	0.10869227D 01
0.79634283D 00	0.98771259D 00
0.81985216D 00	0.89741489D 00
0.84121357D 00	0.81554510D 00
0.86062941D 00	0.74132104D 00
0.87827962D 00	0.67379398D 00
0.89431911D 00	0.61195787D 00
0.90887752D 00	0.55483551D 00
0.92206088D 00	0.50154391D 00
0.93395471D 00	0.45134293D 00
0.94462803D 00	0.40366754D 00
0.95413808D 00	0.35814752D 00
0.96253540D 00	0.31461129D 00
0.96986865D 00	0.27307525D 00
0.97618916D 00	0.23371748D 00
0.98155452D 00	0.19683474D 00
0.98603090D 00	0.16278786D 00
0.98969391D 00	0.13193937D 00
0.99262737D 00	0.10459159D 00
0.99492349D 00	0.80934712D-01
0.99667460D 00	0.61012238D-01
0.99797411D 00	0.44709144D-01
0.99891006D 00	0.31763659D-01
0.99956229D 00	0.21798815D-01
0.10000000D 01	0.11833971D-01

```
TOUT EQUALS              0.10000000D 01
STEP SIZE USED             0.99999966D 00
ORDER USED               1
NUMBER OF STEPS             13
NUMBER OF FUNCTION EVALUATIONS        29
NUMBER OF JACOBIAN EVALUATIONS         5
```

T	TDOT
0.12630016D-13	0.31304404D 01
0.77964133D-01	0.31154475D 01
0.15577237D 00	0.31004547D 01
0.23298686D 00	0.30636286D 01
0.30895379D 00	0.29978627D 01
0.38287999D 00	0.28992579D 01
0.45391668D 00	0.27672587D 01
0.52124291D 00	0.26044615D 01
0.58413974D 00	0.24160797D 01
0.64204689D 00	0.22091435D 01
0.69459691D 00	0.19915709D 01
0.74162543D 00	0.17712637D 01
0.78316009D 00	0.15553588D 01
0.81939337D 00	0.13497149D 01
0.85064583D 00	0.11586566D 01
0.87732619D 00	0.98494800D 00
0.89989323D 00	0.82993940D 00
0.91882316D 00	0.69381744D 00
0.93458410D 00	0.57589708D 00
0.94761801D 00	0.47490664D 00
0.95832943D 00	0.38923613D 00
0.96707982D 00	0.31713391D 00
0.97418613D 00	0.25684974D 00
0.97992231D 00	0.20672890D 00
0.98452257D 00	0.16526657D 00
0.98818563D 00	0.13113207D 00
0.99107918D 00	0.10317198D 00
0.99334423D 00	0.80399846D-01
0.99509917D 00	0.61978331D-01
0.99644315D 00	0.47198211D-01
0.99745908D 00	0.35457295D-01
0.99821601D 00	0.26241343D-01
0.99877114D 00	0.19108194D-01
0.99917142D 00	0.13675572D-01
0.99945492D 00	0.96124748D-02
0.99965205D 00	0.66335494D-02
0.99978660D 00	0.44955364D-02
0.99987682D 00	0.29947619D-02
0.99993634D 00	0.19647350D-02
0.99997506D 00	0.12732269D-02
0.10000000D 01	0.58171883D-03

TOUT EQUALS 0.10000000D 02
STEP SIZE USED 0.99999966D 01
ORDER USED 1
NUMBER OF STEPS 16
NUMBER OF FUNCTION EVALUATIONS 35
NUMBER OF JACOBIAN EVALUATIONS 6

T	TDOT
0.18505710D-13	0.31412804D 01
0.78229256D-01	0.31259429D 01
0.15629714D 00	0.31106054D 01
0.23375952D 00	0.30731771D 01
0.30995598D 00	0.30065497D 01
0.38408700D 00	0.29068451D 01
0.45529823D 00	0.27735500D 01
0.52276449D 00	0.26093204D 01
0.58576424D 00	0.24194392D 01
0.64373644D 00	0.22110086D 01
0.69631466D 00	0.19920132D 01
0.74333710D 00	0.17704104D 01
0.78483518D 00	0.15533775D 01
0.82100597D 00	0.13467978D 01
0.85217506D 00	0.11550047D 01
0.87875620D 00	0.98075898D 00
0.90121301D 00	0.82539744D 00
0.92002607D 00	0.68908718D 00
0.93566736D 00	0.57112021D 00
0.94858208D 00	0.47020109D 00
0.95917742D 00	0.38469672D 00
0.96781692D 00	0.31283418D 00
0.97481913D 00	0.25284416D 00
0.98045912D 00	0.20305522D 00
0.98497189D 00	0.16194811D 00
0.98855653D 00	0.12817964D 00
0.99138087D 00	0.10058566D 00
0.99358581D 00	0.78170495D-01
0.99528939D 00	0.60088993D-01
0.99659026D 00	0.45625481D-01
0.99757067D 00	0.34172740D-01
0.99829890D 00	0.25213011D-01
0.99883132D 00	0.18302177D-01
0.99921401D 00	0.13057584D-01
0.99948420D 00	0.91493346D-02
0.99967147D 00	0.62944596D-02
0.99979892D 00	0.42530579D-02
0.99988413D 00	0.28254100D-02
0.99994019D 00	0.18491938D-02
0.99997659D 00	0.11962167D-02
0.10000000D 01	0.54323960D-03

```
TOUT EQUALS           0.10000000D 03
STEP SIZE USED          0.99999966D 02
ORDER USED            1
NUMBER OF STEPS           19
NUMBER OF FUNCTION EVALUATIONS        41
NUMBER OF JACOBIAN EVALUATIONS         7
```

T	TDOT
0.18506800D-13	0.31412807D 01
0.78229263D-01	0.31259432D 01
0.15629715D 00	0.31106057D 01
0.23375954D 00	0.30731774D 01
0.30995601D 00	0.30065500D 01
0.38408703D 00	0.29068453D 01
0.45529826D 00	0.27735502D 01
0.52276453D 00	0.26093206D 01
0.58576428D 00	0.24194393D 01
0.64373649D 00	0.22110086D 01
0.69631471D 00	0.19920132D 01
0.74333714D 00	0.17704104D 01
0.78483522D 00	0.15533775D 01
0.82100601D 00	0.13467977D 01
0.85217510D 00	0.11550046D 01
0.87875624D 00	0.98075887D 00
0.90121304D 00	0.82539732D 00
0.92002610D 00	0.68908705D 00
0.93566739D 00	0.57112009D 00
0.94858211D 00	0.47020097D 00
0.95917744D 00	0.38469660D 00
0.96781694D 00	0.31283408D 00
0.97481914D 00	0.25284406D 00
0.98045914D 00	0.20305513D 00
0.98497190D 00	0.16194802D 00
0.98855654D 00	0.12817957D 00
0.99138087D 00	0.10058560D 00
0.99358582D 00	0.78170443D-01
0.99528940D 00	0.60088950D-01
0.99659026D 00	0.45625445D-01
0.99757067D 00	0.34172711D-01
0.99829890D 00	0.25212988D-01
0.99883132D 00	0.18302160D-01
0.99921401D 00	0.13057571D-01
0.99948420D 00	0.91493247D-02
0.99967147D 00	0.62944524D-02
0.99979892D 00	0.42530529D-02
0.99988413D 00	0.28254065D-02
0.99994019D 00	0.18491914D-02
0.99997659D 00	0.11962151D-02
0.10000000D 01	0.54323883D-03

```
TOUT EQUALS          0.10000000D 04
STEP SIZE USED         0.99999966D 03
ORDER USED           1
NUMBER OF STEPS          22
NUMBER OF FUNCTION EVALUATIONS        47
NUMBER OF JACOBIAN EVALUATIONS         8
```

T	TDOT
0.18506800D-13	0.31412807D 01
0.78229263D-01	0.31259432D 01
0.15629715D 00	0.31106057D 01
0.23375954D 00	0.30731774D 01
0.30995601D 00	0.30065500D 01
0.38408703D 00	0.29068453D 01
0.45529826D 00	0.27735502D 01
0.52276453D 00	0.26093206D 01
0.58576428D 00	0.24194393D 01
0.64373649D 00	0.22110086D 01
0.69631471D 00	0.19920132D 01
0.74333714D 00	0.17704104D 01
0.78483522D 00	0.15533775D 01
0.82100601D 00	0.13467977D 01
0.85217510D 00	0.11550046D 01
0.87875624D 00	0.98075887D 00
0.90121304D 00	0.82539732D 00
0.92002610D 00	0.68908705D 00
0.93566739D 00	0.57112009D 00
0.94858211D 00	0.47020097D 00
0.95917744D 00	0.38469660D 00
0.96781694D 00	0.31283408D 00
0.97481914D 00	0.25284406D 00
0.98045914D 00	0.20305513D 00
0.98497190D 00	0.16194802D 00
0.98855654D 00	0.12817957D 00
0.99138087D 00	0.10058560D 00
0.99358582D 00	0.78170443D-01
0.99528940D 00	0.60088950D-01
0.99659026D 00	0.45625445D-01
0.99757067D 00	0.34172711D-01
0.99829890D 00	0.25212988D-01
0.99883132D 00	0.18302160D-01
0.99921401D 00	0.13057571D-01
0.99948420D 00	0.91493247D-02
0.99967147D 00	0.62944524D-02
0.99979892D 00	0.425305[illegible]9D-02
0.99988413D 00	0.28254065D-02
0.99994019D 00	0.18491914D-02
0.99997659D 00	0.11962151D-02
0.10000000D 01	0.54323883D-03

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FLUID INJECTION THROUGH ONE SIDE OF A LONG VERTICAL CHANNEL BY QUASILINEARIZATION

by

KENNETH SIDOROWICZ

B. Arch., Kansas State University, 1974

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1976

This report presents numerical solutions for the velocity functions, the temperature distribution function and the heat transfer rate function for a long vertical channel with fluid injected through one side. The Navier-Stokes equations and an energy equation are investigated yielding three coupled, nonlinear, ordinary differential equations. An attempt is made to obtain solutions to the problems governing differential equations by two approximate methods:

1. The quasilinearization technique.
2. The perturbation technique.

The quasilinearization technique is well suited to obtain simultaneous solutions for all the required functions. The perturbation technique has limited applications to this problem. Extensive numerical results are presented for both methods. The solutions obtained by the approximate methods are compared and discussed.

This type of problem has applications in transpiration cooling processes and the cooling of electronic equipment.