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TRAFFIC ASSIGNMENT
BY THE DISCRETE MAXIMUM PRINCIPLE

by

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SUMMARY

This paper is concerned with the "Minimum Total Time Assignment Problem", which may be stated as follows. We have a network which represents the main traffic arterials in an urban area. The number of interzonal movements in the area is assumed known. In this formulation of the problem, travel time is assumed to be the significant factor in determining the merits of a particular route. Therefore, in the optimal assignment, the total travel time will be the distribution criterion. A generalized discrete version of the maximum principle is employed to minimize the total travel time. Link travel times per vehicle are assumed to be constant.

Through the use of the Gravity Model interzonal movements have been predicted for Manhattan, Kansas (18). The maximum principle is used to assign those movements to the existing arterial network on the basis of minimum total travel time, assuming link travel time per vehicle to be independent of volume.

INTRODUCTION

In recent years several attempts have been made to formulate and solve the Urban Traffic Assignment Problem. During the development of transportation planning techniques, the use of an assignment procedure has become increasingly important. In the early use of assignment techniques the volume of vehicles assigned to any particular link in the network often did not correspond with the flows that occurred in the real network. However, the technique did provide the engineer with a general indication of the volumes to be expected and the level of service being provided by the network under evaluation. In 1952, the technique was summed up as follows:

"Traffic assignment is fundamental to the justification of a proposed highway facility and to its structural and geometric design, to spotting points for access and for advance planning of traffic regulation and control measures. As yet, traffic assignment is considered to be more of an art than a science..." (1)

Since 1952, the large urban area transportation studies and other planning agencies have developed assignment procedures to the point where less and less emphasis is placed on personal judgment. The net result of this intensive development activity has been the creation of several different traffic assignment techniques.

REVIEW OF LITERATURE

Distribution criteria can be stated in different ways. In the transportation literature the first attempt to reach an explicit statement of such criteria was made by J. C. Wardrop in 1952 (2). He suggested two different possibilities. The principle of equal travel times says that, in the optimal assignment, the travel time between two points will be the same on all routes used and less than the travel time of even a single vehicle on any other route between the same two points. The principle of overall minimization says that in the optimal assignment the average travel for all travellers in the system has its minimum value.

In his paper in 1956, E. Wilson Campbell (3) presented the coding techniques and machine procedures worked out by the Detroit Staff to facilitate rapid machine assignment. The allocation of trips to expressways was based on combinations of distance and speed ratios. There are several features that contribute to the speed and workability of this system. First, the alternate distances can be rapidly and accurately estimated by a machine technique. Second, the concept of treating an expressway trip in parts and matching parts together to form trips eliminates the necessity of reviewing each zone to zone transfer and makes coding vastly simpler. Finally, the adaptability of this procedure to high speed computing and summarizing makes possible a tremendous conservation of time.

J. Douglas Carroll, Jr. (4) developed in 1959 a method which would rapidly assign travelers to the surface arterial streets as well as to express routes and also to rapid transit and surface transit routes.

In 1961, N. A. Irwin, N. Dodd, and H. G. Von Cube (5) presented a paper describing a systematic model for predicting vehicular traffic flow using high-speed electronic computing techniques. This model contains a direct feedback mechanism by which capacity restraints and the resultant congestion are allowed to affect route generation, trip distribution and vehicle assignment in successive programs to give promising results. The use of capacity restraints and the resultant feedback of congestion effects by means of travel time is felt to be essential in traffic simulation programs.

The diversion curve technique is one in which the total number of trips between an origin and destination are divided between two routes, one of expressway characteristics and the other an arterial or equivalent highway (6), (7), (8). The technique originated as a solution to the problem of locating a single expressway relative to some existing highway. The diversion curves are based on data obtained from observations at other locations where two "similar" facilities exist. Diversion curves have been developed for a variety of parameters such as time saved by using the expressway and ratio of time by expressway to time by alternative. Similar expressions relating to distance have also been used, and in some cases curves have been developed relating the cost differences between using the expressway and some other facility. In each case a curve indicates for a given value of the parameters used, such as time saved, the percentage of drivers that will use the expressway.

B. V. Martin (9) discussed in 1963 the results of a comparison of several minimum path algorithms that may be used in transportation planning. The algorithms considered include those of Moore, the Road Research Laboratory, Shimbel and a modification made by the author of the Road Research Laboratory algorithm. An attempt was made to relate the most suitable algorithm to three characteristics of network: the number of links; the number of nodes;

and, the link-node ratio.

Morton M. Schneider (10) suggested resolving the traffic into trips, each with an origin and a destination, then finding the best path through the network for each trip and noting the aggregate appearances on every network member of trips following these paths.

W. W. Mosher, Jr. (11) presented a "capacity restraint" algorithm which permits the evaluation of network performance based on arbitrarily selected network figures of merit. The values of these figures of merit depend on network loading, and are governed by individually determined link performance functions. Each link performance function may be any linear or nonlinear function relating link flow to cost (e.g., distance) for that flow. Once a performance function is established for each network link, the network can be loaded in an optimum manner either by minimizing the figure of merit for the entire system or by equalizing the path figures of merit over appropriate sets of paths.

Joseph A. Wattleworth and Paul W. Shuldiner (12) introduced some of the basic concepts and computational techniques of linear programming as applied to traffic assignment. The linear programming technique is used to obtain partial solutions to the Charnes-Cooper multi-copy (multi-origin) problem which permits the placing of capacities on individual links and guarantees an optimal (minimum travel time) solution. A method of enlarging the network to place restrictions on turning movements is also presented. Such restrictions as time penalties and prohibiting individual turns were evaluated. These could be incorporated into either the linear programming or minimum time-path models.

Charles Pinnell and Gilbert T. Satterly, Jr. (13) presented an application of a linear programming model (the multi-copy missing model developed by A. Charnes and W. W. Cooper) as a solution to the problem of arterial street system analysis. The example presents a specific use of the technique, that of holding a freeway volume at or below a fixed amount and developing the resulting optimum flow pattern in the system. It is shown that the results of simulation can be used as a guide to determine those control measures necessary for optimum use of the street system. It is also illustrated that the non-linear travel time vs. volume curve (delay curve) can be piecewise linearly approximated and included in the traffic assignment procedure in order to simulate closely the actual traffic behavior in a street network.

In 1964, Brian V. Martin and Marvin L. Manheim (14) described an incremental traffic assignment technique which had been incorporated into a more general traffic assignment computer program, which can be used for the comparison of several of the traffic assignment procedures in use today.

Thus, in its brief history, traffic assignment has been developed from a completely manual task to a highly automated, versatile, and powerful tool for transportation systems planning.

DISCRETE MAXIMUM PRINCIPLE

In recent years, engineers have become more and more concerned with optimization. A simple optimization problem corresponds to the finding of the extreme value of a function in calculus. This can be accomplished by differentiation. As the problem becomes more involved, partial differentiation and the calculus of variations may have to be used. More often than not, optimal problems in engineering and industry cannot be solved by direct applications of these conventional mathematical methods. A variety of approaches more sophisticated than the conventional methods has been proposed to solve complex problems. Among them are Dynamic Programming and the Maximum Principle.

The maximum principle is an optimization technique which was first discovered by a Russian mathematician, Pontryagin in 1956 (16). The first attempt to extend the maximum principle to the optimization of stagewise processes which are linear in the state variables was made by Rozonoer in 1959 (17).

Dr. L. T. Fan, Professor of Chemical Engineering at Kansas State University and C. S. Wang have recently accomplished a major break through in the development of a discrete version of the maximum principle (15).

The transformation of the process at the n-th stage is described by a set of performance equations

$$x_i^n = T_i^n (x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}, \theta_1^n, \theta_2^n, \ldots, \theta_t^n)$$

$$i = 1, 2, \ldots, s$$

or in vector form

$$X^n = T^n (X^{n-1}, \theta^n), \qquad n = 1, 2, \ldots N$$

where T^n is called the transformation operator. X^n is an s-dimensional state variable, and θ^n is a t-dimensional decision variable.

A stage may represent any real or abstract entity, e. g., a space unit, a time period, or an economic activity, in which a certain transformation takes place. Those variables which are transformed in each stage are called state variables. The desired transformation for the state variable is achieved through manipulation of decision variables which remain or may be considered to remain constant within each stage of the process.

A typical optimization problem associated with such a process is to find a sequence of θ^n, n = 1, 2, ... N to maximize, $\sum_{i=1}^{S} C_i X_i^N$. Here C_i, i = 1, 2, ... S are some specified constants. The function, $\sum_{i=1}^{S} C_i X_i^N$, which is to be maximized, is the objective function of the process.

The procedure for solving such an optimization problem by the discrete maximum principle, is to introduce an s-dimensional covariant vector, Z^n, and a Hamiltonian function, H^n, satisfying

$$H^n = \sum_{i=1}^{s} Z_i^n T_i^n (X^{n-1}, \theta^n), \qquad n = 1, 2, \ldots N \tag{1}$$

$$Z_i^{n-1} = \frac{\partial H^n}{\partial X_i^{n-1}}, \qquad i = 1, 2, \ldots, s; \; n = 1, 2, \ldots, N$$

or in vector form

$$Z^{n-1} = \frac{\partial H^n}{\partial X^{n-1}}, \qquad n = 1, 2, \ldots, N \tag{2}$$

and

$Z_i^n = C_i$ (for fixed [illegible] problems), i = 1, 2, ..., s and to determine the optimal sequence of the decisions, θ^n, from the conditions

H^n = maximum, n = 1, 2, ..., N, at the boundary and $\frac{\partial H^n}{\partial \theta^n} = 0$

at interior points.

Both X and Z are considered as fixed in maximizing the Hamiltonian.

If the minimizing sequence instead of the maximizing sequence of the decision vector is to be determined, the procedure remains unchanged except that the condition, H^n = maximum, is replaced by H^n = minimum.

In a word, for a process with all the performance equations and the initial and/or final values of some of the state variables given, find the values of the decision variables at each stage, subject to certain constraints, in such a way that the objective function is maximized or minimized.

GENERAL FORMULATION OF THE PROBLEM

This section defines the step-by-step procedures that occur during a complete traffic assignment. Traffic assignment may be defined as the process of allocating a given set of trip interchanges to a specific transportation system. The traffic assignment procedure in this paper is based essentially on the selection by the maximum principle of a minimum-time path between zones.

Definitions:

1. Objective function - a linear combination of the variables to be optimized by the selected solution. In this model the total travel time accumulated by all vehicles is to be minimized.
2. Zone centroid - a point of trip origin or destination.
3. Node - a point where segments of the arterial street system connect.
4. Link - a connection between two nodes representative of a segment of the arterial street system.
5. Path - a series of connected links representative of a trip route.
6. Network - the combination of all links and nodes.

Notation:

$X_j^{n,m}$ - state variable for flow from node (n, m).

$\theta_j^{n,m}$ - decision variable at node (n,m).

$t_j^{n,m}$ - basic travel time per vehicle via the node (n,m) on link j.

$K_j^{n,m}$ - travel time coefficient at node (n,m).

where $j = 1$, for horizontal link

$j = 2$, for vertical link

$H^{n,m}$ - Hamiltonian function at node (n,m).

$V^{n,m}$ - total desire (Vehicles) via node (n,m)

In Figure 1, the numbers on the links which connect nodes denote the time function between the nodes. Suppose that the problem is to find the least-time path from node (1,1) (origin) to node (3,3) (destination) with an additional restriction that the passages will be allowed along any path only in the directions preassigned. Common sense tells us that we can enumerate all possible paths between node (1,1) and node (3,3), and then pick the one whose travel time is smallest.

(1) path (1,1) - (1,2) - (1,3) - (2,3) - (3,3) = 23

(2) path (1,1) - (1,2) - (2,2) - (2,3) - (3,3) = 18

(3) path (1,1) - (1,2) - (2,2) - (3,2) - (3,3) = 14

(4) path (1,1) - (2,1) - (2,2) - (2,3) - (3,3) = 14

(5) path (1,1) - (2,1) - (2,2) - (3,2) - (3,3) = 10

(6) path (1,1) - (2,1) - (3,1) - (3,2) - (3,3) = 15

The least-time path is the 5th one listed, i. e., (1,1) - (2,1) - (2,2) - (3,2) - (3,3), whose total time is 10.

But enumeration of all the possibilities would be prohibitive in a problem of practical size.

We will now show that this particular traffic assignment problem can be readily solved by means of the discrete maximum principle. This method works so well in solving such a problem that numerical solutions can often be reached by hand calculations.

Suppose that there is an n x m network as shown in Figure 2. The

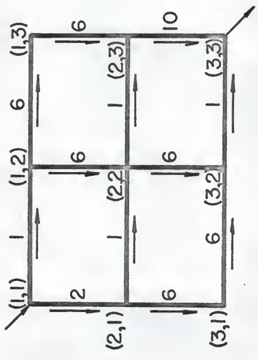


Fig. 1. Diagram of directed network

directions of flows in the links and total vehicles leaving the network at each node, $V^{n,m}$, are prescribed. Let $\theta_j^{n,m}$ = that fraction of the volume entering node (n,m) which is assigned to the jth link as it leaves node (n,m).

$K_L^{n,m}$ = travel time coefficient of left turn penalties at node (n,m).

$K_R^{n,m}$ = travel time coefficient of right turn penalties at node (n,m).

If there are n x m nodes, the problem is to determine the quantities $\theta_j^{n,m}$, j = 1, 2, to minimize the total travel time

$X_3^{N,M}$ (horizontal links) plus

$X_4^{N,M}$ (vertical links).

We assume that link travel time is constant and does not vary with the number of vehicles on the link. We further assume that $V^{n,m}$ is split up in such a way that $V^{n,m}/2$ is on the vertical link and $V^{n,m}/2$ is on the horizontal link as shown in Figure 2. That is we assume there is no input or output right at the node except the destination node. This assumption enables us to determine the number of left or right turns at each node. For each copy there is only one destination with multiple origins.

For the problem described above and considering each node as a stage, we can write the following performance equations

$$X_1^{n,m} = \theta_1^{n,m}\,(X_1^{n,m-1} - V^{n,m}/2) + (1 - \theta_2^{n,m})(X_2^{n-1,m} - V^{n,m}/2) \qquad (3)$$

$$X_2^{n,m} = (1 - \theta_1^{n,m})(X_1^{n,m-1} - V^{n,m}/2) + \theta_2^{n,m}(X_2^{n-1,m} - V^{n,m}/2) \qquad (4)$$

$$X_3^{n,m} = X_3^{n,m-1} + K_1^{n,m}\left[\theta_1^{n,m}(X_1^{n,m-1} - V^{n,m}/2) + (1 - \theta_2^{n,m})\right.$$
$$\left.(X_2^{n-1,m} - V^{n,m}/2)\right] + K_L^{n,m}(1 - \theta_2^{n,m})(X_2^{n-1,m} - V^{n,m}/2) \qquad (5)$$

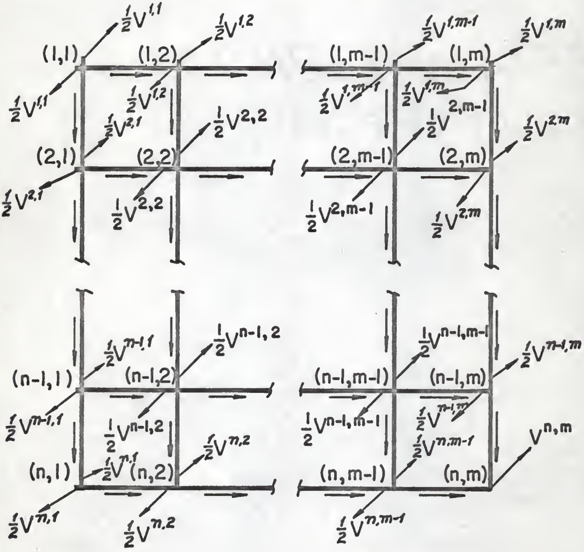


Fig. 2. n x m network.

$$X_4^{n,m} = X_4^{n-1,m} + K_2^{n,m}\left\lfloor(1 - \theta_1^{n,m})(X_1^{n,m-1} - V^{n,m}/2) + \theta_2^{n,m}\right.$$

$$\left.(X_2^{n-1,m} - V^{n,m}/2)\right\rfloor + K_R^{n,m}(1 - \theta_1^{n,m})(X_1^{n,m-1} - V^{n,m}/2) \qquad (6)$$

Here $X_j^{n,m}$, $j = 1, 2$, are the state variables representing the number of vehicles from node (n,m) in the horizontal and vertical links respectively. Where $X_3^{n,m}$ and $X_4^{n,m}$, the sum of which are to be minimized, represent the total cummulative travel time up to node (n,m) in the horizontal and vertical links respectively.

The Hamiltonian function can be written as

$$H^{n,m} = Z_1^{n,m}\left\lfloor\theta_1^{n,m}(X_1^{n,m-1} - V^{n,m}/2) + (1 - \theta_2^{n,m})(X_2^{n-1,m} - V^{n,m}/2)\right\rfloor$$

$$+ Z_2^{n,m}\left\lfloor(1 - \theta_1^{n,m})(X_1^{n,m-1} - V^{n,m}/2) + \theta_2^{n,m}(X_2^{n-1,m} - V^{n,m}/2)\right\rfloor$$

$$+ Z_3^{n,m}\left\{X_3^{n,m-1} + K_1^{n,m}\left\lfloor\theta_1^{n,m}(X_1^{n,m-1} - V^{n,m}/2) + (1 - \theta_2^{n,m})\right.\right.$$

$$\left.\left.(X_2^{n-1,m} - V^{n,m}/2)\right\rfloor + K_L^{n,m}(1 - \theta_2^{n,m})(X_2^{n-1,m} - V^{n,m}/2)\right\}$$

$$+ Z_4^{n,m}\left\{X_4^{n-1,m} + K_2^{n,m}\left[(1 - \theta_1^{n,m})(X_1^{n,m-1} - V^{n,m}/2) + \theta_2^{n,m}\right.\right.$$

$$\left.\left.(X_2^{n-1,m} - V^{n,m}/2)\right\rfloor + K_R^{n,m}(1 - \theta_1^{n,m})(X_1^{n,m-1} - V^{n,m}/2)\right\} \quad (7)$$

where Z = covariant vector. The values of $\theta_j^{n,m}$, $j = 1, 2$, are determined in such a way that $H^{n,m}$ is a minimum.

Applying equation (2), $Z^{n-1} = \frac{\partial H^n}{\partial X^{n-1}}$, n = 1, 2, ..., N to this problem gives

$$Z_1^{n,m-1} = \frac{\partial H^{n,m}}{\partial X_1^{n,m-1}} = Z_1^{n,m}\,\theta_1^{n,m} + Z_2^{n,m}(1 - \theta_1^{n,m}) + K_1^{n,m}\,\theta_1^{n,m}$$

$$+ K_2^{n,m}(1 - \theta_1^{n,m}) + K_R^{n,m}(1 - \theta_1^{n,m}) \qquad (8)$$

and

$$Z_2^{n-1,m} = \frac{\partial H^{n,m}}{\partial X_2^{n-1,m}} = Z_1^{n,m}(1 - \theta_2^{n,m}) + Z_2^{n,m}\,\theta_2^{n,m} + K_1^{n,m}(1 - \theta_2^{n,m})$$

$$+ K_L^{n,m}(1 - \theta_2^{n,m}) + K_2^{n,m}\,\theta_2^{n,m} \qquad (9)$$

where $Z_3^{n,m} = Z_4^{n,m} = 1$. However, it should be noted that the recurrence relation of the state variables is applicable only when the minima of the Hamiltonian functions occur at stationary points and the optimal decisions so obtained lie within the constraints.

For this problem, the minima do not occur at the stationary points since the Hamiltonian function is linear in the decision variables. For such a case, the optimal decisions are either the lower bounds or the upper bounds of the decision variables. Therefore, a minima-seeking method instead of the recurrence relation should be used.

There are several computational procedures which can be used to solve problems to which the recurrence relations of the state variables are not applicable. All such procedures employ one type or another of trial and error technique.

Through the use of the equations for the covariant vectors obtained in the last section (Equations (8) and (9)) it is possible to solve this problem by back and forth calculation. The procedure for this method is as follows:

1. Assume θ's for all stages.
2. Starting at node (1,1) work forward through the network calculating all X's.
3. Start from the destination node and work backward to calculate the covariant vectors at each stage in terms of the θ's at that stage.

4. Minimize the Hamiltonian function at each stage, in turn, thus determining the desired value of the decision variables at each stage.
5. Return to step 2 and repeat until two consecutive sets of decision variables are identical.

The technique may be simply illustrated in the following example.

In Figure 3, suppose that the problem is to determine the optimal path from the origins (1,1) and (2,1) to the destination (3,3), assuming that $V^{11} = -4$, $V^{21} = -4$ and $V^{33} = 8$. The direction of flow in each link is preassigned. For convenience in calculation, let us assume $K_L^{n,m} = 3$ and $K_R^{n,m} = 1$. The link K's are assigned as shown in the figure.

Since the Hamiltonian function is linear in the decision variables, i.e., the minimum occurs at the boundary, the values of the θ's at each stage are one or zero. Hence, let us arbitrarily assume θ's throughout the network as follows:

$$\theta_1^{1,1} = 1, \quad \theta_2^{1,1} = 0; \quad \theta_1^{1,2} = 1, \quad \theta_2^{1,2} = 0;$$

$$\theta_1^{1,3} = 0, \quad \theta_2^{1,3} = 1; \quad \theta_1^{2,1} = 0, \quad \theta_2^{2,1} = 1;$$

$$\theta_1^{2,2} = 1, \quad \theta_2^{2,2} = 0; \quad \theta_1^{2,3} = 0, \quad \theta_2^{2,3} = 1;$$

$$\theta_1^{3,1} = 1, \quad \theta_2^{3,1} = 0; \quad \theta_1^{3,2} = 1, \quad \theta_2^{3,2} = 0.$$

We can now work forward starting from stage (1,1) for X's by application of the equations(3), (4), (5) and (6) as follows.

$$X_1^{1,1} = 4, \quad X_2^{1,1} = 0, \quad X_3^{1,1} = 86, \quad X_4^{1,1} = 0;$$

$$X_1^{1,2} = 4, \quad X_2^{1,2} = 0, \quad X_3^{1,2} = 286, \quad X_4^{1,2} = 0;$$

$$X_1^{1,3} = 0, \quad X_2^{1,3} = 4, \quad X_3^{1,3} = 286, \quad X_4^{1,3} = 204;$$

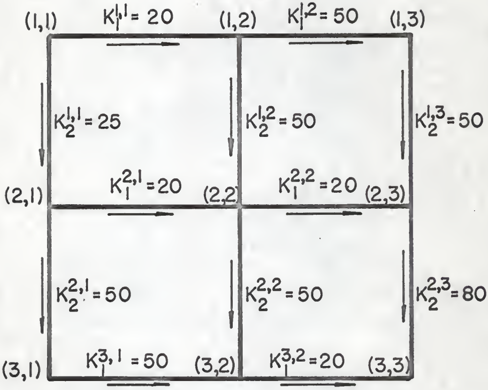


Fig. 3. Network

$$X_1^{2,1} = 0, \quad X_2^{2,1} = 4, \quad X_3^{2,1} = 0, \quad X_4^{2,1} = 202;$$

$$X_1^{2,2} = 0, \quad X_2^{2,2} = 0, \quad X_3^{2,2} = 0, \quad X_4^{2,2} = 0;$$

$$X_1^{2,3} = 0, \quad X_2^{2,3} = 4, \quad X_3^{2,3} = 0, \quad X_4^{2,3} = 524;$$

$$X_1^{3,1} = 4, \quad X_2^{3,1} = 0, \quad X_3^{3,1} = 212, \quad X_4^{3,1} = 202;$$

$$X_1^{3,2} = 4, \quad X_2^{3,2} = 0, \quad X_3^{3,2} = 292, \quad X_4^{3,2} = 0.$$

The total travel time for this assumption is

$$X_3^{1,3} + X_4^{2,3} + X_4^{3,1} + X_3^{3,2} = 1304 \text{ units.}$$

By application of the recurrence equations for covariant vectors (Equations (8) and (9)), we now work backward starting from the destination node (3,3). We obtain $Z_1^{3,2} = Z_2^{2,3} = 0$ and $Z_1^{3,1} = 20$, $Z_2^{1,3} = 80$, $Z_1^{2,2} = 81$, $Z_2^{2,2} = 23$. Then, by equation (7), we obtain $H^{2,2} = 0$. Therefore $H^{2,2}$ is independent of $\theta^{2,2}$, so let us arbitrarily choose $\theta_1^{2,2} = 1$ and $\theta_2^{2,2} = 0$.

Then, at stage (2,1) $Z_1^{2,1} = 101$ and $Z_2^{2,1} = 73$. Therefore, if $\theta_1^{2,1} = 0$ and $\theta_2^{2,1} = 1$, $H^{2,1} = 494$. Now let us assume $\theta_1^{2,1} = 1$ and $\theta_2^{2,1} = 0$, then $H^{2,1} = 490$. Let us then assume that all cars have no turns at node (2,1), i.e., $\theta_1^{2,1} = \theta_2^{2,1} = 1$, then $H^{2,1} = 488$. Now let us assume that all cars have turns at node (2,1), i.e., $\theta_1^{2,1} = \theta_2^{2,1} = 0$, then $H^{2,1} = 496$. Obviously, $H^{2,1}$ for all cars having no turns at node (2,1) is less than that for the other three cases. We, therefore, choose the least Hamiltonian function, i.e., $\theta_1^{2,1} = \theta_2^{2,1} = 1$.

At stage (1,3) the boundary conditions dictate the value of $\theta_1^{1,3}$ and $\theta_2^{1,3}$.

Then at stage (1,2) $Z_1^{1,2} = 131$ and $Z_2^{1,2} = 104$. Therefore, $H^{1,2} = 810$ for $\theta_1^{1,2} = 1$ and $\theta_2^{1,2} = 0$. Now let us assume $\theta_1^{1,2} = 0$ and $\theta_2^{1,2} = 1$, we obtain $H^{1,2} = 706$. Let us then assume that all cars have no turns at node (1,2), i.e., $\theta_1^{1,2} = \theta_2^{1,2} = 1$, then $H^{1,2} = 810$. Now let us assume that all cars have turns at node (1,2), i.e., $\theta_1^{1,2} = \theta_2^{1,2} = 0$, then $H^{1,2} = 706$. Hence, we determine either $\theta_1^{1,2} = 0$ and $\theta_2^{1,2} = 1$ or $\theta_1^{1,2} = \theta_2^{1,2} = 0$. We will arbitrarily choose $\theta_1^{1,2} = 0$ and $\theta_2^{1,2} = 1$.

Finally at stage (1,1), $Z_1^{1,1} = 181$ and $Z_2^{1,1} = 123$. Therefore $H^{1,1} = 810$ for $\theta_1^{1,1} = 1$ and $\theta_2^{1,1} = 0$. Let us assume $\theta_1^{1,1} = 0$ and $\theta_2^{1,1} = 1$, we find $H^{1,1} = 594$. Let us then assume $\theta_1^{1,1} = \theta_2^{1,1} = 1$, then $H^{1,1} = 698$. Now let us assume $\theta_1^{1,1} = \theta_2^{1,1} = 0$, then $H^{1,1} = 706$. Hence we choose the least Hamiltonian and determine $\theta_1^{1,1} = 0$ and $\theta_2^{1,1} = 1$ at stage (1,1).

Hence, we obtain a new set of θ's in the first iteration as follows:

$$\theta_1^{1,1} = 0, \quad \theta_2^{1,1} = 1; \quad \theta_1^{1,2} = 0, \quad \theta_2^{1,2} = 1;$$

$$\theta_1^{1,3} = 0, \quad \theta_2^{1,3} = 1; \quad \theta_1^{2,1} = 1, \quad \theta_2^{2,1} = 1;$$

$$\theta_1^{2,2} = 1, \quad \theta_2^{2,2} = 0; \quad \theta_1^{2,3} = 0, \quad \theta_2^{2,3} = 1;$$

$$\theta_1^{3,1} = 1, \quad \theta_2^{3,1} = 0; \quad \theta_1^{3,2} = 1, \quad \theta_2^{3,2} = 0.$$

Likewise, for the second iteration, we obtain all X's by application of equations (3) to (6) as follows.

$$X_1^{1,1} = 0, \quad X_2^{1,1} = 4, \quad X_3^{1,1} = 0, \quad X_4^{1,1} = 102;$$

$$X_1^{1,2} = 0, \quad X_2^{1,2} = 0, \quad X_3^{1,2} = 0, \quad X_4^{1,2} = 0;$$

$$X_1^{1,3} = 0, \quad X_2^{1,3} = 0, \quad X_3^{1,3} = 0, \quad X_4^{1,3} = 0;$$

$$X_1^{2,1} = 2, \quad X_2^{2,1} = 6, \quad X_3^{2,1} = 40, \quad X_4^{2,1} = 252;$$

$$X_1^{2,2} = 2, \quad X_2^{2,2} = 0, \quad X_3^{2,2} = 80, \quad X_4^{2,2} = 0;$$

$$X_1^{2,3} = 0, \quad X_2^{2,3} = 2, \quad X_3^{2,3} = 80, \quad X_4^{2,3} = 162;$$

$$X_1^{3,1} = 6, \quad X_2^{3,1} = 0, \quad X_3^{3,1} = 318, \quad X_4^{3,1} = 252;$$

$$X_1^{3,2} = 6, \quad X_2^{3,2} = 0, \quad X_3^{3,2} = 438, \quad X_4^{3,2} = 0.$$

Hence the total travel time for the second iteration is $X_3^{2,3} + X_4^{2,3} + X_4^{3,1} + X_3^{3,2} = 932$ units.

By application of equations (8) and (9), we obtain $Z_1^{3,2} = Z_2^{2,3} = 0$ and $Z_1^{3,1} = 20$, $Z_2^{1,3} = 80$, $Z_1^{2,2} = 81$, $Z_2^{2,2} = 23$. Then, by equation (7), we obtain $H^{2,2} = 262$ for $\theta_1^{2,2} = 1$ and $\theta_2^{2,2} = 0$. Let us assume $\theta_1^{2,2} = 0$ and $\theta_2^{2,2} = 1$, then $H^{2,2} = 148$. Let us then assume that all cars have no turns at node (2,2), i.e., $\theta_1^{2,2} = \theta_2^{2,2} = 1$, then $H^{2,2} = 242$. Now let us assume that all cars have turns at node (2,2), i.e., $\theta_1^{2,2} = \theta_2^{2,2} = 0$, then $H^{2,2} = 188$. Therefore, the minimum Hamiltonian $H^{2,2} = 148$. Hence we choose the least Hamiltonian and obtain $\theta_1^{2,2} = 0$ and $\theta_2^{2,2} = 1$ at stage (2,2).

At stage (2,1), we find that the minimum Hamiltonian is obtained when $\theta_1^{2,1} = 1$ and $\theta_2^{2,1} = 0$. Thus $Z_1^{2,1} = 74$ and $Z_2^{2,1} = 73$.

At stage (1,2), minimization of $H^{1,2}$ occurs when $\theta_1^{1,2} = 0$ and $\theta_2^{1,2} = 1$. Thus $Z_1^{1,2} = 131$ and $Z_2^{1,2} = 73$.

Finally, at stage (1,1), we obtain the minimum Hamiltonian when $\theta_1^{1,1} = 0$ and $\theta_2^{1,1} = 1$. Thus $Z_1^{1,1} = 124$ and $Z_2^{1,1} = 97$.

Hence, we obtain another set of θ's in the second iteration as follows:

$$\theta_1^{1,1} = 0, \quad \theta_2^{1,1} = 1; \quad \theta_1^{1,2} = 0, \quad \theta_2^{1,2} = 1;$$

$$\theta_1^{1,3} = 0, \quad \theta_2^{1,3} = 1; \quad \theta_1^{2,1} = 1, \quad \theta_2^{2,1} = 0;$$

$$\theta_1^{2,2} = 0, \quad \theta_2^{2,2} = 1; \quad \theta_1^{2,3} = 0, \quad \theta_2^{2,3} = 1;$$

$$\theta_1^{3,1} = 1, \quad \theta_2^{3,1} = 0; \quad \theta_1^{3,2} = 1, \quad \theta_2^{3,2} = 0.$$

Similarly, for the third iteration, we obtain all X's by equations (3) to (6).

$$X_1^{1,1} = 0, \quad X_2^{1,1} = 4, \quad X_3^{1,1} = 0, \quad X_4^{1,1} = 102;$$

$$X_1^{1,2} = 0, \quad X_2^{1,2} = 0, \quad X_3^{1,2} = 0, \quad X_4^{1,2} = 0;$$

$$X_1^{1,3} = 0, \quad X_2^{1,3} = 0, \quad X_3^{1,3} = 0, \quad X_4^{1,3} = 0;$$

$$X_1^{2,1} = 8, \quad X_2^{2,1} = 0, \quad X_3^{2,1} = 178, \quad X_4^{2,1} = 102;$$

$$X_1^{2,2} = 0, \quad X_2^{2,2} = 8, \quad X_3^{2,2} = 178, \quad X_4^{2,2} = 408;$$

$$X_1^{2,3} = 0, \quad X_2^{2,3} = 0, \quad X_3^{2,3} = 178, \quad X_4^{2,3} = 0;$$

$$X_1^{3,1} = 0, \quad X_2^{3,1} = 0, \quad X_3^{3,1} = 0, \quad X_4^{3,1} = 102;$$

$$X_1^{3,2} = 8, \quad X_2^{3,2} = 0, \quad X_3^{3,2} = 184, \quad X_4^{3,2} = 408.$$

Hence the total travel time for the third iteration is $X_3^{2,3} + X_4^{3,1} + X_3^{3,2} + X_4^{3,2} = 872$ units.

Likewise, by equations (8) and (9), we obtain $Z_1^{3,2} = Z_2^{2,3} = 0$ and $Z_1^{3,1} = 20$, $Z_2^{1,3} = 80$, $Z_1^{2,2} = 81$, $Z_2^{2,2} = 23$. When we repeat the iterative process, we find that the last two consecutive sets of decision variables are identical. Therefore, we determine that for $V^{1,1} = -4$, $V^{2,1} = -4$ and $V^{3,3} = 8$, the least-time path for $V^{1,1}$ is (1,1) - (2,1) - (2,2) - (3,2) - (3,3); and the least-time path for $V^{2,1}$ is (2,1) - (2,2) - (3,2) - (3,3).

It took three iterations to solve the 2 x 2 network problem. The number of iterations may increase with increasing dimension of the network. Figure 4 shows the number of iterations vs. the total travel time for the example demonstrated above. However, the process may be further simplified if we observe that there is only one possible value for each covariant vector $Z_1^{n,m-1}$ if $\theta_1^{n,m}$ has been selected. Likewise there is only one value of $Z_2^{n-1,m}$ if $\theta_2^{n,m}$ has been selected. Therefore if we start at the destination node and work backward from it as we did in the previous example, we find $Z_1^{3,2} = Z_2^{2,3} = 0$ and $Z_1^{3,1} = 20$.

At node (2,2) it is necessary now to determine the relevant value of the covariant vectors $Z_1^{2,2}$ and $Z_2^{2,2}$. This can be accomplished for $Z_1^{2,2}$ by simply adding $Z_2^{2,3} + K_2^{2,3} + K_R^{2,3}$ (see equation (8)). The value of $Z_2^{2,2}$ can be found by adding $Z_1^{3,2} + K_1^{3,2} + K_L^{3,2}$ (see equation (9)). If we add $Z_1^{2,2} + K_1^{2,2}$ and compare that value to $Z_2^{2,2} + K_2^{2,2} + K_R^{2,2}$ we can determine by inspection the desired values for $\theta_1^{2,2}$. If $Z_1^{2,2} + K_1^{2,2}$ is smaller than $Z_2^{2,2} + K_2^{2,2} + K_R^{2,2}$ then $\theta_1^{2,2}$ should equal 1. If the reverse is true, $\theta_1^{2,2}$ should equal 0 in order to minimize the Hamiltonian. If we add $Z_2^{2,2} + K_2^{2,2}$ and compare that value to $Z_1^{2,2} + K_1^{2,2} + K_L^{2,2}$ we can determine the desired value of $\theta_2^{2,2}$. Likewise, at node (2,1) it is necessary now to determine the relevant values of the covariant vectors $Z_1^{2,1}$ and $Z_2^{2,1}$. This can be accomplished by use of equations (8) and (9). If we add $Z_1^{2,1} + K_1^{2,1}$ and compare that value to $Z_2^{2,1} + K_2^{2,1} + K_R^{2,1}$ we determine by inspection the desired value for $\theta_1^{2,1}$. If we add $Z_2^{2,1} + K_2^{2,1}$ and compare that value to $Z_1^{2,1} + K_1^{2,1} + K_L^{2,1}$ we determine by inspection the desired value for $\theta_2^{2,1}$. Similarly, we proceed to node (1,3). Since the boundary conditions dictate the value of $\theta_1^{1,3}$ and $\theta_2^{1,3}$, we determine $Z_2^{1,3}$ simply by adding $Z_2^{2,3} + K_2^{2,3}$. At node (1,2), the value of $Z_1^{1,2}$ can be found by use of equation (8). The

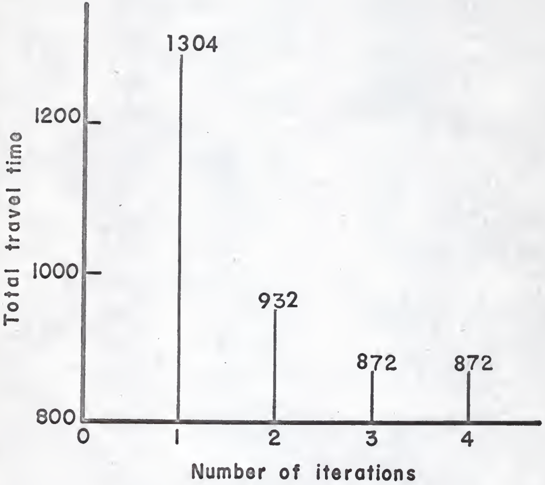


Fig. 4. Number of iterations vs. total travel time.

value of $Z_2^{1,2}$ can be determined by use of equation (9). If we add $Z_1^{1,2}+K_1^{1,2}$ and compare that value to $Z_2^{1,2}+K_2^{1,2}+K_R^{1,2}$ we determine by inspection the desired value for $\theta_1^{1,2}$. If we add $Z_2^{1,2}+K_2^{1,2}$ and compare that value to $Z_1^{1,2}+K_1^{1,2}+K_L^{1,2}$ we determine the desired value for $\theta_2^{1,2}$. Finally, we proceed to the final stage at node (1,1). The value of $Z_1^{1,1}$, $Z_2^{1,1}$ can be found by use of equations (8) and (9). If we add $Z_1^{1,1}+K_1^{1,1}$ and compare that value to $Z_2^{1,1}+K_2^{1,1}+K_R^{1,1}$ we determine by inspection the desired value for $\theta_1^{1,1}$. If we add $Z_2^{1,1}+K_2^{1,1}$ and compare that value to $Z_1^{1,1}+K_1^{1,1}+K_L^{1,1}$ we can determine the desired value of $\theta_2^{1,1}$.

Therefore, by this technique, we can determine the minimum time path by just one iteration. Now, let us apply this technique to the previous example.

We start at the destination node (3,3) and work backward from it, we find $Z_1^{3,2}=Z_2^{2,3}=0$ and $Z_1^{3,1}=20$. At node (2,2), $Z_1^{2,2}=Z_2^{2,3}+K_2^{2,3}+K_R^{2,3}=81$, and $Z_2^{2,2}=Z_1^{2,3}+K_1^{2,3}+K_L^{2,3}=23$. If we add $Z_1^{2,2}+K_1^{2,2}=101$ and compare that value to $Z_2^{2,2}+K_2^{2,2}+K_R^{2,2}=74$, we determine that $\theta_1^{2,2}=0$. If we add $Z_1^{2,2}+K_1^{2,2}+K_L^{2,2}=104$ and compare that value to $Z_2^{2,2}+K_2^{2,2}=73$, we determine the value of $\theta_2^{2,2}=1$. Likewise, proceeding to node (2,1), we find $Z_1^{2,1}=Z_2^{2,2}+K_2^{2,2}+K_R^{2,2}=74$, and $Z_2^{2,1}=Z_1^{3,1}+K_1^{3,1}+K_L^{3,1}=73$. If we add $Z_2^{2,1}+K_2^{2,1}+K_R^{2,1}=124$ and compare that value to $Z_1^{2,1}+K_1^{2,1}=94$, we see that $\theta_1^{2,1}=1$. If we add $Z_1^{2,1}+K_1^{2,1}+K_L^{2,1}=97$ and compare that value to $Z_2^{2,1}+K_2^{2,1}=123$, we find that $\theta_2^{2,1}=0$. Similarly, we proceed to node (1,3), we obtain $Z_2^{1,3}=Z_2^{2,3}+K_2^{2,3}=80$. The boundary conditions dictate the value of $\theta_2^{1,3}=1$. Then, we proceed to node (1,2), we find $Z_1^{1,2}=Z_2^{1,3}+K_2^{1,3}+K_R^{1,3}=131$, and $Z_2^{1,2}=Z_2^{2,2}$ $K_2^{2,2}=73$. If we add $Z_1^{1,2}+K_1^{1,2}=181$ and compare that value to $Z_2^{1,2}+$

$K_2^{1,2} + K_R^{1,2} = 124$, we determine the value of $\theta_1^{1,2}$ to be zero. If we add $Z_2^{1,2} + K_2^{1,2} = 123$ and compare that value to $Z_1^{1,2} + K_1^{1,2} + K_L^{1,2} = 184$, we find that $\theta_2^{1,2} = 1$. Finally, we proceed to node (1,1), we find $Z_1^{1,1} = Z_2^{1,2} + K_2^{1,2} + K_R^{1,2} = 124$ and $Z_2^{1,1} = Z_1^{2,1} + K_1^{2,1} + K_L^{2,1} = 97$. If we add $Z_1^{1,1} + K_1^{1,1} = 144$ and compare that value to $Z_2^{1,1} + K_2^{1,1} + K_R^{1,1} = 123$, we determine the value of $\theta_1^{1,1}$ to be zero. If we add $Z_1^{1,1} + K_1^{1,1} + K_L^{1,1} = 147$ and compare that value to $Z_2^{1,1} + K_2^{1,1} = 122$, we find that $\theta_2^{1,1} = 1$. Therefore, we determine the least-time path which is identical with the previous example.

This indicates that for a given set of constant link travel times in a network, we may build all the trees to seek a minimum time routing between the points of origin and the point of destination for each individual trip transfer.

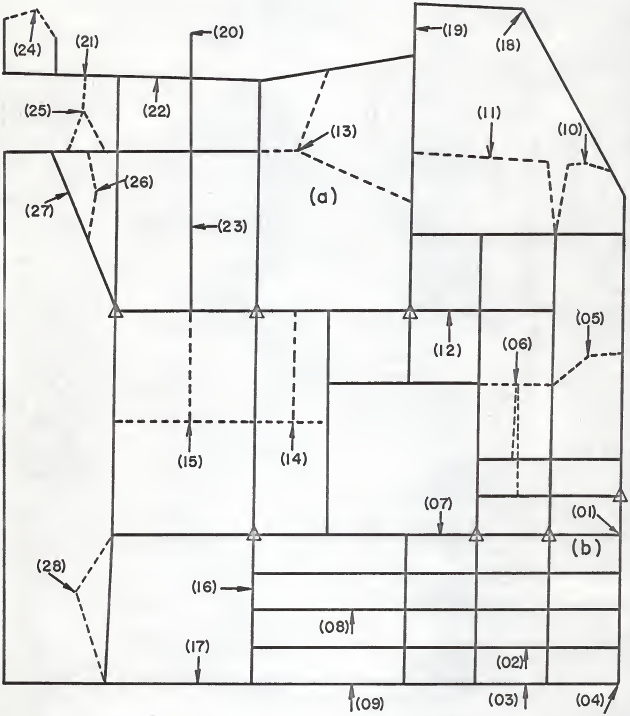
EXAMPLE

The use of the maximum principle as a tool for traffic assignment will be demonstrated by the following example. The existing street network of Manhattan, Kansas was chosen as the example (18).

The street system was simplified to that shown in Fig. 5. This simplification was made for the purpose of eliminating nodes that were not essential to a workable description of the network, so that calculations could be kept to a minimum.

The search procedure begins at a point which corresponds to the destination zone centroid for the copy in question. From this beginning point the procedure carries out a search of the network proceeding systematically outward from that zone to every other zone and identifies the route which requires the least travel time. This procedure insures that the shortest travel route in terms of the measure used -- in this case travel time -- is shown. Basically, the procedure is to search outward from a zone (as destination) identifying minimum path routes successively until the most distant zones have been reached. This is known as building a "minimum path tree". The descriptive term "tree" is used because, starting from one zone centroid (destination), there is one and only one path from that beginning point to each next zone centroid reached. As successively more distant zones are found, more branches are developed, so that the entire trace of minimum paths identified would seem like a tree with the trunk at the zone centroid (destination) and with ever increasing numbers of branches leading to the outermost zones.

After the minimum path from all origins to a destination (that is, a "copy") is completed, the trips from the origin zones to a destination zone can be loaded onto the network routes that have been identified. Each zone



△ Signal light
------- Artificial links
(12)→ Centroid of the 12th zone
(a) University
(b) Central Business District

Fig. 5. Simplified street network.

thus is taken in sequence (i.e., as a destination) and the process is repeated until all zones have been treated and all trips have been loaded onto the network.

To be able to insert a turn penalty for right and left turns, it is necessary to preassign the direction of movement on each link. Movements between links of the same direction involve no turning movements. In this example, the right turn penalty is assumed to be zero. For left turn penalties, those with signal light are assumed as 0.20 minutes, and those without signal light are assumed as 0.30 minutes. These turn penalties were inadvertently reversed but illustrate the procedure adequately.

The speed from zone centroids to the major street network is assumed as 12 MPH. The speeds for major streets are taken from the report "Manhattan Guide Plan, 1964-1985" (18).

The external interview stations and internal zones are shown in Figure 6. The centroid of each zone is determined by judging the present traffic conditions. The internal zone to internal zone nondirectional average daily vehicle trip data are shown in Appendix A in Table 1, and the external station to internal zone average daily vehicle trip data are shown in Appendix A in Table 2. The interchange of through traffic between interview stations is shown in Appendix A in Table 3. The average travel times are shown in Figure 7. The average travel time from the central business district (C.B.D.) in minutes is shown in Figure 8. The average travel time from the university in minutes is shown in Figure 9.

In order to demonstrate that the least-time paths are very sensitive to the location of the zone centroids, two sets of results were calculated. Case I was calculated using the best available estimate of the actual zone centroids.

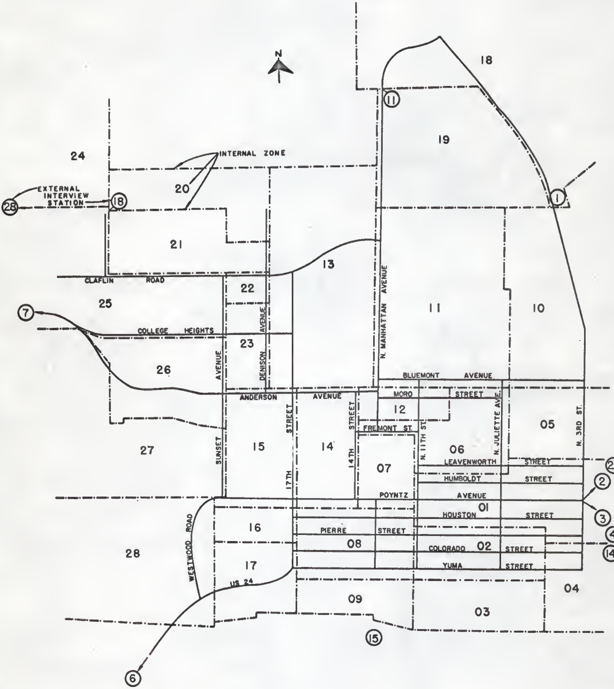


Fig. 6. The external interview stations and internal zones.

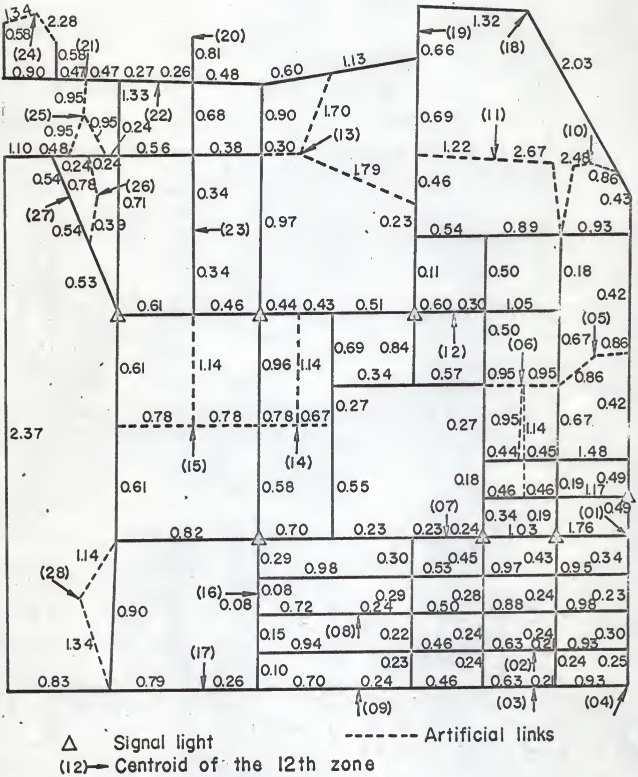


Fig. 7. Average travel times in minutes.

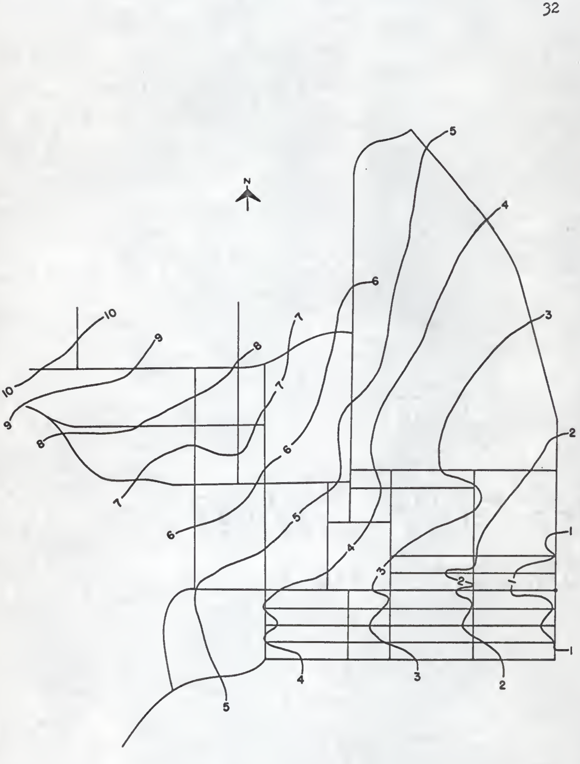


Fig. 8. The average travel time from business district in minutes.

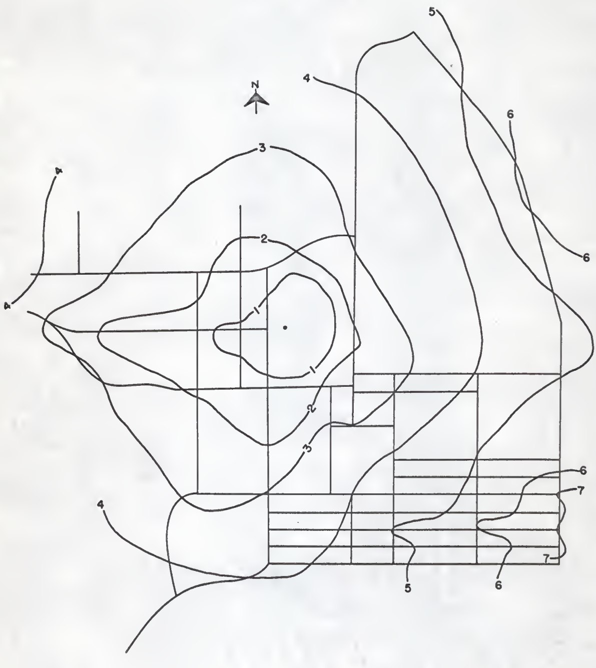


Fig. 9. The average travel time from university in minutes.

Case II was calculated after changing the average link travel times on two artificial links within zone 13. The travel time on the west artificial link in zone 13 was changed from .30 to .95 minutes, while the travel time on the east artificial link in zone 13 was changed from 1.79 to 1.14 minutes. This change has the effect of moving the zone centroid of zone 13 to the east of its actual location.

DISCUSSION OF EXAMPLE

The street system was simplified so that only the major streets were involved in the computation. The traffic volume of some zones, (zones 5, 6, 10, 11, 13, 14, 15, 25, 26, and 28), was not put into the network at a single point but rather was given a choice of several points of entry into the network in order to obtain a more reasonable distribution. This was made possible by adding artificial links, running from the zone centroids to the main network. These artificial links are shown as dashed lines in Figures 5, 7, and 13. In order to make a comparative analysis, the distribution of traffic as it would occur without capacity restraints was developed. For this type of distribution, the traffic would flow over the minimum path trees as shown in Figure 10.

Let us compare the 1963 average daily traffic volumes assigned to the major street network by the maximum principle method (Figure 10) with the traffic volume assigned in the Manhattan Guide Plan, 1964-1985 (Figure 11). It is not possible to check either method against actual traffic counts on the various streets as they are not available. It is possible to look at the two assignments and compare them in a rather general way. For the north-south direction, there are more cars for the set by the maximum principle method on Sunset, 14th, Manhattan, and 3rd streets but fewer cars on 17th, 11th and Juliette. For the east-west direction, there are more cars for the set by the maximum principle method on Claflin, Bluemont, Anderson, Fremont, part of Pierre and Yuma and fewer cars on all one way streets, Moro, Poyntz, Houston, and Colorado. In the Manhattan Guide Plan, 1964-1985 three screen lines were chosen as a means of comparing actual traffic counts with the volumes predicted in the above-mentioned report. The screen lines are shown in Figure 13 and the screen line checks are shown in Figure 14. From Figure 13

Fig. 10. Average daily traffic volumes assigned to the major street network by the maximum principle method (Case 1).

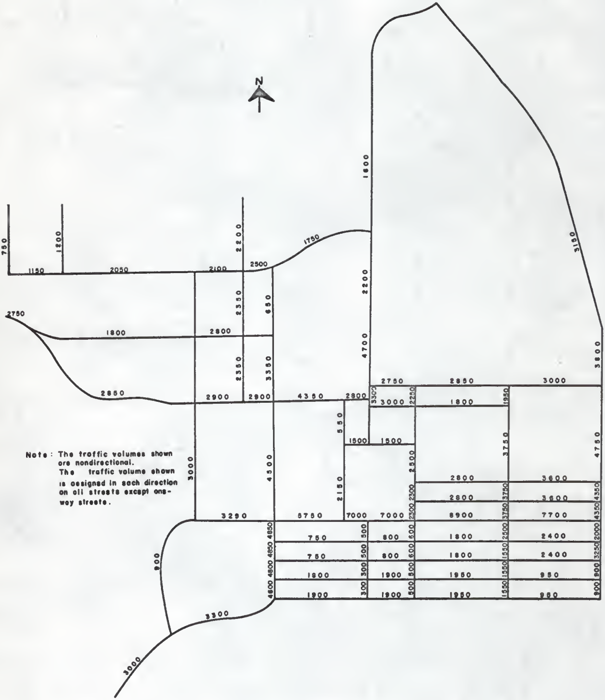


Fig. 11. Average daily traffic volumes assigned to the major street network as shown in the Manhattan Guide Plan (18).

it can be seen that the screen lines in some cases cross some of the artificial links which were added to the network in order to move vehicles from the zone centroids to the major street network. In those cases the number of vehicles on the artificial links which are crossed must be included in the total for the screen line involved.

The results across screen line No. 1 and screen line No. 2 by the maximum principle method are well checked by either the actual counts or those in the Wilson report. The principal reason there is so much difference in the count on screen line No. 3 is that there are approximately 4500 more cars on N. Third street (not included in screen No. 3) for the set by the maximum principle than for the Manhattan Guide Plan, 1964-1985. Screen line No. 3 was terminated at Juliette in order to eliminate the circulating traffic in the central business district.

As can be seen from Figures 10 and 11, there are considerably fewer cars on Poyntz and on the one-way system in the C.B.D. for the set by the maximum principle method. The primary reason for the very significant difference in volume on Poyntz is believed to be in the fact that, for the set calculated by the maximum principle method, the zone centroid for zone 1 was taken as the intersection of Third and Poyntz. This choice enables cars to come to the C.B.D. centroid without ever being counted on Poyntz. That is, they can come from either North or South Third Streets and never appear in the volume shown on Poyntz. This choice of zone centroid also enables the vehicles from external stations 1, 2, 3, 4, 14, and 22 (approx. 6,000) to reach the centroid of zone 1 without being counted on Poyntz.

One of the very significant problems involved in traffic assignment is the selection of the correct location for the zone centroids. It is not difficult to see that the high volues obtained, by the maximum principle method, on Third Street as well as Yuma and Pierre would be decreased materially by moving the centroid of zone 1 westward along Poyntz. This movement would add travel time, for cars going to the C.B.D., to most of the paths which include Third Street, Yuma, Pierre or Bluemont and would decrease the travel time for cars coming to C.B.D. on Poyntz. West of the downtown area, for instance where screen line No. 1 cuts Poyntz, the volume on Poyntz is only 5603 as compared to 11,500 in the Manhattan Guide Plan, however, screen line No. 1 checks rather well indicating that this volume difference is made up on other links in the system (primarily Yuma and Pierre Streets).

The minimum-time-paths are quite sensitive to any changes in the location of the zone centroids. This fact is demonstrated by the traffic volumes shown in Figure 12. Only those volumes which were different from the volumes shown in Figure 10 are shown in Figure 12. The times on two of the artificial links in zone 13 were changed to obtain this distribution. The time on the west link was changed from .30 minutes to .95 minutes, while the time on the east link was changed from 1.79 minutes to 1.14 minutes. These changes had the effect of moving the zone centroid eastward within the zone. It can be seen from Figure 12 that some of the predicted volumes are different by as much as 100%. This gives some indication of the magnitude of the changes that could be anticipated if the centroid of zone 1 were moved westward along Poyntz.

The maximum principle method outlined in this thesis should obtain exactly the same answers as linear programming or minimum-path tree building assuming that all three start with the same points of reference (i.e. link

Fig. 12. Average daily traffic volumes assigned to the major street network by the maximum principle method (Case II).

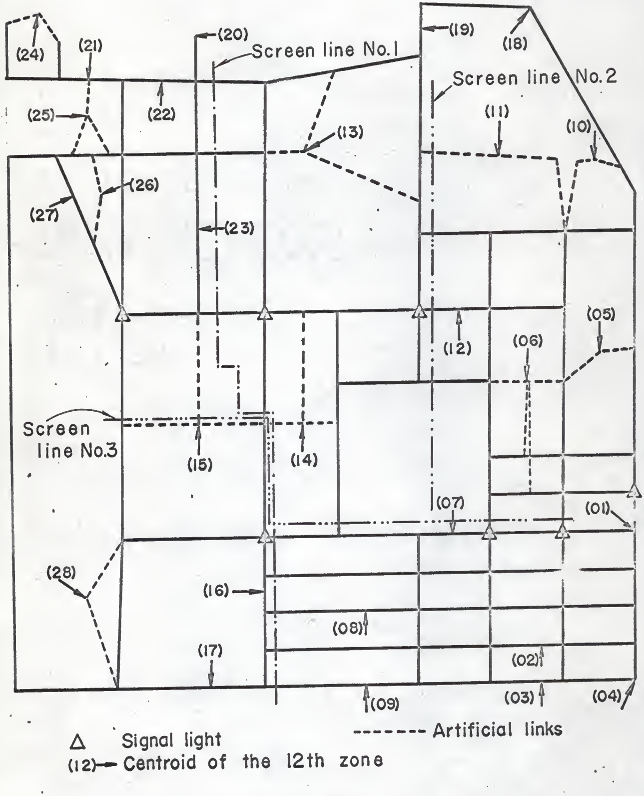


Fig. 13. Screen lines.

Notes:

(A) - 1963 actual counts.

(B) - 1963 assignment in the Wilson report.

(C) - 1963 assignment by the maximum principle method (Case I).

(D) - 1963 assignment by the maximum principle method (Case II).

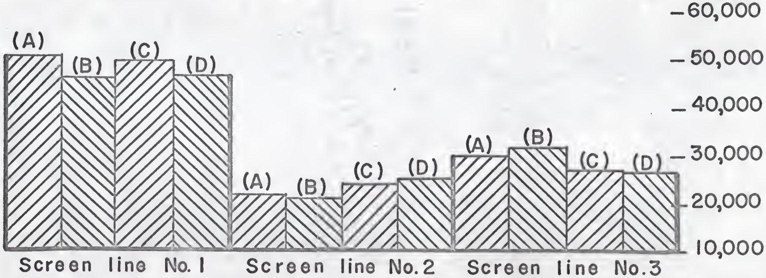


Fig. 14. Screen line checks.

travel times, zone centroid locations, etc.). Since the above is true, the only possible explanation of the differences in predicted volumes, is that the two assignments were not carried out under identical sets of assumptions concerning the above parameters.

CONCLUSIONS

The results of this study have indicated that the discrete maximum principle technique is a useful tool in transportation systems analysis. However, there is no particular advantage for the maximum principle in dealing with linear time function in traffic assignment. On the other hand, when dealing with nonlinear time functions, it shows great promise and further research is indicated.

There is no reason to believe that the assignment by the maximum principle method is any more accurate than the assignment in the Manhattan Guide Plan, 1964-1985. The differences in them must be attributed to using different basic assumptions as to the location of the zone centroids, link travel times, turn penalties, etc.

It was demonstrated that a small change in the location of the zone centroids, or in the average link travel time assigned to an individual link, may change rather drastically the route for the least-time paths in traffic assignment by the maximum principle method when linear link travel time - link traffic volume relationships are used. This would also be true if linear programming or minimum time path tree building were used. This is one of the principle disadvantages of dealing with linear time functions. Moreover, there is no convenient way to predict traffic congestion when linear time functions are used. Therefore it appears that the use of non-linear link travel times, made possible with the development of the maximum principle technique, is a necessity if adequate methods of traffic assignment are to be developed.

RECOMMENDATIONS FOR FURTHER RESEARCH

There is no particular time advantage in traffic assignment by the maximum principle method, for linear time functions, since in that case the method of solution is to build all minimum time trees. Also, there is no convenient way to predict traffic congestion with linear time functions. This is a serious drawback to any method which uses linear time functions. However, it is possible to introduce non-linear time functions with the maximum principle techniques thereby making it possible to introduce traffic delay due to congestion and the resulting selection of alternate, less congested, routes thus more realistically reflecting traffic movement. Although it was demonstrated as being feasible to use non-linear time functions during the course of this study, time did not permit the complete development of the logic to be used. It is important to the efficient application of the maximum principle for transportation planning use that further research be undertaken with the objective being to develop the procedures to be used with non-linear link travel time - traffic volume relationships in assigning trips to an arterial street network.

Another area which needs further research is that of locating the zone centroids. As was demonstrated in this thesis, a relatively small change in the location of a centroid can cause a considerable change in the traffic assignment. Because this is true it is important that better methods of determining the locations of the zone centroids be developed. This is especially true for the zones which have comparatively high volumes. This normally would include any zones in the C.B.D. and also other zones which contain high volume traffic generators.

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APPENDIX A

TABLE 1. INTERNAL ZONE TO AND FROM INTERNAL ZONE AVERAGE DAILY VEHICLE TRIP DATA

ZONE	1	2	3	4	5	6	7
1	914	638	372	250	614	1,337	86
2		67	107	93	103	324	42
3			43	74	96	222	26
4				41	102	184	15
5					109	394	37
6						402	106
7							2
8							
9							
10							
11							
12							
13							
14							
15							
16							
17							
18							
19							
20							
21							
22							
23							
24							
25							
26							
27							
28							

ZONE	8	9	10	11	12	13	14
1	553	143	511	1,152	398	2,147	517
2	154	41	114	328	189	966	171
3	102	28	85	225	120	539	115
4	85	23	67	186	86	232	91
5	136	33	170	393	218	896	199
6	330	69	335	772	494	2,033	401
7	46	14	27	55	29	97	42
8	99	59	121	298	201	1,084	216
9		6	31	78	55	294	57
10			76	368	180	1,041	146
11				461	463	2,226	362
12					91	505	234
13						1,526	1,042
14							140
15							
16							
17							
18							
19							
20							
21							
22							
23							
24							
25							
26							
27							
28							

Table 1 (cont.)

ZONE	15	16	17	18	19	20	21
1	616	49	89	676	6	362	238
2	270	9	27	65	3	88	56
3	162	8	21	43	3	55	39
4	154	6	21	33	3	50	36
5	249	10	25	132	3	116	73
6	506	14	44	184	3	212	133
7	53	5	9	39	3	32	22
8	301	13	39	80	3	111	77
9	78	4	10	19	3	28	19
10	202	9	22	133	3	89	69
11	510	16	58	240	4	271	197
12	323	17	38	187	3	180	123
13	957	132	175	1,719	13	1,339	879
14	485	14	43	110	3	182	130
15	348	19	61	131	3	286	204
16		0	3	8	3	8	5
17			4	11	3	22	14
18				124	4	72	57
19						3	3
20						101	143
21							50
22							
23							
24							
25							
26							
27							
28							

Table 1 (cont.)

ZONE	22	23	24	25	26	27	28	TOTAL
1	166	118	247	374	612	229	145	13,559
2	40	30	63	131	228	67	33	4,447
3	28	18	41	84	146	45	23	2,870
4	19	16	34	60	106	44	20	2,131
5	50	34	75	142	223	72	34	4,738
6	93	62	148	305	473	134	77	9,791
7	15	11	19	32	45	24	14	947
8	51	37	83	163	260	94	51	4,847
9	14	10	21	43	68	20	12	1,280
10	37	31	80	134	189	61	30	4,361
11	108	90	232	356	579	184	90	10,302
12	78	65	112	167	278	126	57	5,017
13	525	382	747	1,068	1,133	541	359	24,597
14	73	60	130	231	367	135	59	5,755
15	116	96	210	357	601	257	108	7,663
16	5	4	7	16	28	7	4	423
17	11	9	17	32	58	20	15	901
18	46	27	72	169	199	32	21	4,633
19	3	3	3	3	3	3	3	96
20	66	50	125	253	316	81	38	4,679
21	55	35	94	202	215	63	30	3,261
22	15	24	50	96	137	38	17	1,976
23		8	41	86	128	29	15	1,519
24			85	248	239	91	31	3,345
25				234	487	169	64	5,706
26					372	253	100	7,843
27						49	32	2,900
28							10	1,492
						TOTAL TRIP ENDS		141,079

TABLE 2. EXTERNAL STATION TO INTERNAL ZONE AVERAGE DAILY VEHICLE TRIP DATA

zone station	1	2	3	4	5	6	7
1	2,617	113	39	195	518	103	22
2	1,590	91	36	139	146	70	23
3	702	42	22	82	120	27	3
4	803	39	21	78	68	58	6
6	1,366	174	91	185	229	303	20
7	502	26	14	42	27	34	15
11	124	6	9	5	12	54	5
14	221	14	5	38	19	16	5
15	223	43	19	19	28	8	2
18	538	57	26	33	43	68	3
22	63	5	0	2	47	8	2
28	165	15	4	7	8	15	0
TOTAL	8,914	625	286	825	1,265	764	106

Table 2 (cont.)

zone station	8	9	10	11	12	13	14
1	123	21	399	225	106	215	115
2	228	30	75	75	48	183	82
3	54	16	59	40	32	123	19
4	56	9	79	59	31	189	71
6	206	52	242	292	108	390	209
7	34	12	42	50	61	275	59
11	53	3	17	117	151	1,037	113
14	36	5	15	18	11	63	5
15	35	31	36	19	14	33	15
18	85	14	28	57	106	444	69
22	5	3	52	35	31	123	11
28	33	3	2	30	38	210	17
TOTAL	948	199	1,046	1,017	737	3,285	785

Table 2 (cont.)

zone station	15	16	17	20	21	22	23
1	187	20	23	23	31	14	33
2	95	13	32	34	51	19	46
3	73	0	17	2	5	11	11
4	86	8	14	21	29	14	26
6	188	35	64	39	94	29	67
7	64	0	5	5	27	23	12
11	141	10	10	49	61	57	66
14	28	2	8	5	8	2	14
15	43	2	19	0	9	2	2
18	156	6	13	71	105	50	28
22	5	0	3	10	5	4	8
28	50	3	5	15	68	3	20
TOTAL	1,116	99	213	274	493	228	333

Table 2 (cont.)

zone station	25	26	27	28
1	25	18	29	98
2	40	14	58	55
3	8	5	13	31
4	32	14	24	24
6	83	32	41	94
7	151	5	63	41
11	46	26	40	38
14	11	4	2	3
15	7	4	2	15
18	77	26	78	15
22	6	5	3	4
28	279	23	30	13
TOTAL	765	176	383	431

From Kansas State Highway Commission 1962 Survey Data.

TABLE 3 SUMMARY OF THROUGH TRAFFIC

INTERVIEW STATIONS*

(1)	212.14	79.20	191.99	318.06	25.80		39.74	24.52	22.95	175.24	6.33	1,095.97
	(2)	67.40	176.89	425.27	178.95	2.37	51.24	18.45	15.98		8.14	1,156.83
		(3)	63.26	61.76	8.20	3.34	23.82	7.69	11.56	2.28	8.00	336.51
			(4)	96.00	58.06	1.18	23.31	5.56	21.62	2.28	12.52	652.67
				(6)		71.24	5.47	24.97		1.52		1,004.29
					(7)	10.55	7.70	2.61	31.03	11.52	62.26	396.68
						(11)		1.74	3.84		2.61	96.51
							(14)	8.34	3.45		3.22	166.29
								(15)	0.89		1.74	96.51
									(18)	3.80		114.76
										(22)		196.64
											(28)	104.82
										Total Termini		5,418.48

* These trips have both an entering and departing station

From Kansas State Highway Commission 1962 Survey.

Table 3 (cont.)

INTERVIEW STATIONS**	(7)	(18)	(28)	
(1)	1.36			1.36
(2)	161.44			161.44
(3)	8.44			8.44
(4)	44.32			44.32
(6)	73.64	44.80	78.54	196.98
(7)				290.56
(11)				
(14)				1.36
(15)				
(18)				44.80
(22)				
(28)				78.54
Total Termini				827.80

** Using the US 24 By-pass. These trips have both an entering and departing station.

From Kansas State Highway Commission 1962 Survey.

LIST OF SYMBOLS

DISCRETE MAXIMUM PRINCIPLE

c	constant in objective function
H	the Hamiltonian
n	the n-th stage
N	the N-th stage or the total number of stages
s	total number of state variables in each stage
t	total number of decision variables in each stage
T	transformation operator
X	state variable
Z	an s-dimensional covariant vector
θ	decision variable

GENERAL FORMULATION OF THE PROBLEM

$X_j^{n,m}$	state variable for flow at node (n,m)
$\theta_j^{n,m}$	decision variable at node (n,m)
$t_j^{n,m}$	basic travel time per vehicle via the node (n,m) on link j
$K_j^{n,m}$	travel time coefficient at node (n,m) where $j = 1$, for horizontal link $j = 2$, for vertical link
$H^{n,m}$	Hamiltonian function at node (n,m)
$V^{n,m}$	total desire (vehicles) via node (n,m)
$K_L^{n,m}$	travel time coefficient of left turn penalties at node (n,m)

$K_R^{n,m}$ travel time coefficient of right turn penalties at node (n,m)

$X_3^{n,m}$ total travel time for horizontal links

$X_4^{n,m}$ total travel time for vertical links

Z covariant vector

TRAFFIC ASSIGNMENT
BY THE DISCRETE MAXIMUM PRINCIPLE

by

TSUNG CHANG YANG

Diploma, Provincial Taipei Institute of Technology, 1957

AN ABSTRACT OF A THESIS

submitted in partial fulfillment of the

requirements for the degree

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This work consists of four parts. In the first part, the context and scope of traffic assignment by different methods are outlined. The literature review has been presented from 1952 up to 1964. A brief historical development of traffic assignment is sketched. The second part is concerned with the discrete maximum principle. A brief description of the application of the principle has been introduced.

Part three deals with the general formulation of the problem. That is, how a traffic network problem solved by the discrete maximum principle is formulated. The maximum principle is used to assign traffic movements to an arterial network on the basis of minimum total travel time, assuming link travel time per vehicle to be independent of volume.

The recurrence relation of the state variables is applicable only when the minima of the Hamiltonian functions occur at stationary points and when the optimal decisions so obtained lie within the constraints. For this problem, the minima do not occur at the stationary points since the Hamiltonian function is linear in the decision variables. For such a case, the optimal decisions are either the lower bounds or the upper bounds of the decision variables. Therefore, the minima seeking method instead of the recurrence relation should be used.

The procedures of the minima seeking method used were described in this work. A short cut has been found for determining the least-time path. The short cut indicates that, for a given set of constant times in a network, we may build all the trees to seek a minimum time routing between the points of origin and the point of destination for each individual trip transfer.

In the final part, the existing street network of Manhattan, Kansas was chosen as an example to demonstrate the use of the maximum principle as a tool for traffic assignment. A turn penalty for right and left turns was inserted into the problem. Also, in order to show the sensitivity of average travel times in affecting the least-time paths chosen, two solutions of the problem have been calculated. The problem was successfully attacked by the method developed in this work.