METRIZATION OF UNIFORM SPACES

by

SAMUEL LELAND LESSEIG

B. S., Fort Hays Kansas State College, 1961

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

Approved by:

Neal E. Foland

Major Professor

TABLE OF CONTENTS

INTRODUCTION

Fréchet first considered abstract spaces in his thesis in 1906 [4]. Much of the developement of the concept of a topological space may be found in Hausdorff's Grundzüge der Mengenlehre [6] and in the early volumes of Fundamenta Mathematica. From this research two fundamental concepts have developed: that of a toplogical space and that of a uniform space. André Weil was the first to formalize the notion of a uniform space in a paper in 1937 [13]. The concept of a uniform space developed from the study of topological groups and has presented many new ideas to the study of metric spaces. One of these is the concept that the finest uniformity for a metrizable space is the one induced by a metric [12] . J. R. Isbell has done work using uniform spaces in the study of the algebra of functions [7] .

This report is devoted to the examination of several properties of uniform spaces and the relationship between uniformities and pseudo-metrics.

RELATIONS

In the study of uniform spaces the subsets of a cartesian product $X \times X$ of a set X with itself are considered. These subsets are relations or ordered pairs. If U is a relation, then U^{-1} is the set of all (x,y) such that (y,x) is in U. The process of taking inverses is involutory. If $U = U^{-1}$, then U is said to be symmetric. If U and V are relations, then $U \circ V$ is defined as the set of all pairs (x,z) such that for some y it is true that (y,z) is in U and (x,y) is in V. This operation between relations is called composition. It is easily shown that composition is associative. The inverse of a composition is the composition of the inverses in the reverse order. The identity relation is defined to be the set of all (x,x) such that x is in X. This identity relation will be referred to as the diagonal and denoted by $\triangle$. For each subset A of X, the set $U[A]$ is defined to be the set of all y such that (x,y) is in U for some x in A. If x is a point of X, then $U[x]$ is defined to be $U[\{x\}]$. It can be shown that for each U and V and each A in X it is true that $U \circ V[A]$ is equal to $U[V[A]]$.

To complete the list of needed properties of relations, an important lemma is included.

<u>LEMMA I</u>: If V is symmetric, then $V \circ U \circ V = \bigcup \{V[x] \times V[y] : (x,y)$ is in $U\}$.

Proof:

By definition $V \circ U \circ V$ is the set of all pairs (u,v) such that for some x and y in X, (u,x) is in V, (x,y) is in U and (y,v) is in V. Since V is symmetric, this is the set of all (u,v) such that u is in $V[x]$ and v is in $V[y]$ for some (x,y) in U. However, u is in $V[x]$ and v is in $V[y]$ if and only if (u,v) is in $V[x] \times V[y]$. Thus $V \circ U \circ V = \{(u,v) : (u,v)$ is in

$V[x] \times V[y]$ for some (x,y) in $U\}$. Hence $V \circ U \circ V$ is equal to $\bigcup \{V[x] \times V[y] : (x,y)$ is in $U\}$.

DEFINITION OF UNIFORM SPACE

A uniformity for a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ such that:

(a) each member of $\mathcal{U}$ contains the diagonal Δ ;

(b) if U is in $\mathcal{U}$, then U^{-1} is in $\mathcal{U}$;

(c) if U is in $\mathcal{U}$, then $V \circ V \subset U$ for some V in $\mathcal{U}$;

(d) if U and V are members of $\mathcal{U}$, then $U \cap V$ is in $\mathcal{U}$;

(e) if U is in $\mathcal{U}$ and $U \subset V \subset X \times X$, then V is in $\mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a uniform space.

For a given space X there exists many different uniformities, ranging from the set of all subsets of $X \times X$ which contain Δ to the set whose only member is $X \times X$. It follows from condition (a) that each member of $\mathcal{U}$ is a neighborhood of Δ, but the converse is not true. If X is the set of real numbers with the usual uniformity for X of the family of all subsets of $X \times X$ such that $\{(x,y) : |x-y| < r\}$ is contained in U for some positive number r, then consider the $\{(x,y) : |x-y| < 1/(1 + |y|)\}$. This set is a neighborhood of the diagonal but is not a member of $\mathcal{U}$ since it fails to satisfy condition (c).

DEFINITION OF BASE AND SUBBASE

A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ if and only if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. From this definition if is evident that $\mathcal{B}$ determines $\mathcal{U}$ entirely, for each member of $\mathcal{B}$ is in $\mathcal{U}$ and a subset U of $X \times X$ belongs to $\mathcal{U}$ if and only if U contains a member of $\mathcal{B}$. A subfamily $\mathcal{S}$ is a subbase for $\mathcal{U}$ if and only if the set of all finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{U}$.

To characterize a base for a uniformity the following theorem is given.

<u>THEOREM I</u>: A family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if:

(a) each member of $\mathcal{B}$ contains the diagonal Δ;

(b) if U is in $\mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$;

(c) if U is in $\mathcal{B}$, then $V \circ V \subset U$ for some V in $\mathcal{B}$;

(d) the intersection of two members of $\mathcal{B}$ contains a member of $\mathcal{B}$.

Proof:

It is obvious that given a family $\mathcal{B}$ of subsets of $X \times X$ satisfying these four conditions, then $\mathcal{B}$ forms a base for some uniformity $\mathcal{U}$. Therefore it suffices to show that if $\mathcal{B}$ is a base for some uniformity $\mathcal{U}$, then the four conditions are satisfied.

Let $\mathcal{B}$ be a base for some uniformity $\mathcal{U}$, then by definition each member of $\mathcal{U}$ contains a member of $\mathcal{B}$ and each member of $\mathcal{B}$ is in $\mathcal{U}$. Since each member of $\mathcal{U}$ contains the diagonal, it is evident that each member of $\mathcal{B}$ contains the diagonal. If U is in $\mathcal{B}$, then U is in $\mathcal{U}$. Since $\mathcal{U}$ is a uniformity, U^{-1} is in $\mathcal{U}$ and hence U^{-1} contains a member of $\mathcal{B}$. If U is in $\mathcal{B}$, then U is in $\mathcal{U}$ and there exists a V in $\mathcal{U}$ such that $V \circ V \subset U$.

However, V in $\mathcal{U}$ implies there exists a V_1 in $\mathcal{B}$ such that V_1 is contained in V. Then $V_1 \circ V_1 \subset V \circ V$, which implies that $V_1 \circ V_1$ is contained in U. For the last condition, let B_1 and B_2 be members of $\mathcal{B}$. Then B_1 and B_2 are in $\mathcal{U}$ and hence $B_1 \cap B_2$ is in $\mathcal{U}$, and thus $B_1 \cap B_2$ contains a member of $\mathcal{B}$.

The property of being a subbase is not as easy to characterize as the property of being a base. However, the following theorem will be needed in the study of uniform spaces.

THEOREM II: A family $\mathcal{S}$ of subsets of $X \times X$ is a subbase for some uniformity of X if:

(a) each member of $\mathcal{S}$ contains the diagonal Δ;

(b) for each U in $\mathcal{S}$ the set U^{-1} contains a member of $\mathcal{S}$;

(c) for each U in $\mathcal{S}$ there is a V in $\mathcal{S}$ such that $V \circ V \subset U$. In particular, the union of any collection of uniformities for X is a subbase for a uniformity for X.

Proof:

It suffices to show that the family $\mathcal{B}$ of finite intersections of members of $\mathcal{S}$ satisfy the conditions of theorem I.

If $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_m$ are subsets of $X \times X$ such that U_i and V_i are in $\mathcal{S}$ and if $U = \bigcap_{i=1}^{n} U_i$ and $V = \bigcap_{i=1}^{m} V_i$, then it is obvious that the first condition is satisfied. By observing that $V \subset U^{-1}$ (or $V \circ V \subset U$) whenever $V_i \subset U_i^{-1}$ (or $V_i \circ V_i \subset U_i$) for each i, then the remainder of the conditions in Theorem I follow immediately.

DEFINITION OF UNIFORM TOPOLOGY

If $(X, \mathcal{U})$ is a uniform space, then the topology $\mathcal{J}$ of the uniformity $\mathcal{U}$ or the uniform topology, is the family of all subsets T of X such that for each x in T there is a U in $\mathcal{U}$ such that $U[x] \subset T$. To verify that $\mathcal{J}$ is a topology it suffices to show that $\mathcal{J}$ is closed under an arbitrary union of its members and the intersection of any two members.

Let T_α be any arbitrary set of elements of $\mathcal{J}$ and let $T = \bigcup_\alpha T_\alpha$. Let x be an element of T; then x is in some T_α. Hence there exists a U in $\mathcal{U}$ such that $U[x] \subset T_\alpha$ and therefore $U[x]$ is in T and T is in $\mathcal{J}$.

Let T_1 and T_2 be elements of $\mathcal{J}$ and let x be an element of $T = T_1 \cap T_2$. Since x is in T, then x is in T_1 and x is in T_2. However x in T_1 implies that there exists a U in $\mathcal{U}$ such that $U[x] \subset T_1$ and since x is in T_2 there exists a V in $\mathcal{U}$ such that $V[x] \subset T_2$. It follows that $(U \cap V)[x]$ is contained in $T_1 \cap T_2$; consequently, T is an element in $\mathcal{J}$ and $\mathcal{J}$ is a topology.

The relation between a uniformity and the uniform topology will now be examined.

<u>THEOREM III</u>: The interior of a subset A of X relative to the uniform topology is the set of all points x such that $U[x] \subset A$ for some U in $\mathcal{U}$.

Proof:

Let $B = \{x : U[x] \subset A \text{ for some } U \text{ in } \mathcal{U}\}$. If x is in B, then there exists a U in $\mathcal{U}$ such that $U[x] \subset A$. Since U is in $\mathcal{U}$ there exists a V in $\mathcal{U}$ such that $V \circ V \subset U$. Thus if y is in $V[x]$, then $V[y] \subset V \circ V[x] \subset U[x] \subset A$ and therefore y is in B. Hence $V[x] \subset B$ and B is open. By definition B must contain every open subset of A and therefore B is the interior of A.

<u>COROLLARY</u>: $U[x]$ is a neighborhood of x for each U in the uniformity $\mathcal{U}$.

Proof:

The proof follows directly from the results of the previous theorem.

For a topological space $(X, \mathcal{J})$, a collection of open sets $\mathcal{B}$ is called a neighborhood system of x if x in A, A a member of $\mathcal{J}$, implies that there exists a U in $\mathcal{B}$ such that x is in $U \subset A$. Therefore, with the preceding corollary, the following theorem shows the correspondence between a base for a uniformity and one for a neighborhood system.

<u>THEOREM IV</u>: If $\mathcal{B}$ is a base (or subbase) for the uniformity $\mathcal{U}$, then for each x the family of sets $U[x]$ for U in $\mathcal{B}$ is a base (subbase respectively) for the neighborhood system of x.

Proof:

It was stated in the previous corollary that $U[x]$ was a neighborhood of x for each U in $\mathcal{U}$. Therefore if $\mathcal{B}$ is a base (or subbase) of $\mathcal{U}$, then $U[x]$ is a neighborhood of x for all U in $\mathcal{B}$. Hence the family of sets $U[x]$ for each x and for U in $\mathcal{B}$ is a base (or subbase) for the neighborhood system of x.

The following theorem is used later to assure the existence of a symmetric base for $\mathcal{U}$.

<u>THEOREM V</u>: If U is a member of the uniformity $\mathcal{U}$, then the interior of U is also a member; consequently the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

Proof:

From the results of Theorem III it is evident that the interior of a subset M of $X \times X$ is the set of all (x,y) such that for some U and some V in $\mathcal{U}$, $U[x] \times V[y] \subset M$. Since $\mathcal{U}$ is a uniformity, $(U \cap V)$ is in $\mathcal{U}$ and hence the interior of M is $\{(x,y) : V[x] \times V[y] \subset M$ for some V in $\mathcal{U}\}$. If U is in $\mathcal{U}$, then the symmetric member $V = (U \cap U^{-1})$ of $\mathcal{U}$ is such that $V \circ V \circ V \subset U$ and, according to lemma I, $V \circ V \circ V = \bigcup \{V[x] \times V[y] : (x,y)$ is in $V\}$. Hence every point of V is an interior point of U and, since the interior of U contains V, the interior of U is a member of $\mathcal{U}$.

As a result of this theorem, many of the proofs for the remaining theorems will use the fact that there exists a symmetric base for each uniformity $\mathcal{U}$.

The concept of closure is considered next since this property provides a method for comparing topological spaces.

A point x is a point of accumulation of a set $A \subset X$ if and only if $U[x]$ intersects $A - \{x\}$ for each U in $\mathcal{U}$. Thus x is in the closure of a set $A \subset X$ if and only if $U[x]$ intersects A for each U in $\mathcal{U}$.

<u>THEOREM VI</u>: The closure, relative to uniform topology, of a subset A of X is $\bigcap \{U[A] : U$ in $\mathcal{U}\}$. The closure of a subset M of $X \times X$ is $\bigcap \{U \circ M \circ U : U$ in $\mathcal{U}\}$.

Proof:

Let x be an element of $\bar{A}$ and U a member of $\mathcal{U}$. This implies that $U[x] \cap A$ is non-null. If y is in $U[x] \cap A$, then (x,y) is in U. $U[A] = \{x : (x,y)$ is in U for some y in $A\}$ and since (x,y) is in U and y is in A, x is an element of $U[A]$. Thus x in $\bar{A}$ implies that x is in $U[A]$. If x is in $\bigcap U[A]$, it is evident that x is in the closure of A.

For the proof of the second part of the theorem, if U is a symmetric member in $\mathcal{U}$, then $U[x] \times U[y]$ intersects a subset M of $X \times X$ if and only if (x,y) is in $U[u] \times U[v]$ for some (u,v) in M. Thus $U[x] \times U[y]$ intersects M if and only if (x,y) is in $\bigcup \{U[u] \times U[y] : (u,v)$ is in $M\}$.

By lemma I, this is the set $U \circ M \circ U$ and the following result is obtained; that (x,y) is in M if and only if (x,y) is in $\bigcap \{ U \circ M \circ U \quad : \quad U \text{ is in } \mathcal{U} \}$. Therefore the closure of M is $\bigcap \{ U \circ M \circ U \quad : \quad U \text{ is in } \mathcal{U} \}$.

Now the following theorem which shows the existence of another base for $\mathcal{U}$ can be proved.

<u>THEOREM VII</u>: The family of closed symmetric members of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$.

Proof:

If U and V are members of $\mathcal{U}$ such that $V \circ V \circ V \subset U$, then by preceding theorem $V \circ V \circ V$ contains the closure of V. Thus U contains a closed member W of $\mathcal{U}$ and $W \cap W^{-1}$ is a closed symmetric member of $\mathcal{U}$ contained in U. Hence the set of all closed symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

From the previous two theorems it is evident that a uniform space is always a regular, for each neighborhood of a point x contains a neighborhood $V[x]$ such that V is a closed member of $\mathcal{U}$, and therefore $V[x]$ is closed. Hence a space with a uniform topology is a Hausdorff space if and only if each set consisting of a single point is closed. However, the closure of the set $\{x\}$ is $\bigcap \{ U[x] \quad : \quad U \text{ is in } \mathcal{U} \}$, and thus a space is Hausdorff if and only if $\bigcap \{ U \quad : \quad U \text{ in } \mathcal{U} \}$ is the diagonal Δ. In this case the space is said to be Hausdorff or separated.

UNIFORM CONTINUITY

If f is a function on a uniform space $(X, \mathcal{U})$ with values in a uniform space $(Y, \mathcal{V})$, then f is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ if and only if for each V in $\mathcal{V}$ the set $\{ (x,y) \quad : \quad (f(x), f(y)) \text{ is in } V \}$ is a member of $\mathcal{U}$. This definition can be stated in several ways. For each function on X to Y, let f_2 denote the induced function on $X \times X$ to $Y \times Y$ which is defined by $f_2(x,y) = (f(x), f(y))$.

Then f is uniformly continuous if and only if for each V in $\mathcal{V}$ there is a U in $\mathcal{U}$ such that $f_2\left[U\right] \subset V$. Another way of stating this definition is that if X = Y is the set of real numbers and $\mathcal{U} = \mathcal{V}$ is the usual uniformity of X, then f is uniformly continuous if and only if for each positive number r there is a positive number s such that $\left|f(x) - f(y)\right|$ is less than r whenever $\left|x - y\right|$ is less than s. The analogy between this definition and the usual one in analysis is quite apparent.

If f is a function on X to Y and g is a function on Y, then $(g\, f)_2 = g_2 \circ f_2$ where the subscript 2 denotes the induced function. From this it can be seen that the composition of two uniformly continuous functions is a uniformly continuous function. If f is a one-to-one map of X onto Y and both f and f^{-1} are uniformly continuous, then f is called a <u>uniform isomorphism</u>, and the spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are said to be uniformly equivalent. The composition of two uniform isomorphisms, the inverse of a uniform isomorphism, and the identity map of a space onto itself are all uniform isomorphisms, and therefore the set of all uniform spaces is divided into equivalence classes composed of uniformly equivalent spaces. A property which is possessed by every uniform space in a given equivalence class is called a <u>uniform invariant</u>. This definition is entirely analogous to the definition of a topological invariant.

The following theorem states the relationship between uniform isomorphisms and homeomorphisms.

<u>THEOREM VIII</u>: Each uniformly continuous function is continuous relative to the uniform topology, and hence each uniform isomorphism is a homeomorphism.

Proof:

Let f be a uniformly continuous function on $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ and let U be a neighborhood of f (x). Then there is a V in $\mathcal{V}$ such that $V[f(x)] \subset U$, and $f^{-1}[V[f(x)]] = \{y: f(y) \text{ is in } V[f(x)]\} = \{y : (f(x), f(y)) \text{ is in } V[f(x)]\}$. Hence the set $f^{-1}[V[f(x)]]$ is equal to $f_2^{-1}[V](x)$ and is a neighborhood of x. Therefore continuity is established.

The final statement in the theorem follows directly from the definition of a homeomorphism.

If f is a function on a set X to a uniform space $(Y, \mathcal{V})$, then it is not generally true that the family of all sets $f_2^{-1}[V]$ for V in $\mathcal{V}$ is a uniformity for X. The problem is that there may exist a subset of $X \times X$ which contains some set $f_2^{-1}[V]$, but is not the inverse of any subset of $Y \times Y$. However the following theorem provides a means for developing a uniformity from this set.

<u>THEOREM IX</u>: If f is a function on a set X to a uniform space $(Y, \mathcal{V})$, then the family of all sets $f_2^{-1}[V]$ for V in $\mathcal{V}$ is a base for a uniformity for X.

Proof:

It is evident that f_2 preserves inclusions, intersections, and inverses. Therefore it is only necessary to show that for each member U of $\mathcal{V}$ there is a V in $\mathcal{V}$ such that $f_2^{-1}[V] \circ f_2^{-1}[V] \subset f_2^{-1}[U]$. If $V \circ V \subset U$ and (x,y) and (y,z) belong to $f_2^{-1}[V]$, then both the points (f(x), f(y)) and (f(y), f(z)) belong to V, and hence (f(x), f(z)) is in $V \circ V$. Thus for each U in $\mathcal{V}$ there exists a V in $\mathcal{V}$ such that $f_2^{-1}[V] \circ f_2^{-1}[V] \subset f_2^{-1}[U]$ and therefore the family of inverses of $\mathcal{V}$ is a base for a uniformity $\mathcal{U}$ for X.

It can be shown that f is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$, and in fact that $\mathcal{U}$ is smaller than any other uniformity for which f is uniformly continuous [12].

If (X, $\mathcal{U}$) is a uniform space and Y is a subset of X, then by the previous statements there is a smallest uniformity such that the identity map of Y into X is uniformly continuous. In this case the members of $\mathcal{V}$ are simply the intersections of the members of $\mathcal{U}$ with $Y \times Y$. This is sometimes called the trace of $\mathcal{U}$ on $Y \times Y$. The uniformity $\mathcal{V}$ is called the relativization of $\mathcal{U}$ to Y, or the relative uniformity for Y, and (Y, $\mathcal{V}$) is said to be a uniform subspace of (X, $\mathcal{U}$).

PRODUCT UNIFORMITIES

In the preceding discussion it is stated that there is always a unique smallest uniformity which makes a map of a set X into a uniform space uniformly continuous. This idea may be extended to a family F of functions such that each member f of F maps X into a uniform space (Y_f, $\mathcal{U}_f$). The family of all sets of the form $f_2^{-1}[U] = \{(x,y) : (f(x), f(y)) \text{ is in } U\}$, for f in F and U in $\mathcal{U}_f$, is a subbase for a uniformity $\mathcal{U}$ for X, and $\mathcal{U}$ is the smallest uniformity such that each map f, and element of F, is uniformly continuous. This leads to the following definition of the product uniformity.

If (X_a, $\mathcal{U}_a$) is a uniform space for each number a of an index set A, then the <u>product uniformity</u> for $\times\{X_a : a \text{ is in } A\}$ is the smallest uniformity such that the projection into each coordinate space is uniformly continuous.

<u>THEOREM X</u>: The topology of the product uniformity is the product topology.

Proof:

The family of all sets of the form $\{(x,y) \ : \ (x_a,y_a) \text{ in } U\}$, for a in A and U in $\mathcal{U}_a$ is a subbase for the product uniformity. If x is a member of the product space, by application of theorem IV a subbase for the neighborhood system of x (relative to the uniform topology) may be obtained from this subbase for the product uniformity. Therefore the family of all sets of the form $\{y \ : \ (x_a,y_a) \text{ in } U\}$ is a subbase for the neighborhood system of x. Hence the family of all finite intersections of the sets of this form provide a base for the neighborhood system of x. However, a base for the neighborhood system of x relative to the topology of the product uniformity is the family of finite intersections of sets of the form $\{y \ : \ y_a \text{ is in } U\,[x_a]\}$ for a in A and U in $\mathcal{U}_a$, and these sets may also be written as $\{y \ :(x_a,y_a) \text{ is in } U\}$. Thus the base for the neighborhood system of x relative to the topology of the product uniformity is the same as the base for the neighborhood system of x relative to the product topology, and hence the product topology is the topology of the product uniformity.

<u>THEOREM XI</u>: A function f on a uniform space to a product of uniform spaces is uniformly continuous if and only if the composition of f with each projection into a coordinate space is uniformly continuous.

Proof:

By definition of the product uniformity, if f is uniformly continuous with values in the product $\times\{X_a \ : a \text{ in } A\}$, then each projection P_a is uniformly continuous and hence the composition $P_a \circ f$ is uniformly continuous and the necessity of the theorem is proved.

The sufficiency part of the theorem states that if the composition of f with each projection is uniformly continuous, then f is uniformly continuous.

Therefore if $P_a \circ f$ is uniformly continuous for each a in A and U a member of the uniformity for X_a, then the set $\{(u,v) : (P \circ f(u), P \circ f(v)) \text{ is in } U\}$ is a member of the uniformity $\mathcal{V}$ of the domain of f. This set can be written as follows:

$$f_2^{-1}\left[\{(x,y) : x_a,y_a) \text{ in } U\}\right]$$

where $\{(x,y) : (x_a,y_a) \text{ in } U\}$ is a subbase for the product uniformity. Hence the inverse under f_2 of each member of a subbase for the product uniformity belongs to $\mathcal{V}$ and therefore f is uniformly continuous.

The developement of the relationship between uniformities and pseudo-metrics for X will be considered next.

A pseudo-metric is a distance function on the cartesian product $X \times X$ which satisfies the conditions for a metric except that it may equal zero for for two distinct points.

<u>THEOREM XII</u>: Let $(X, \mathcal{U})$ be a uniform space and let d be a pseudo-metric for X. Then d is uniformly continuous on $X \times X$ relative to the product uniformity if and only if the set $\{(x,y) : d(x,y) < r\}$ is a member of $\mathcal{U}$ for each positive r.

Proof:

Let $V_{d,r} = \{(x,y) : d(x,y) < r\}$. If U is a member of $\mathcal{U}$, then the sets $\{((x,y), (u,v)) : (x,u) \text{ is in } U\}$ and $\{((x,y), (u,v)) : (y,v) \text{ is in } U\}$ are members of the product uniformity for $X \times X$, and the family of all sets of the form $\{((x,y), (u,v)) : (x,u) \text{ in } U \text{ and } (y,v) \text{ in } U\}$ is a base for the product uniformity. Therefore if d is uniformly continuous on $X \times X$, then for each positive r there exists a U in $\mathcal{U}$ such that, if (x,u) and (y,v) are in U, then

$$|d(x,y) - d(u,v)| < r.$$

In particular, letting $(u,v) = (y,y)$, it follows that, if (x,y) is in U then $d(x,y) < r$. Hence $U \subset V_{d,r}$ and therefore $V_{d,r}$ is in $\mathcal{U}$.

To prove the sufficiency of the theorem, let (x,u) and (y,v) belong to $V_{d,r}$, $V_{d,r}$ an element of $\mathcal{U}$. This implies the following two inequalities:

(a) $d(x,y) \leqslant d(y,v) + d(v,u) + d(u,x)$;

(b) $d(u,v) \leqslant d(v,y) + d(y,x) + d(x,u)$.

Now (a) implies that $d(x,y) \leqslant 2r + d(u,v)$ and (b) implies that $d(u,v) \leqslant 2r + d(x,y)$. Therefore $\left|d(x,y) - d(u,v)\right| \leqslant 2r$ and it follows that if $V_{d,r}$ is $\mathcal{U}$ for each positive r, then d is uniformly continuous on $X \times X$.

METRIZATION

Every pseudo-metric d for a set X generates a uniformity in the following way. For every positive number r let $V_{d,r}$ equal $\left\{(x,y) \; : \; d(x,y) < r\right\}$. Then $(V_{d,r})^{-1} = V_{d,r}$, $V_{d,r} \cap V_{d,s}$ equals $V_{d,t}$ where t is the minimum of r and s, and the composition of $V_{d,r}$ with itself is equal to $V_{d,2r}$. From this it is apparent that the family of all sets of the form $V_{d,r}$ form a base for a uniformity for X. This uniformity is called the <u>pseudo-metric uniformity</u>, or the uniformity generated by d.

The uniformity generated by d can be described in an other way. By theorem XII a psuedo-metric d is uniformly continuous relative to a uniformity $\mathcal{V}$ if and only if $V_{d,r}$ is in $\mathcal{V}$ for each positive r. Therefore the uniformity $\mathcal{U}$ generated by d can be described as the smallest uniformity for which d is uniformly continuous on $X \times X$. This leads to the statement that a uniform space $(X, \mathcal{U})$ is said to be pseudo-metrizable (or metrizable) if and only if there is a pseudo-metric (metric, respectively) d such that $\mathcal{U}$ is the uniformity generated by d.

Before progressing to the metrization theorem it is necessary to prove the following lemma.

METRIZATION LEMMA: Let $\{U_n,\ n = 0,1,2,\cdots\}$ be a sequence of subsets of $X \times X$ such that $U_0 = X \times X$, each U_n contains the diagonal, and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for each n. Then there is a non-negative real-valued function d on $X \times X$ such that

(a) $d(x,y) + d(y,z) \geqslant d(x,z)$ for all x,y,z:

(b) $U_n \subset \{(x,y) : d(x,y) < 2^{-n}\} \subset U_{n-1}$ for each positive integer n.

If each U_n is symmetric, then there is a pseudo-metric d satisfying condition (b).

Proof:

Define a real-valued function f on $X \times X$ as follows:

$$f(x,y) = \begin{cases} 1/2^n & \text{if } (x,y) \text{ is in } U_{n-1} - U_n. \\ 0 & \text{if } (x,y) \text{ is in } U_n \text{ for all } n. \end{cases}$$

For each x and each y in X let d(x,y) be the greatest lower bound of $\sum \{f(x_i,x_{i+1}) : i = 0, 1,2,\cdots,n\}$ over all finite sequences x_0, $x_1,\cdots,x_{n+1}$ such that $x = x_0$ and $y = x_{n+1}$. To show that the triangle inequality holds consider any three points x,y,z in X. Then d(x,y) = greatest lower bound of the summation of all finite chains connecting x and y. However, when considering all finite chains connecting x and y, it is obvious that the one passing from x to z to y is included in the summation. This would be true for any three points in $X \times X$. Hence it is quite obvious that the triangle inequality holds.

The proof of condition (b) will be done in two parts. The first part, $U_n \subset \{(x,y) : d(x,y) < 2^{-n}\}$, follows from the fact that since d(x,y) is always less than or equal to f(x,y), then $U_n \subset \{(x,y) : d(x,y) < 1/2^n\}$.

The second part of condition (b) states that if $d(x,y)$ is less than $1/2^n$, then (x,y) is in U_{n-1}. This reduces to showing that $f(x_0, x_{n+1}) \leqslant 2 \sum \left\{ f(x_i,x_{i+1}) \right\}$. This last inequality is easily shown to be valid for $n = 0$. For the rest of the proof consider the $\sum \left\{ f(x_i,x_{i+1}) \ : \ i = r, \ldots, s \right\}$ as the length of the chain from r to s+1, and let a be the length of the chain from 0 to n+1. Let k be the largest integer such that the chain from 0 to k is of length at most a/2, which implies that the length from k+1 to n+1 is at most a/2. By the hypothesis, each of the two, $f(x_0, x_k)$ and $f(x_k,x_{n+1})$ is at most $2(a/2) = a$ and certainly it is true that $f(x_k, x_{k+1})$ is at most a. If m is the smallest integer such that $2^{-m} \leqslant a$, then (x_0, x_k), (x_k, x_{k+1}) and (x_{k+1}, x_{n+1}) all belong to U_m and therefore (x_0, x_{n+1}) is in U_{m-1}. Hence $f(x_0, x_{n+1}) \leqslant 2^{-m+1} \leqslant 2a$ and the proof is completed.

If U_n is symmetric, then $f(x,y) = f(y,x)$ for each pair (x,y) and hence there exists a pseudo-metric which satisfies condition (b).

The metrization theorem first appeared in the paper by P. Alexandroff and Urysohn in 1923, in which the authors were seeking a solution to the general metrization problem. Their results stated that a topological Hausdorff space $(X, \mathcal{J})$ is metrizable if and only if there is a uniformity with a countable base such that $\mathcal{J}$ is the uniform topology. While this does not give a satisfactory solution to the topological metrization problem, it does satisfy the metrization problem for uniform spaces. The following form of the theorem first appeared in André Weil's monograph [13]. The proof given here is the arrangement of A.H. Frink's proof as given in Bourbakie.

Metrization THEOREM : A uniform space is pseudo-metriable if and only if its uniformity has a countable base.

Proof:

If a uniformity $\mathcal{U}$ for X has a countable base $V_0, V_1, V_2 \cdots$, then it is possible to construct by induction a family of sets $U_0, U_1, U_2, \cdots$, such that each U_n is symmetric, and the composition of U_n with itself twice is contained in U_{n-1} and also $U_n \subset V_n$ for each positive integer n.

The family of sets U_n is then a base for $\mathcal{U}$, and upon applying the metrization lemma it follows that the uniform space $(X, \mathcal{U})$ is pseudo-metrizable.

This theorem implies that a uniform space is metrizable if and only if its uniformity has a countable base and it is Hausdorff.

A uniformity for a set X may be derived from a family P of pseudo-metrics in the following fashion. Letting $V_{p,r}$ equal $\{(x,y) : p(x,y) < r\}$, the family of all sets $V_{p,r}$ for p in P and r positive is a subbase for a uniformity $\mathcal{U}$ for X. This uniformity is defined to be the uniformity generated by P and can be described in several ways. By theorem XII the uniformity generated by P is the smallest uniformity which makes each member of P uniformly continuous on $X \times X$ [9, p. 187] .

Corresponding to the metrization theorem for topological spaces is the classification of these uniformities generated by families of pseudo-metrics. This might be referred to as the generalized metrization problem for uniform spaces. The solution is given by the following theorem.

<u>THEOREM XIII</u>: Each uniformity for X is generated by the family of all pseudo-metrics which are uniformly continuous on $X \times X$.

Proof:

Let $(X, \mathcal{U})$ be a uniform space and let P be the family of all pseudo-metrics for X which are uniformly continuous on $X \times X$.

The uniformity generated by P is smaller than $\mathcal{U}$ by theorem XII. However, the metrization lemma shows that for each member U of $\mathcal{U}$ there is a member p of P such that the set $\left\{(x,y) \; : \; p(x,y) < \frac{1}{4}\right\}$ is contained in U, and therefore $\mathcal{U}$ is smaller than the uniformity generated by P. Thus each uniformity is generated by P and the proof is completed.

The following theorem yields a characterization of those topologies which can be the uniform topology for some uniformity.

<u>THEOREM XIV</u>: Each uniform space is uniformly isomorphic to a subspace of the product of pseudo-metric spaces and each uniform Hausdorff space is isometric to a subspace of the product of metric spaces.

Proof:

Let X be a space and P be the set of all pseudo-metrics on X X. If $X_p = \left\{X, p\right\}$, then let $Z = \times\left\{X_p \; : \; p \text{ is in } P\right\}$. Let f be the map of X into Z where f is defined by $f(x)_p = x$ for each x in X and p in P. Let the p-th coordinate space of this product be assigned the uniformity of the pseudo-metric p, and let Z have the product uniformity. Then the projection of Z into the p-th coordinate space is the identity map of X onto the pseudo-metric space (X, p), and it follows from Theorem XI that the uniformity generated by P is the smallest having the property that the map of X into Z is uniformly continuous. Since f is one-to-one, this is a uniform isomorphism of X onto a subspace of the product of pseudo-metric spaces. The proof for the uniform Hausdorff space is ommitted.

As a result of this theorem which states that each uniform space is homeomorphic to a subspace of a product of pseudo-metrizable spaces, the following corollary can be stated.

<u>COROLLARY</u>: A topology $\mathcal{J}$ for a set X is the uniform topology for some

uniformity for X if and only if the topological space $(X, \mathcal{J})$ is completely regular.

Proof:

The proof uses the fact that a space is completely regular if and only if it is homeomorphic to a subspace of a product of pseudo-metric spaces [9, p. 134].

The remainder of this paper is devoted to the clarification of the relationship between uniformities and pseudo-metrics. A family P of pseudo-metrics for a set X is said to be a <u>gage</u> if and only if there is a uniformity $\mathcal{U}$ for X such that P is the family of all pseudo-metrics which are uniformly continuous on $X \times X$ relative to the product uniformity derived from $\mathcal{U}$. A direct description of the gage generated by a family P of pseudo-metrics can be given. The family of all sets of the form $V_{p,r}$ for p in P and r positive is a subbase for the uniformity of the gage. Therefore a pseudo-metric q is uniformly continuous on the product space if and only if for each positive number s the set $V_{p,s}$ contains some finite intersection of sets $V_{p,r}$ for p in P. This remark establishes the following theorem. A detailed proof can be found in Nagata's paper [10].

<u>THEOREM XV</u>: Let P be a family of pseudo-metrics for a set X and Q be the gage generated by P. Then a pseudo-metric q belongs to Q if and only if for each positive number s there is a positive number r and a finite subfamily $p_1, p_2, \cdots, p_n$ of P such that $\bigcap \{V_{p_i,r} : i = 1,2,\cdots, n\}$ is contained in $V_{q,s}$.

Each concept which is based on the notion of a uniformity can be described in terms of a gage because each uniformity is completely determined by its gage. The following theorem simply summarizes the major ideas and is given without proof.

<u>THEOREM XVI</u>: Let $(X, \mathcal{U})$ bo a uniform space and let P be the gage of . Then:

(a) Tho family of all sets $V_{p,r}$ for p in P and r positive is a base for the uniformity $\mathcal{U}$.

(b) The closure relative to the uniform topology of a subset A of X is the set of all x such that the p-distance (x, A) = 0 for each p in P.

(c) The interior of a set A is the set of all points such that for some p in P and some positive number r the sphere $V_{p,r}[x] \subset A$.

(d) Suppose P is a subfamily of P which generates P. A net* S_n, n in D in X converges to a point s if and only if $p(S_n, s)$, n in D converges to zero for each p in P.

(e) A function f on X to a uniform space $(Y, \mathcal{V})$ is uniformly continuous if and only if for each member q of the gage Q of $\mathcal{V}$ it is true that $q \circ f_2$ is in P. Equivalently, f is uniformly continuous if and only if for each q in Q and each positive number s there is a p in P and an r positive such that, if $p(x,y) < r$, then $q(f(x), f(y)) < s$.

(f) If $(X_a, \mathcal{U}_a)$ is a uniform space for each member a of an index set A and P_a is the gage of $\mathcal{U}_a$, then the gage of the product uniformity for $\times\{X_a : a \text{ in } A\}$ is generated by all pseudo-metrics of the form $q(x,y) = p_a(x_a, y_a)$ for a in A and p_a in P_a.

* A <u>net</u> is a pair $(S, \leqslant)$ such that S is a function and $\leqslant$ directs the domain of S [9, p. 65].

ACKNOWLEDGMENT

The writer wishes to express his sincere appreciation to Dr. N. E. Foland for his patient guidance and supervision given during the preparation of this report.

REFERENCES

1. Atsuji, Masahiko.
"Uniform Continuity of Continuous Functions of Metric Spaces," Pacific Journal of Mathematics, Vol. 8, 1958. pp. 11-16.

2. Bing, R. H.
"Metrization of Topological Spaces," Canadian Journal of Mathematics, Vol. 3, 1951. pp. 175-186.

3. Doss, R.
"On Uniform Spaces with a Unique Structure," American Journal of Mathematics, Vol. 71, 1949. pp. 19-23.

4. Fréchet, M.
"Sur Quelque Points du Calcul Fonctionnel," Rendiconti di Palermo, 1906. pp. 1-74.

5. Gal, I. S.
"Uniformizable Spaces with a Unique Structure," Pacific Journal of Mathematics, Vol. 9, 1959. pp. 1053-1060.

6. Hausdorff, F.
Grundzüge der Mengenlehre, Leipzig, 1914.

7. Isbell, J. R.
"Algebras of Functions," Annals of Mathematics, Vol. 68, 1958. pp. 96-124.

8. ---------------
"On Finite Dimensional Uniform Spaces," Pacific Journal of Mathematics, Vol. 9, 1959. pp. 107-121.

9. Kelley, John L.
General Topology. New York: D. Van Nostrand Company, Inc., 1955. 281 p.

10. Nagata, J.
"On a Necessary and Sufficient Condition of Metrizability," Osaka Mathematics Journal, Vol. 1, 1950. pp. 93-100.

11. Nakano, Hidegoro.
Topology and Linear Topological Spaces. Chapt. 4, Tokyo, Maruzen Co., Ltd., 1951. pp. 62-100.

12. Rainwater, John.
"Spaces Whose Finest Uniformity is Metric," Pacific Journal of Mathematics, Vol. 9, 1959. pp. 567-570.

13. Weil, André.
"Sur les Espaces a Structure Uniforme et sur la Topologie Generale," Actualites Sci. Ind., Vol. 551, 1937.

METRIZATION OF UNIFORM SPACES

by

SAMUEL LELAND LESSEIG

B. S., FORT HAYS KANSAS STATE COLLEGE, 1961

AN ABSTRACT OF A REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

Fréchet first considered abstract spaces in his 1906 thesis. The early development of this concept can be found in Hausdorff's Grundzuge der Mengenlehre. From this early research two fundamental ideas have developed: that of a topological space and that of a uniform space. André Weil was the first to formalize the idea of a uniform space in a paper in 1937. The study of uniform space developed from the study of topological groups as is evident by the similarity of the defining properties for a uniformity and a topological group.

The defining properties and characterizations of a base and a subbase for a uniformity are given since a uniformity is determined entirely by its base or subbase. These two concepts are then extended to the uniform topology of a space $X \times X$.

The definition and necessary and sufficient conditions for uniform continuity are stated for a uniform space. These conditions provide a different base for a given uniformity $\mathcal{U}$, and enable one to show the existence of a smallest uniformity for space.

The concept of a smallest uniformity is extended to the product space and is used to prove that the topology of the product uniformity is the product topology. This result leads to the necessary and sufficient conditions for a pseudo-metric or metric on the space X to be uniformly continuous on $X \times X$.

The last section is concerned with the generalized metrization problem for uniform spaces. A solution of this problem is given by the Metrization Theorem. This solution enables one to state most of the previous results in terms of a pseudo-metric on the space.