SEARCH TECHNIQUES
AND WAGE INCENTIVE PLANS

by

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ABSTRACT

This thesis has tow sections. Part 1 deals with the literature survey and the development of new techniques to handle search problems. Since the effectiveness of the search procedure is characterized by its rate of convergence, much of research work has been and are still being done to reduce the computation time. An attempt was made to solve one-dimensional search problems for convex functions by bisecting the enveloping cone of the function and then rotating it till the bisector becomes vertical. The generalization of this new method for any unimodal function by coupling with Fibonacci search was also discussed. This approach essentially cuts down the total number of experiments required to reach at optimum. A new method for multi-dimensional search problems based on the intersection of quadratics passing through the line-optimums in co-ordinate directions was developed and exemplified along with the comparison with other standard methods to show its efficiency.

In the second section, a case study was made with a view to show how operations research technique can be applied to formulate and solve certain wage incentive problems. Since the basic problem in an incentive scheme is to define the base level efficiency from which the incentive should start and also the incentive rates, the problem was formulated with the objective as to minimize the variance between the optimum base

level efficiency and the current different efficiencies of various departments. A constraint was that the total incentiv to be paid to the workers must not exceed the current overtime expenses. This problem was solved by Generalized Reduced Gradient method and separable programming.

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CHAPTER I

LITERATURE SURVEY ON SEARCH TECHNIQUES

1.1 Introduction

Search for the optimum is the main objective of all decision problems, whether constrained or unconstrained. In fact, constrained problems can be converted into unconstrained ones. The simplest form of search is that for a function having only one variable. In general, two policies, viz, sequential and simultaneous searches are normally used. Some research work has also been done combining these two policies. All these techniques are available in different literatures. Section 1.2 deals with the literature survey on one-dimensional search. In addition, the quadratic and cubic interpolation methods are also briefly discussed. The literature review on multi-dimensional search is provided in section 1.3.

Although there exist different types of searches depending on the nature and the objective of the search, this chapter does not cover all of them.

1.2 One Dimensional Search

For a maximum of an unimodal function, the second order search in the sense that the information is given by pairs of observation, was attempted by Kieffer[1]. He determined the interval containing this maximum without postulating any regularity conditions involving continuity, derivatives, etc. This is essentially known as Fibonacci search method which has theoretical connections with problems dating all the way back to Euclid.

A less powerful method known as Dichotomous Search[2] reduces the interval of uncertainity by placing pairs of experiments successively in the remaining interval. The effectiveness of this method grows exponentially with the number of experiments. In a Fibonacci scheme each new experiment serves to reduce the interval of uncertainity but in a dichotomous scheme it takes two new experiments to cut down the interval of uncertainity.

Another technique which is nearly as effective as the Fibonacci does not require any knowledge in advance the number of experiments to be carried out. This is familiar as Golden Section Method[2] which essentially divides a segment into two unequal parts so that the ratio of the whole to the larger is equal to the larger to the smaller. Euclid, himself, did this simply by a ruler and a compass.

When the variable does not assume continuous values within a given interval of uncertainly but instead is confined to a finite number of discrete points, the Lattice Search Technique as Kieffer[2] calls them is used. In this case the number of points are to be finite and arrangable in some order that will make the criterion of effectiveness unimodal.

Oliver and Wilde[3] pointed out that Kiefer's original technique (Fibonacci Search) is assymmetric in the sense that the last two experiments are not located symmetrically with respect to each other. The modified procedure developed by Oliver et. al. is symmetric since it permits the last experiment to be placed symmetrically with respect to the most effective previous experiment.

Avriel and Wilde[4] developed a minimum search plan using 'Block Search Strategy' technique that can be used for any number of experiments and for any number of blocks in the sequence. For one experiment per block, it reduces to the Fibonacci strategy. The 'Block Search Strategy' is optimal in the sense that for a required final interval of uncertainity and for any given number of simultaneous experiments and blocks, it has the largest possible starting interval.

According to Berman[5] his method which uses Fibonacci numbers is optimal because 1) it does not postulate any regularity conditions, 2) it is simpler, 3) it often requires fewer number of evaluations, 4) it is self-correcting i.e. an error in any particular evaluation will not affect the final result. He also exemplified some possible application of his method.

When an arbitrary probability density function for the distribution of maximum is given, the problem of estimating the optimal interval containing the location of the maximum of a unimodal function was investigated by Heymann[6]. The statistical information gained by the search is used for such estimation. He found that the strategies had to be different in accordance with the odd or even number of experiments.

The minimax block search strategy presented by Avriel and Wilde[4] was further improved by them in a latter publication[7] and it was shown that this method is an excellent approximation of the previous one. This nearly optimal minimax golden block search method has the advantage that the number of function evaluations need not be specified in advance.

When some bound on the rate of change of function of one variable is available, Shubert's[8] method can be used to locate the maximum of the function defined over a closed interval.

Wilde and Beamer[9] presented a minimax search strategy for locating the boundary point of a region on a line joining a feasible point to an infeasible point. These strategies, as it were claimed, could be useful subroutines for many multi-dimensional optimization algorithims.

Gottfried[10] showed that for a given interval of uncertainity , the minimax separation between two points considering the distinguishibility of the function values, the search should be terminated when the interval of uncertainity is less than $(\epsilon\rho)/(2-\rho)$ where ρ is the golden ratio.

One of the new additions in the development of search procedures for one dimensional problems was made by Fox, et. al.[11] Their method finds three points bracketting the minimum, fits a quadratic through them to yield a fourth point, then fits successive cubic through four points discarding one at each time, until certain stop criteria are met. No gradient evaluations are required. This procedure is claimed to take 1/2 to 3/4th less computer time than others.

When all experiments must be run at the same time, it is necessary to use a simultaneous search plan. It is less effective than the sequential plans but the experimenter, at times, is forced to use a simultaneous plan. The interval is divided into '$(n+1)$' equal interval and the function is evaluated at 'n' points. The best value of function is picked up, the interval bracketing this best value is again divided into $(n+1)$ divisions and the process is repeated till it meets the stopping criterion. For two experiments only, simultaneous plan is just as good as a sequential one.

Wilde[2] suggested that for even number of experiments, search by uniform pairs which is, essentially comes under simultaneous search plan is the best way as far as the deployment of the experiments are concerned.

If the objective function is continuous and convex in the interval of uncertainity, it is often possible to obtain a good estimate of the optimum value (12) of the objective function by using a quadratic approximation of the function to locate the optimum point. But if in addition to the above, the derivate of the function is available then cubic interpolation provides a good estimate of the location of the optimum point.

1.3 Multi Dimensional Search

The problem of locating the optimum on a multi-dimensional response surface is more important than uni-dimensional search since the problems encountered in the real world usually involve multi-dimensions.

With the object of finding the optimum in these types of problems Cauchy[13] first introduced the method of steepest descent which, as a matter of fact, forms the basis for all the searches currently in use. It was an intitutively attractive idea of climbing the steepest path but because of the inherent difficulties (slow convergence due to interaction among the variables) associated with each new direction being normal to the old direction, the method is not very efficient.

In the modified steepest descent method[14] the step size of the steepest descent is multiplied by 0.9 and the process is continued. After, say, four such repetitions of this procedure, one step of full length is taken. In this case, the successive directions will not be mutually orthogonal.

The method of rotating the co-ordinates, as devised by Rosenbrock[15] is very effective at finding the optimum of a function. Instead of taking a fixed step in each direction, Rosenbrock rotates the co-ordinate system so that one axis points along the direction of ridge as estimated by the previous trial. The other axes are arranged in directions normal to the first.

For straight type of ridges Partan Method[16] would appear efficient. This technique which does not use gradients can be extended to ellipsoidal functions of any number of independent variables. For non ellipsoidal functions, the Partan will work but if the function is not radially similar on every possible cross section, it will not work.

A variant of this method was discovered independently by Powell[17]. It is based on the theorem which is that because the function $f(\underline{x})$ is quadratic in the independent variables, any line which passes through the optimum point 'ξ', intersects the members of the family of contours $f(\underline{x})=c$

(constant) at equal angles. The corollary is that of the normal at '$\underline{t}$' to the contour $f(\underline{x}) = f(\underline{t})$ is parallel to the normal at '$\underline{t}'$' to $f(\underline{x}) = f(\underline{t}')$, then the lines joining $\underline{t}$ to $\underline{t}'$ pass through 'γ'. This method gives second order convergence.

The method of sectioning or one at a time method[18] will not always reach the maximum, even when the contours are convex. Its practical value is extremely limited. It is good for circular contours only.

The pattern search technique of Hooke and Jeeves[19] has had reasonable practical success, probably due to its ability to follow a curved ridge when necessary. Mugele's "poor man's optimizer"[20] scheme also is able to track the curved ridges. In these methods gradient evaluations are not needed.

In case of defined gradient, Fletcher and Powell's method[21] which essentially is a simplified version of Davidon's (1959) method, provides quadratic convergence and it is superior to Powell's and Partan method both in that it uses the information determined by previous iterations and also in that each iteration is quick and simple to carry out. Further more it yields the curvature of the function at the optimum.

Fletcher and Reeves[22] conjugate gradient method is as effective as that of Fletcher and Powell's method. In the latter method, storage space for H (Hessian) matrix is to be provided while in Fletcher and Reeve's method, storage is required for only three vectors and time for manipulating H matrix is saved. So in problems, where 'n', the number of variables is large, this method may be preferred to Fletcher and Powell's method.

When derivatives are not available, Powell's method[23] furnishes faster convergence in dealing with many variables. The first iteration is same as that for changing one parameter at a time. This latter method is next modified to generate conjugate directions by making each iteration define a new direction ξ and choosing the linearly independent directions for next iterations.

Sequential simplex method is also useful to handle these types of problems. It was introduced by Nelder and Mead[24]. It has the same convergence rate as that of Powell's method.

For minimizing a sum of squares of non-linear functions Powell's generalized least square method[25] does not require evaluation of derivatives. This method has the comparable convergence with the classical procedure and the number of times the individual terms of the sum of squares have to be calculated is approximately proportional to the number of variables.

In a review paper[26] Fletcher discussed the efficiency of the three different methods, viz, Davis, Swam and Campey method (DSC method), Powell's method and Smith's method using some standard test functions as a basis for comparisons. All these three methods do not require any calculation of derivatives. DSC method is simple and effective for large numbers of variables and when the minimum cannot be represented adequately by a quadratic where as on the basis of function evaluations the most efficient method is that of Powell. However, for large number of variables it is less favorable than DSC method. The Smith's method is generally inferior to other methods and is acceptable only when 'n' is small (2, 3, 4).

Brannen[27] showed that a return function with a given probability distribution can be maximized using an iterative method which is somewhat analogous to Newton's iterative method.

Box[28] proved that as the number of variables increases, Fletcher and Powell's method is most consistently successful when the gradient is available. Powell's method and Fletcher and Powell's method work substantially better with 5, 10, 20 dimension test functions than other methods though it assumes quadratic optimum characteristic. He has also pointed out that simplex method perform better than Powell's method in case of two-dimensions but lesser and lesser successful as the dimension increases.

Curtis and Powell[29] discussed in detail on exchange algarithims for calculating minimax approximation with a view to provide a deep insight into the convergence of this method.

Powell's method has been criticized by Zangwill[30] who in his counter-example showed that Powell's method not only does not converge to the minimum of a quadratic in a finite number of iterations but it will not converge in any number of iterations. He made some modification of Powell's method which can be useful strictly for convex function.

The variation matrix method developed by Davidon[31] which uses the inverse matrix of second derivative of any function is the generalized form of variable metric method (Davidon, 1959). The algorithim is simpler and in quadratic cases, gradient evaluations are half the number made in variable metric algorithim.

An algorithim for non-linear minimax approximation was described by Osborne and Watson[33] in 1969 . This algorithim was illustrated by the evaluation of several approximation to the solution of Blasias equation.

An improved procedure presented by Palmer[34] to generate orthogonal search vectors for use in Rosenbrock's(1960) and Swann's (1964) optimization method was shown to make considerable savings in time and in storage requirements. It also deals more satisfactorily with certain cases in which the original method fails.

Pearson[35] did an extensive numerical comparison among Newton-Raphson Method, Fletcher and Reeves method and the DFP method. His conclusion was that for well-behaved function Fletcher and Reeves method is simple and fast while for the penalty function methods, the variable metric algorithims are much better and operate more efficiently with reset. The generalized Newton-Raphson algorithim always required fewer iterations and when it can be used, it proves to be the quickest method.

Based on Davidon's method, Mielle et.al.[36] proposed a new accelerated gradient for finding the minimum of a function. He included one extra form $\langle \delta \hat{x} \rangle$ in the step length calculation that takes into account the change in position vector from the iteration preceding that under consideration. He showed that, as compared to Fletcher and Reeves method, his method takes 25% to 40% less computation time and uses 50% to 60% less number of iterations.

The DFP method uses the approximate form of inverse of the Hessian H matrix of objective function 'f' using only the gradient of 'f'. Greenstadt[37] showed that by solving certain variational problems, formulas for successive correction to H matrix can be developed that closely resembles Davidon's and satisfies DFP's condition.

Using the Greenstadt's variational approach, Goldfarb[38] developed a new rank - two variable metric method. Like DFP method it preserves the positive definiteness of the H - matrix.

Extension of Davidon's method for minimization problem in Hilbert space was demonstrated by Tokumaru, et. al.[39] by solving optimal control problems.

Chazan and Miranker[40] described an algorithim which is suitable for execution on a parallel computer. A non-gradient method similar to Powell's method was used and was shown that the algorithim terminates at minimum for quadratics and converges for strictly convex twice continuously differentiable function.

The variable metric algorithim was further simplified by Fletcher[41] and it was claimed to be superior to Fletcher and Powell's method since it requires less number of gradient and function evaluations. In this method an approximation of H matrix to G^{-1} matrix is kept and is updated in each iteration.

To account for the efficiency of different techniques, Huang and Levy[42] tested two different quadratically convergent algorithims (viz, DFP, Mccormick, Pearson, generalized Fletcher and Powell etc) through several numerical examples. All algorithims behave identically in case of quadratic function if high-precision arithmatic together with high accuracy in the one-dimension search is employed. They give same sequence of points, same minimum point and require same number of iterations. For the non-quadratic functions, the results show that some of the algorithims behave identically and so any of them can be considered as a representative of the entire class.

A new method for minimizing a sum of squares of non-linear functions was devised by Peckham[43]. It was claimed to be more efficient than other methods in that fewer function evaluations are required.

In DFP method the objective function F (x) is assumed strictly convex but Powell pointed out in his survey[44] of recent development of unconstrained minimization that some better algorithims have now been developed. The most useful work is that which explores algorithims that avoid subproblem of minimizing a function of one variable on every iteration (e.g. large computation time, more number of function evaluations, may not have function improvement and may go beyond the constraints in case of constrained optimization). The algorithims that provide the above features are due to independent work of Davidon[31], Fiacco and McCormick[45], Murtagh and Sargent[70], Wolfe[71], Bard[72] and Powell[73].

In 1970 Hoshino[46] found that Davis, Swann and Campey minimizaiion process may generate undesirable zig-zag searches. He proposed a simple modified algorithim and tested it on some standard test functions. The number of linear searches required were found less.

A general convergence theorem for iterative methods for unconstrained minimization problem was provided by Ortega and Rheinboldt[47]. The key point is the concept of an essentially gradient related sequence which includes the previously studied gradient-related sequences as well as sequences that arise from univariate relaxation methods.

Cohen[48] discussed the rate of convergence of several conjugate gradient algorithims to minimize non-linear, non-quadratic real valued function and pointed out that in a neighborhood of the minimum that the error, when starting from a point of reinitialization decreases by order 2 after 'n' steps.

Under the assumption of strict convexity, the projection method of conjugate direction for solving unconstrained minimization was presented by McCormick and Ritter[49]. It was shown that it converges with $(n-1)$ step superlinear rate.

Without making an initial estimate of the Go (the current estimate of the inverse of H matrix), the matrix used in variable metric algorithims, Mament, et. al.[50] presented a method that uses $x_i z_i x_i^T$ matrix where Z_i is a diagonal matrix and x_i has maximal rank. The rank of x_i increases by one at each iteration. This pseudo-Newton-Raphson algorithim as called by the authors, was shown to have finite convergence for quadratic functions and asymptotic convergence for a fairly general class of functions.

Unconstrained optimal control problem can be solved using a gradient algorithim in terms of numerical integration formula, the precision of which is controlled adaptively by a test that ensures convergence. This was shown by Klessig and Polak[51]. Their empirical results exhibits that their algorithim is considerably faster than its precision counterpart.

The rate of convergence of Zoutendijk's[52] two procedures were studied and hence two modified methods were developed by Pinonneau and Polak[53]. It is shown that under convexity assumption their method converge linearly while Zoutendijk's procedure converge sublinearly.

The method of changing one variable at a time is not an efficient method since the searches are made along the coordinate directions in sequence and the search path tends to a closed loop. On this loop the gradient of the objective function is bounded away from zero. According to Powell[54] this field alone is rather unimportant. What is important is the success of the algorithims depend on the properties that are not shared by the method that changes one variable at a time.

The conditions under which Huang's conjugate gradient method generates descent directions were discussed by Spedicato[55]. Bounds for the condition number of the inverse Hemian matrix were estimated for the case of a symmetric matrix.

Adachi[56] also found the same thing, i.e., for quadratic functions, search directions are same for all algorithims and they are independent of parameters. They generate unique sequence of minimizing points for the given initial conditions if the objective function is quadratic.

In minimizing interior penalty function, most of the computational time is spent on one-dimensional search. Lasdom et. al.[57] presented a method that performs this search on barrier function which is significantly faster than current techniques. This method exploits the special structure of varrier functions.

Algorithims for changing the step size efficiently was proposed by Krogh[58] in the year 1973. He compared the good and bad features of approximately 10 different ways for changing the step size. He also provided an efficient algorithim for the difference formulations of a frequently used halving and doubling process.

Sayama and Takamatsu[59] found that with the increase in dimensions, the disadvantage of DFP method is the computer storage problem that increases with number of iterations. In this paper this disadvantage was shown to overcome by formulating the direction of one-dimensional search by means of integral kernels to have a new computation scheme. This may be used for large number of dimensions as well as to obtain high precision for problems having ten number of dimensions.

Bertsekas and Mitter[60] proposed a new algorithim, the E - subgradient method, a large step, double iterative algorithim which converges rapidly under very general assumption for optimization problems with non-differentiable cost-functions. They discussed the application of this algorithim in some non-linear problems and optimum control and showed that E - subgradient method contains as a special case of a mini-max algorithim.

Numerical experiments on Dual Matrix algorithims are done by Huang and Chambliss[61] for function minimization. The four algorithims were characterised by the simultaneous use of two matrices and by the property that the one-dimensional search for the optimal step size is not needed for convergence. For quadratic function with n variables it needs at most $(n+1)$ number of iterations. These algorithims were tested on four non-quadratic test-functions and exhibited satisfactory convergence properties and compare favorably with the corresponding quadratically convergent algorithims using one-dimensional search procedure to obtain optimal step size. The reverse one out of 4 algorithims was found best. It requires least number of iterations and least sensitive to step size.

Larichev and Gorvits[62] carried out similar kind comparison test among different search methods vix, steepest descent, accelerated Partan method, conjugate gradient and Davidon's method using several test functions. Davidon's was the best found in terms of minimum function value and number of iterations.

The modified one-at-a-time optimization procedure introduced by Findlay[63] is based on assuming that a partial optimal value of one variable is a linear function of the other independent variables. The essence of this method is to observe the effects of each variable combined with some interactions of that variable. The number of trials required was found more than Rosenbrock method but less than gradient method in his study.

Baranger and Temam[64] in 1975 discussed at length about non-convex optimization problems. The main result is that for almost all values of the parameter, the optimization problem possesses at least one solution.

The algorithim for unconstrained optimization that do not use line searches was developed by Davidon[65]. This method uses the JJ^T instead of using H matrix and only store and update the Jacobian matrix J .

Exact solution of one-dimensional search for solving problems using DFP method is not always necessary to cover the practical situation where only approximate solutions to the line searches can be found. Lenard[66] discovered a class of methods which have n - step quadratic convergence rate when restarted even if the line search is not exact.

An algorithim for unconstrained minimization of a function of n variables that does not require the evaluation of partial derivatives was presented by Mifflin[67]. It is a second order extension of the method of local variations which makes the algorithim an approximate Newton method. Its convergence is superlinear for a twice continuously differentiable strongly convex function.

Best[68] recently developed a method that was claimed to have cubic rate of convergence. The procedure involves 'n' step optimization using any appropriate optimization prodedure which is followed by a special step and then another 'n' iterations of the underlying algorithim followed by a second special step. This pattern is then repeated. The special step is interpreted as an approximation to Newton step. After a certain number of iteration this step size procedure will always use a step size of one.

With the object of comparing the different techniques of unconstrained optimization effectively Shanno and Phua[69] took into account the overhead as well as function evaluations. This new method eliminates much of the machine dependency of earlier criteria.

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CHAPTER II

A NEW SEARCH TECHNIQUE

2.1 Introduction

The problems involving optimization of only one dimension are rarely encountered in real world. On the other hand, almost all search oriented multi-dimensional optimization problems, whether constrained or unconstrained need one dimensional search for its solution. In fact a large part of the computation time of solving multi-dimensional problems is taken by the one-dimensional search. So cutting down the computation time of one-dimensional search has the direct bearing on the reduction of computation time of multi-dimensional problems since these types of problems use one-dimensional search more than once.

Of the many techniques currently used for one dimensional search, Fibonacci search is the most powerful technique followed by Golden section because they do not assume any regularity conditions i.e. convexity, continuity, existence of derivative of function etc. Fibonacci method converges faster than any other method. It is apparent from Fig. 2.1 which shows the relationship[1] between the interval of uncertainity and the number of experiments, that for the first few experiments, the rate of convergence is very fast but after that (say, about 9 experiments), as the interval of uncertainity

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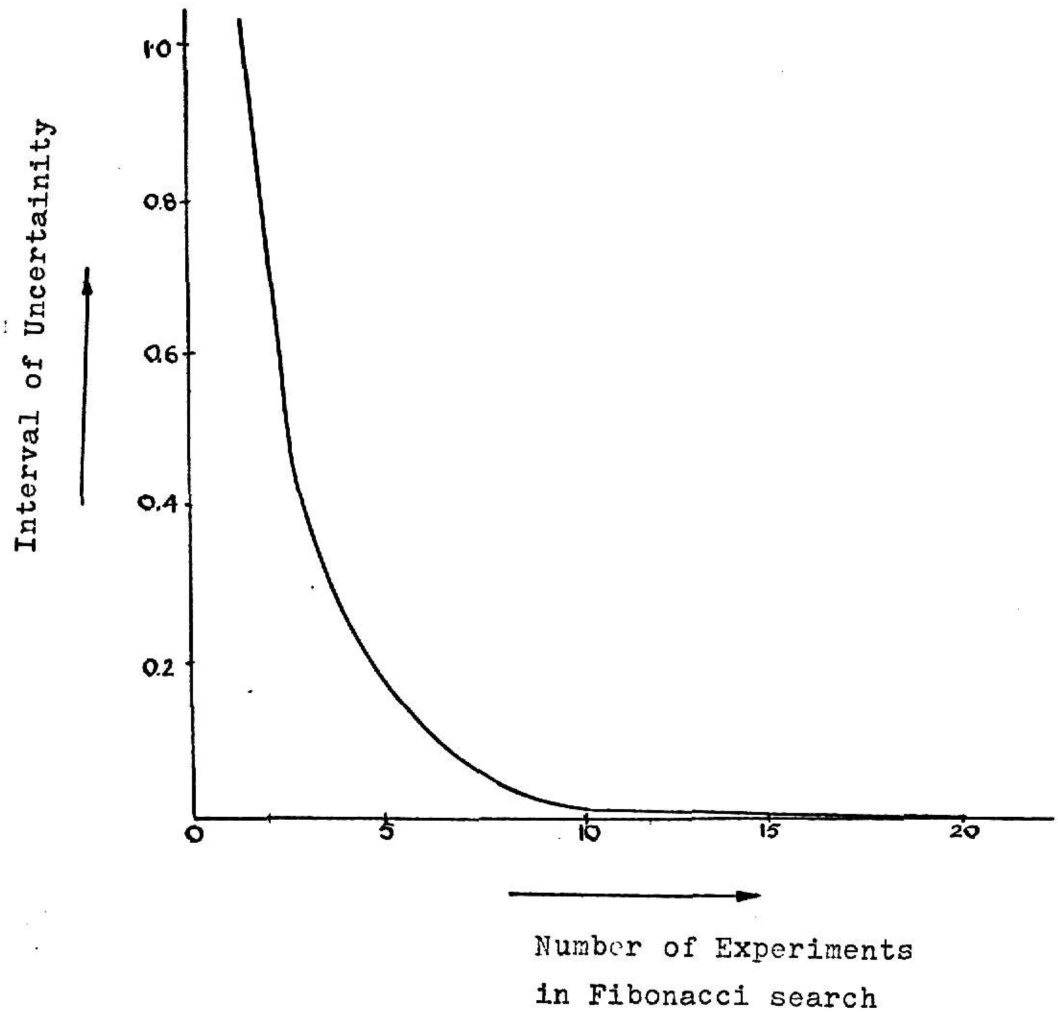

Figure 2.1 **Interval of uncertainity versus number of experiments in Fibonacci Search.**

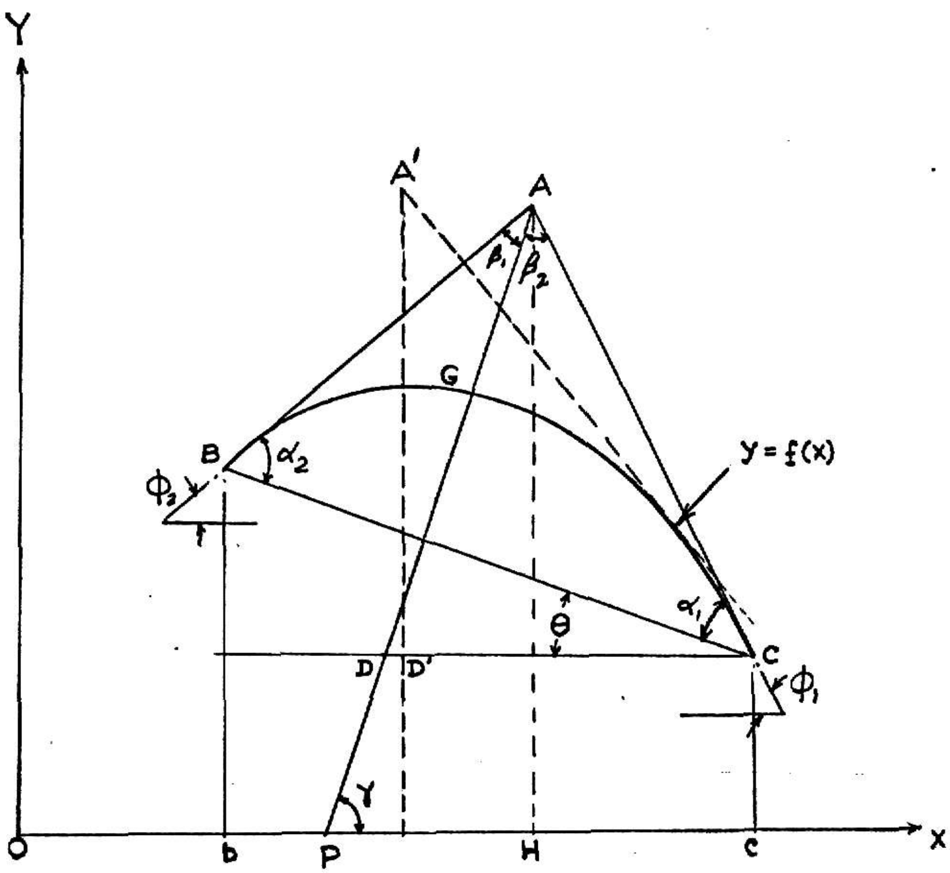

Figure 2.2 One Dimensional search for finding the maximum by this Method.

becomes smaller and smaller, the rate of convergence becomes asymptotic.

A new approach to solve one-dimensional problem is discussed in section 2.2 and how this asymptotic convergence rate can be overcome using the combination of the new method and Fibonacci method, has been discussed in section 2.3. The new method for solving multi-dimensional problems is provided in section 2.4 with two examples and the comparison of this method with the standard methods has been made and presented in section 2.5.

2.2 Method of Bisecting the Envelope of One-Dimensional Function

In case of convex function, peak value can be obtained by bisecting the envelope i.e. the tangent cone of the function and rotating the cone along the curve till the bisector becomes vertical. The point of intersection of the bisector and the abscissa gives the optimum point since the tangent at the point of intersection of the bisector with the curve becomes horizontal. Even when the function is unknown, this method can be used to determine the optimum.

Method:

Let $y = f(x)$ be the convex function as shown in Fig. 2.2 by the curve BGC in an interval bc.

AB and AC are the two tangents at B & C respectively to form the enveloping cone. Thus tan ϕ_2 and tan ϕ_1, are known when the functions are known or they can be calculated numerically by running two experiments one at b and the other at b+ Δx and other two experiments one at c and the other at c+ Δ x.

Now, $\tan\theta = \dfrac{f(b)-f(c)}{c-b}$ $\quad\therefore\ \theta = \tan^{-1}\left[\dfrac{f(b)-f(c)}{c-b}\right]$

Using f(b) , f(c) and slopes of AB and AC, the eqns. of AB and AC can be determined and solving them co-ordinate of A can be calculated.

Now $\alpha_2 = \phi_2 + \theta$ and $\alpha_1 = \phi_1 - \theta$ and since AP bisects $\angle A$, $\beta_1 = \beta_2 = 90^\circ - \left(\dfrac{\alpha_1 + \alpha_2}{2}\right)$

$\therefore$ The inclination of AP = γ = $[180^\circ - (\phi_1 + \beta_1)]$

The angle to be rotated = $\Delta\gamma$ = $[90^\circ - \gamma]$

When slope of AB is less than slope of AC i.e. when α_2 is greater than α_1, the optimum lies in the obtuse angle side of AP when $\angle\gamma$ is acute. On the other hand if α_2 is less than α_1 i.e. when slope of AB is greater than slope of AC with the $\angle\gamma$ being acute, the optimum lies on the acute angle side of the AP.

<u>Case I</u> $\angle\alpha_2 > \angle\alpha_1$

This case is shown in Fig. 2.2. The cone ABC is rotated along the curve maintaining AC always tangent to the curve through an angle $\Delta\gamma$ to make the bisector AD vertical. In that case, A will be shifted to A$'$.

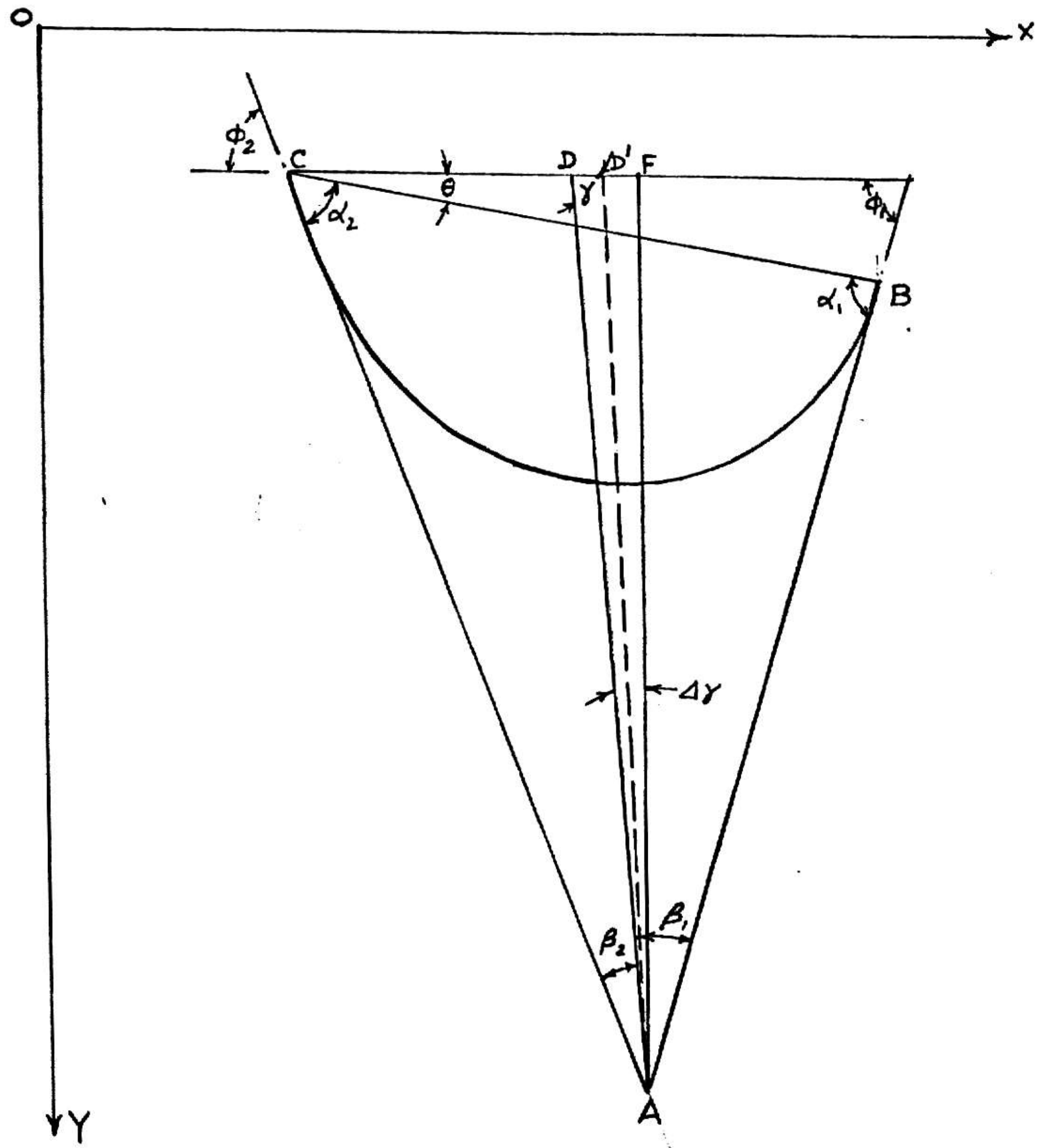

Figure 2.3A One Dimensional search for finding the minimum by this Method.

The amount of shift from point A is given by

$$S = \text{Shift} = AC \cos\phi_1 - AC \cos(\phi_1 - \Delta\gamma)$$

$$= AC \left[\cos\phi_1 - \cos(\phi_1 - \Delta\gamma)\right.$$

$$= \frac{Y_A - f(c)}{\sin\phi_1}\left[\cos\phi_1 - \cos(\phi_1 - \Delta\gamma)\right)$$

Since X_A and Y_A, the co-ordinates of A are known

$\therefore$ Xoptimum $= X_A - S$

Yoptimum = f(Xoptimum)

Case II

When α_2 is less than α_1, and $\angle\gamma$ is acute, the optimum lies within the inner triangle. Using the same procedure i.e. knowing the points B and C, the parameters $\phi_1, \phi_2, \alpha_1, \alpha_2, \beta_1, \beta_2$, Co-ordinate of A and the angle $\Delta\gamma$ are determined.

It is important to note here that, as in the previous case, if the cone is rotated through $\Delta\gamma$, the optimum will be obtained within the triangle ACD which is not true. In this case the optimum lies within the Δ ADF and the angle of rotation required is $\Delta\gamma/2$ (i.e. the rotation required by the bisector AD' of the angle DAF of Δ ADF till this new bisector becomes vertical).

$$\text{Shift} = AB \cos(\alpha_1 - \theta) - AB \left[\cos(\alpha_1 - \theta - \frac{\Delta\gamma}{2})\right]$$

$$\text{Xoptimum} = X_A - \text{Shift}$$

Examples:

Two problems were solved to illustrate the application of this methodology for both the cases. Example 1 is a maximization problem and example 2 is a minimization problem. The detail calculations etc. are provided in Appendix 1.

The results are summarized below.

Function	Optimum Point by the New Method	Optimum Point by Other Method	Difference in Function Evaluation
Max: $5x^2+4xy+8y^2-16x+8y-16=0$	$x=1.5939$ $y=1.2005$	$x=1.6$ $y=1.2$	.0005
Min: $y=e^x-5x$	$x=1.5645$ $y=-3.0422$	$x=1.6$ $y=-3.0469$	0.0047

It may be noted here that this method gives optimum in one step while the other standard methods requires several iterations to reach the optimum.

2.3 Generalization of this Method

It has been shown that for convex functions this method works well but for non-convex unimodal functions having inflection points, this method can be effectively used in combination with Fibonacci method.

It is truc that near the optimum, the function is convex. Outside this convex region bracketting the optimum, noise

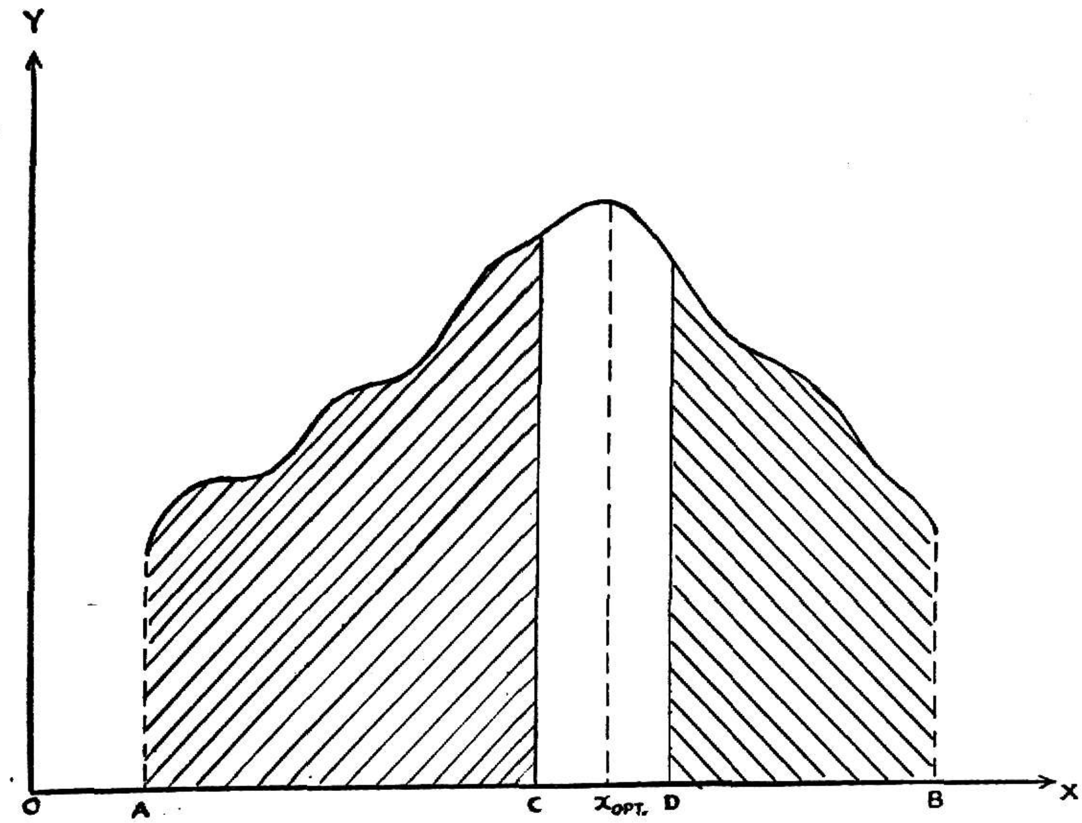

Figure 2.3B   Combination of Fibonacci Search and this Method to reach optimum.

Note:

1) AB is the original interval of uncertainity. Initial reduction from AB to CD can be done by Fibonacci Search.

2) To reach at the optimum point, from CD, this Method can be used.

in terms of inflection points exists. So the problem can be divided into two parts. In the first part noise can be eliminated using Fibonacci search which provides fast rate of convergence before it becomes asymptotic. In the second part, the new method can be used to calculate the optimum.

In the above discussion the problem is how to ascertain the domain of convex region that brackets the optimum. The researcher from his experience and the knowledge of the experiment can assume a certain percentage (say 10% to 15%) of the interval of uncertainity for this purpose leaving the rest of it 85%-90%) for Fibonacci Search (Ref. Fig. 2.3B).

Thus, by the process of coupling this method with Fibonacci search, we can overcome the asymptotic disadvantage of Fibonacci Search and cut down the total number of experiments. This is essentially an economic advantage.

APPENDIX 1

Example 1:

Maximization problem.

Max. $5x^2+4xy+8y^2-16x+8y-16=0$

Differentiating $\quad 10x+4y+4xy'+16yy'-16+8y'=0$

$$\text{or} \quad y' = \frac{dy}{dx} = \frac{8-5x}{2x\ 8y\ 4}$$

$$\left(\frac{dy}{dx}\right)_{4,0} = \frac{8-20}{8\ 4} = -1 = 45^\circ$$

$$\left(\frac{dy}{dx}\right)_{0,1} = \frac{8}{8\ 4} = \frac{2}{3} = 33.69^\circ$$

$$\tan\theta = \frac{0-1}{4-0} = 0.25 = 14.036^\circ$$

$$\therefore \alpha_2 = 33.69^\circ + 14.036^\circ$$

$$= 47.726^\circ$$

$$\therefore \alpha_1 = 45^\circ - 14.036^\circ = 30.964^\circ$$

$$\beta_1 = \beta_2 = 90^\circ - \frac{47.726\ 30.964}{2} = 50.655^\circ$$

$$\therefore \gamma = 180^\circ - (45^\circ\ 50.655^\circ) = 84.345^\circ$$

$\therefore$ Angle to be rotated $= 5.655^\circ$

Co-ordinate of A : $\frac{y}{x-4} = -1$ ..... eqn. of AC
[Fig 2.2]

& $\frac{y-1}{x} = \frac{2}{3}$ ..... eqn. of AB

solving: $x_A = 1.8$

$y_A = 2.2$

$\therefore$ Due to rotation along the curve maintaining the tangency the lateral (L.H.side) shift $= AC\cos 45^\circ - AC\cos(45^\circ - 5.655^\circ)$

$$= AC\left[\cos 45^\circ - \cos 39.345\right]$$

$$= AC \times -.0662357$$

$$\frac{y_A}{\sin 45^\circ} \times -.0662357 = -0.20607$$

$X_o = X_{optimum} = X_A - .20607 = 1.59393$

exact $X_A = 1.6$

Example 2

Minimization problem.

Min. $y = e^x - 5x$

at $x = -1$ $\qquad y = e^{-1} + 5 = 5.36788$

at $x = 3$ $\qquad y = e^3 - 15 = 5.08554$

$$\frac{dy}{dx} = e^x - 5$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1,5.3\ldots} = e^{-1} - 5 = -4.63212 = \tan 77.818^\circ$$

$$\& \left(\frac{dy}{dx}\right)_{x=3,5.08\ldots} = e^3 - 5 = 15.08554 = \tan 86.207^\circ$$

Ref. Fig. 2.3A, Slope of BC $= \dfrac{5.08554 - 5.36788}{3+1} = 0.070585$

$= \tan 4.0375^\circ$

$\alpha_1 = 86.207 + 4.038 = 90.245^\circ$

$\alpha_2 = 77.818 - 4.038 = 73.78^\circ$

$\because \alpha_1 > \alpha_2$

$\therefore$ The optimum will be on the same side of the inclination of the center line AD of cone.

$$\beta_1 = \beta_2 = \frac{180^\circ - (\alpha_1 + \alpha_2)}{2} = 90^\circ - 82.012 = 7.988$$

$$\phi_1 = 180^\circ - (\alpha_1 + \beta_1) = 180^\circ - 90.245 - 7.988 = 81.767^\circ$$

$\therefore$ Slope of AD $= \phi + 4.038 = 85.805$

$\tan 85.805^\circ = 13.6337$

Equation of AB : $y - 5.08554 \quad 15.08554 \quad x - 3$

Equation of AC : $y - 5.36788 \quad -4.63212 \quad x\ 1$

Solving $\quad X_A = 2.0746 \quad$ and $\quad y_A = -8.8746$

When AB is rotated and translated through $\frac{\gamma}{2}$, 'A' moves away from AF and 'G' moves toward AF.

$\therefore$ Horizontal shift $=$ AB Cos $86.207^\circ -$ AB Cos $(86.207^\circ - 2.0975^\circ)$

Now AB $= \sqrt{(3-2.0746)^2\ (5.0855\ \ 8.8746)^2} = 13.985$

Hence Horizontal shift required$=13.985\,[.066152-.10262]$

$=0.51$

$\therefore X_{opt} = 2.0746 - 0.51 = 1.5645$

and $Y_{opt} = -3.0422$

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CHAPTER III

LITERATURE SURVEY ON WAGE INCENTIVE PLANS

3.1 Introduction

Although there is no dearth of literature available on wage incentive plans, I have not found any literature that deals with the application of operations research on the formulation and solution of decision problems with regard to wage incentive. The probable reason may be that the decisions like the base level efficiency or the incentive rate etc. are, in most of the cases, settled between the management and the union accross the table. In section 3.2 the standard techniques of wage incentives are discussed briefly. Section 3.3 provides the general literature survey on different types of incentive plans.

Since each plan has to be tailored to suit a particular condition of each organization and it should be such as to satisgy other objectives of the organization like the employment condition, wage structure and quality of the product. There is a scope for application of standard optimization techniques for the optimum choice of the incentive plan.

3.2 Standard Techniques of Wage Incentive - Payment by Results.

Usually, payment by results, are classified in four main groups in accordance with whether worker's earnings vary 1) in the same proportion as output

2) proportionally less than output

3) proportionally more than output

4) in proportions which differ at different levels of output.

The most common system of payment is the straight piece-work system that comes under the category 1. It may be applied to individuals or to group of workers, the worker is paid at a specified rate per unit of output. Direct labor cost per unit of output remains constant when output increases above standard but the total unit costs decrease because fixed and semi-variable overhead unit costs decrease[1]. Variations in workers earnings and direct labor costs are shown in Fig. 3.1.

When it is difficult to set the job standards accurately, the worker usually shares with his employer the gains or losses that result due to change of output. All schemes under category 2 have this characteristic, i.e., they all possess less motivating reward than straight piece work system. Under the Halsey System, the worker is guaranteed a minimum wage even when his output falls below standard. But if the job is completed in less than standard time, the worker is paid at his time rate for the actual time taken and, in addition, receives a bonus payment at his time rate

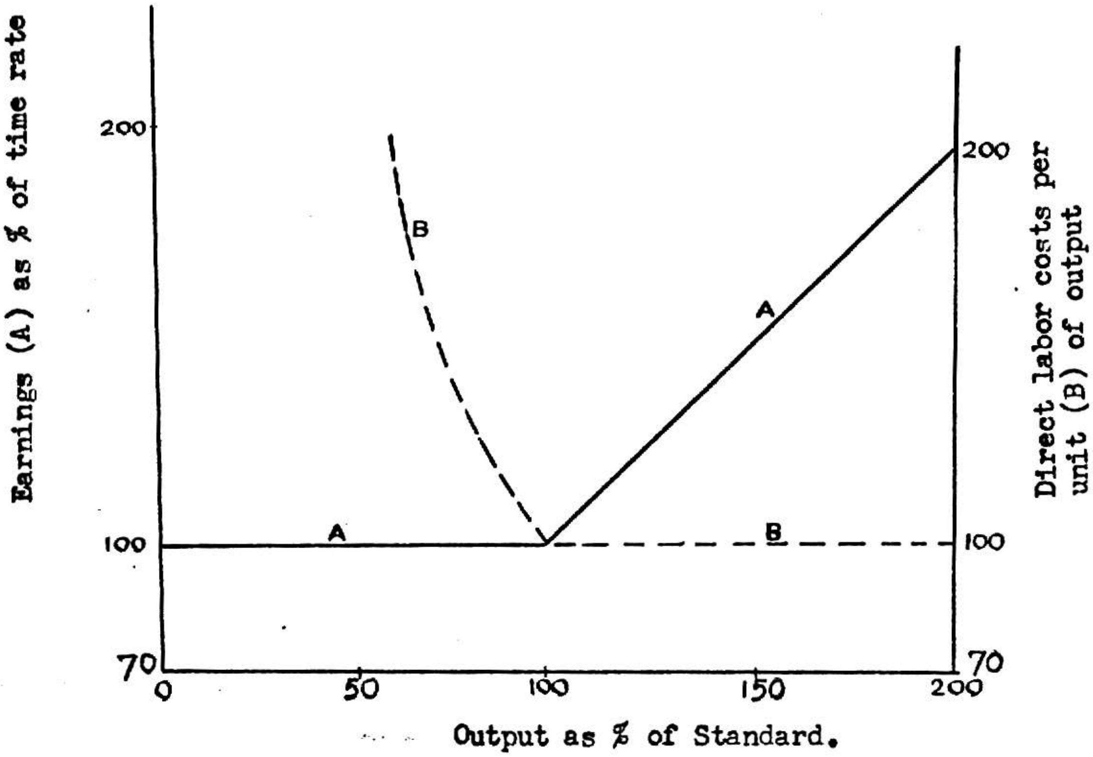


Figure 3.1 Straight piece-work system.

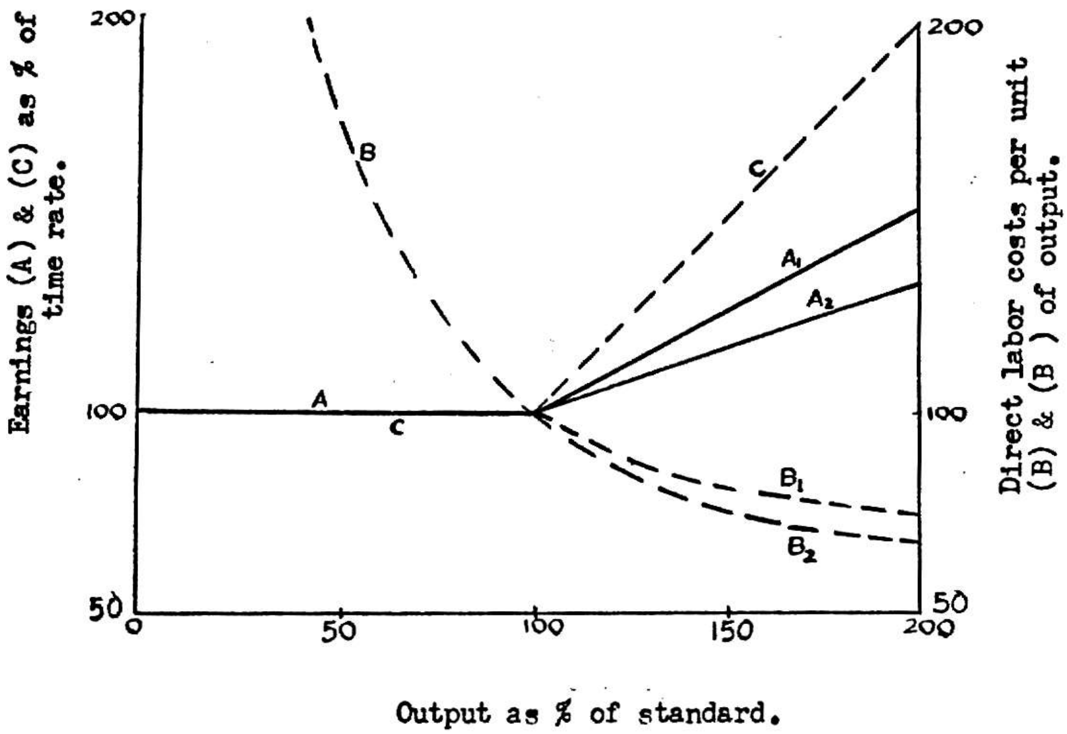
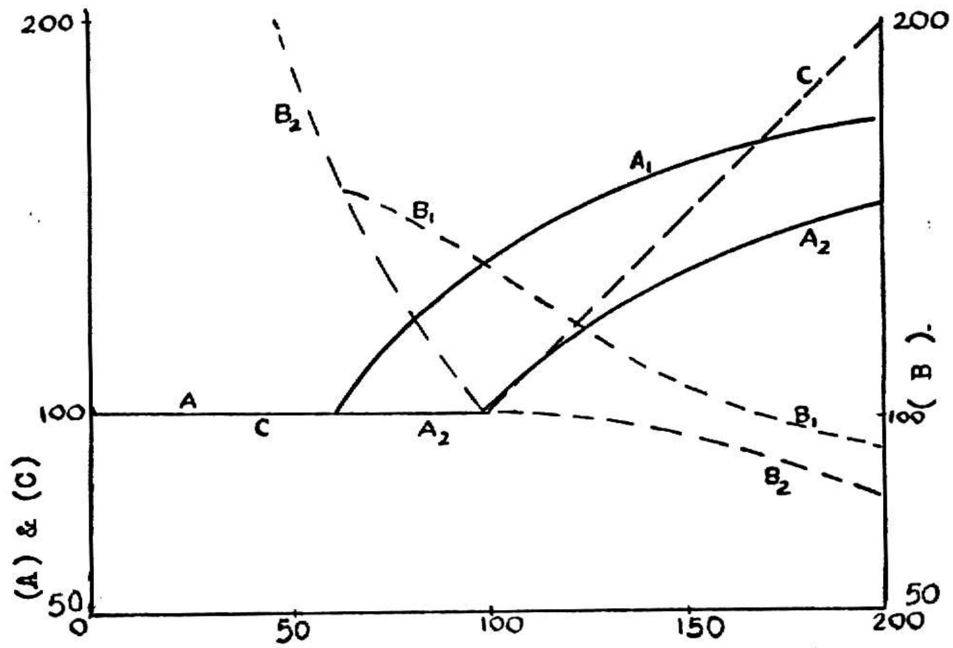


Figure 3.2 Halsey System.

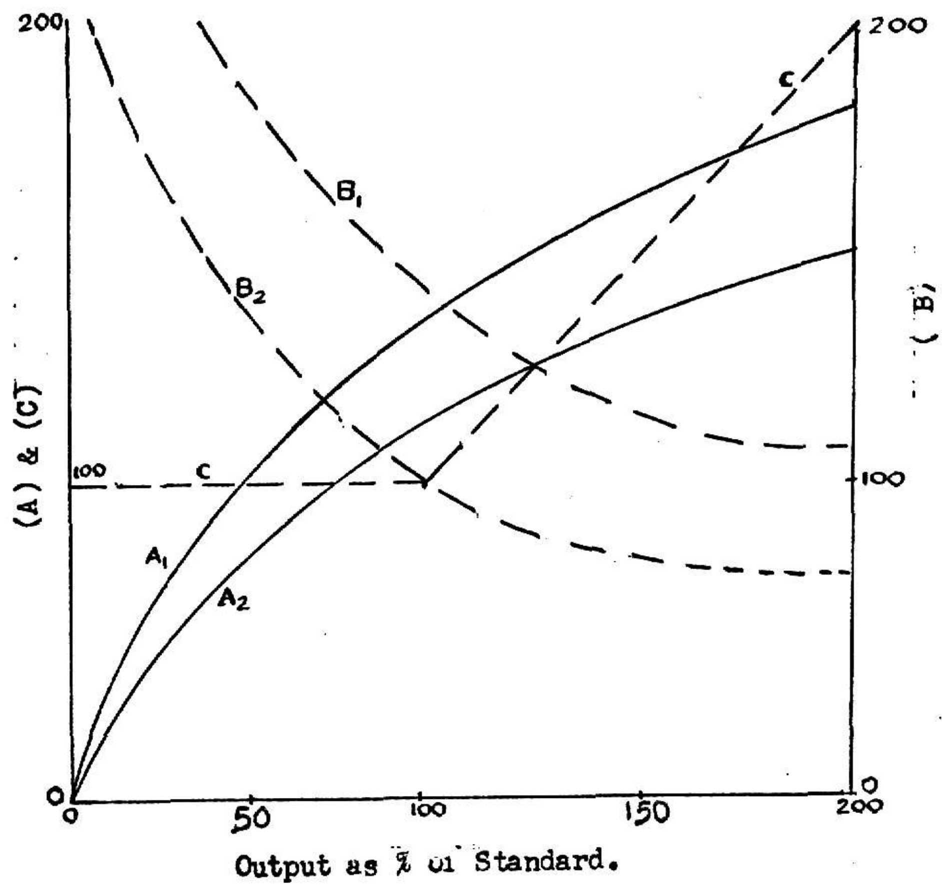


A Earnings (A_1 for low task and A_2 for standard task)

B Direct labor costs per unit of output (B_1 for low task and B_2 for standard task)

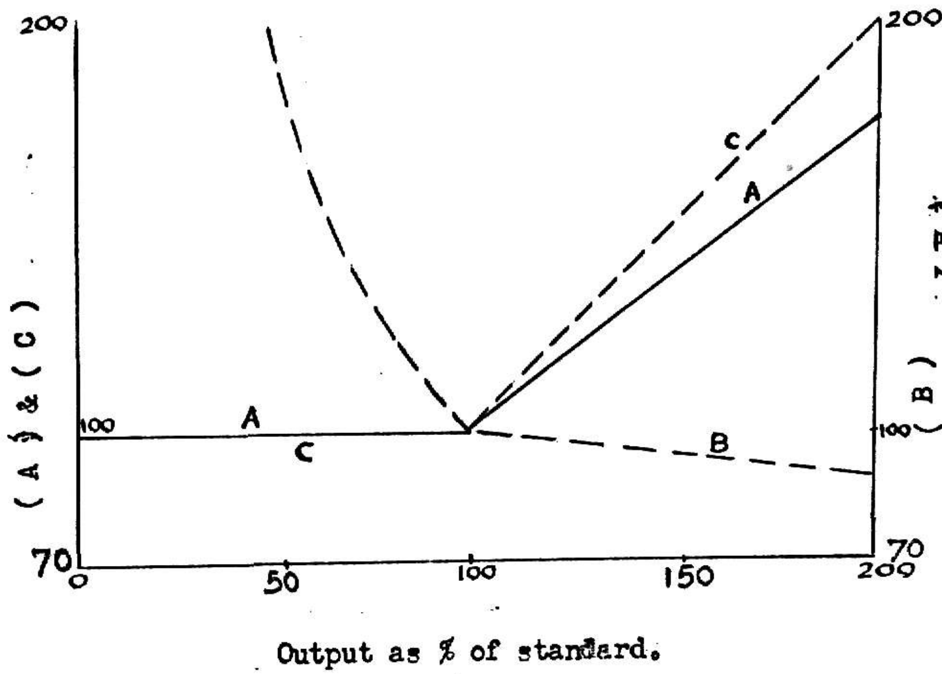
C Earnings on straight piece-work system with guaranteed time rate.

Figure 3.3 The Rowan System.



A Earnings (A_1 for low task and A_2 for standard task)
B Direct labor costs per unit of output
(B_1 for low tasks and B_2 for standard tasks)
C Earnings on straight piece-work with a
ranteed Time rate.

Figure 3.4 The Birth Variable Sharing System.

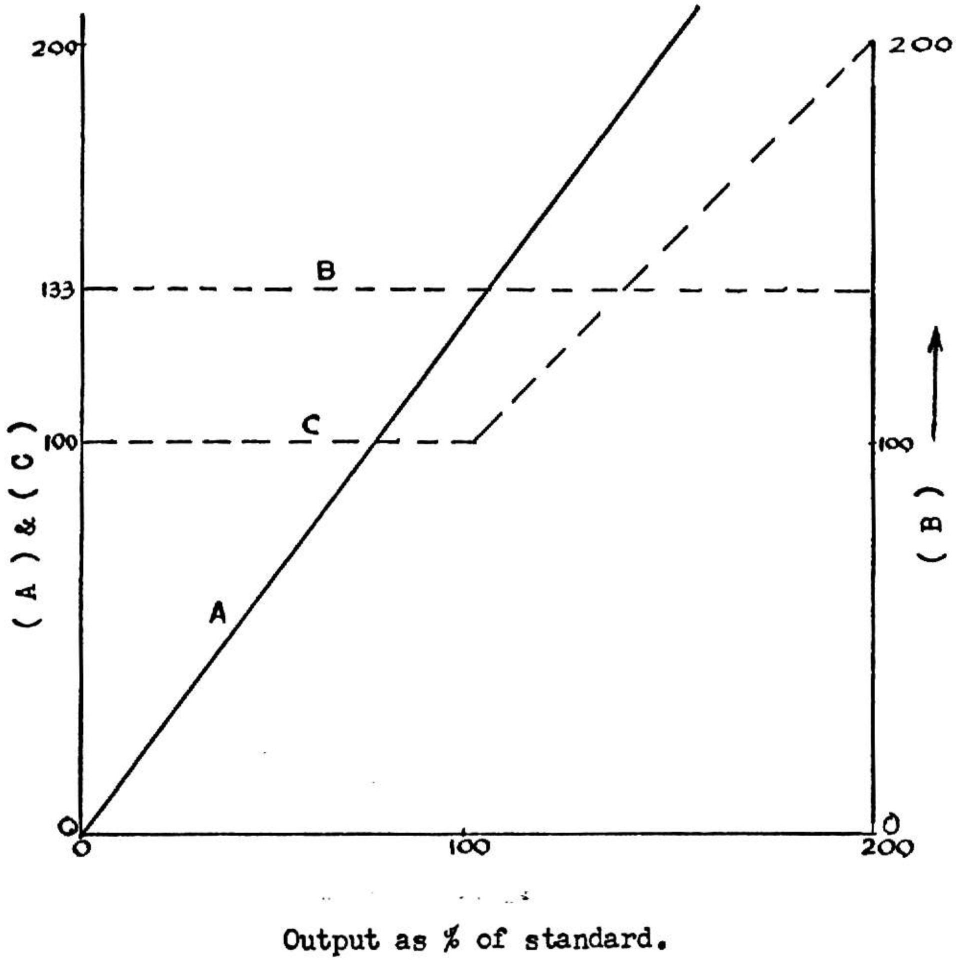


A Earnings as a % of standard.

B Direct labor costs per unit of output.

C Earnings on a straight piece-work system with a guaranteed time rate.

Figure 3.5 The Bedaux System.



A Earnings as % of time rate.

B Direct labor costs per unit of output.

C Earnings on a straight piece-work system with a guaranteed time rate .

Figure 3.6 High piece rate System.

for a specified percentage of time saved (usually varies from 30% to 70%). The variations in worker's earning and direct labor costs are shown in Fig. 3.2. In the Rowan System bonus is similarly paid for any time saved. The bonus takes the form of a percentage of the worker's time rate. This percentage is equal to the proportion which the time saved forms of standard time. The characteristics of the earnings and direct labor cost curves for the low task and standard task under this system is shown in Fig. 3.3. The Birth variable sharing system is similar to the Halsey and Rowan Systems but does not provide for a guaranteed time rate. The worker's pay is ascertained by multiplying the standard hour by the number of hours actually taken to do the job, taking the square root of the product and multiplying by the worker's hourly rate. The characteristics of the earnings and direct labor cost curves for low task and for standard task are shown in Fig. 3.4. Under the Bedaux system, each minute of allowed time is called a point, thus making in all 480 points in an 8-hour day. A standard number of points is specified for the completion of each job. The worker receives, in addition to his hourly or daily rate, a bonus which is, under the original Bedaux system, equal to 75% of the number of points earned in excess of 60 per hour multiplied by one sixtieth of the worker's hourly rate. Fig. 3.5 shows the variations in earnings and direct labor costs under this system.

In category 3, the high piece-rate system provides the worker's earnings in proportion to output as under straight piece-work but the increment in earnings for each increase in output is greater. The characteristics of this system are shown in Figure 3.6.

A great many varieties of systems under category 4 have been developed. The most important ones are a) the Taylor Differential Piece-Rate System, b) the Merrick Differential Piece-Rate System, c) the Gantt Task System, and d) the Emerson Empiric or Efficiency System.

In all these systems earnings vary from minimum to maximum at different levels of output. Earnings for part of the range may vary proportionally less than output and for another part proportionally more, or more usually in the same proportion as output.

3.3 Review on Different Types of Plans

Increased labor productivity is the fundamental requirement for an increased material standard of living. Holt[2] showed a simple mathematical model that there exists a definite relationship between overall efficiency and labor productivity. Other input factors held constant, efficiency rises with the increase in labor productivity. By this model, it is also possible to calculate the amount of investment to be made for the replacement of equipment when the rise in labor productivity is known.

The basis for the incentive scheme for the restricted work (i.e. restricted by the process or the machine performance) should be quite different from that for the unrestricted work. Schieb[3] pointed out that variation in the performance time is precluded by the nature of the operation. He suggested five approaches that should be followed by the Industrial Engineer for the design of incentive schemes in such situations.

Seidel[4] demonstrated a simple technique how much the increase in labor wage incentive can be paid in the next year if the sales, labor force requirement and other cost data are known for the current year and next year. Using this method decisions relating to the incentive rate or increase in labor wages a replacement of equipment can be taken very easily and effectively.

Like Seheib, the disadvantages associated with the straight standard hour system as a basis of incentive plan were also shown by Halty[5] who developed a new system that gives us a mathematical equation to calculate the earning index, taking into account a variable machine incentive allowance.

O'Connor[6] stresses on the unique position of standard time as the most important part of the incentive plan. He explained the merits and demerits of straight piece work and geared linear plans for incentive plan. When there exists some doubt about the accuracy of the time standard, his recommendation was to adopt his curvilinear type of incentive plan.

Usually labor productivity varies with respect to time in a particular organization. Nassi[7] showed how these indices with respect to time which are known as 'index of Laspeyres' and 'index of Paasche' can be calculated. He has also shown how to measure the performance index of Method study and standard department in terms of work saved per unit of time.

Incentives also can be applied for quality improvement. This was shown by Mehra, et al,[8] by linking the scheme with the acceptance sampling incentive plan. The wage, inclusive of incentive would be computed using game theory approach.

With this same objective, Nandi and Nair[9] presented a quality incentive plan for an operator which was designed based on cost equations of the sampling plan and management policy without increasing the total cost per lot.

Success of the incentive scheme depends on the consistency of time data among some other factors. Groff[10] pointed out that the standard output rate obtained by time study is not always optimal since best output rate for standard is simply influenced by the incentive plan for which the data is intended. He presented an incentive plan considering the efefct of selected output response patterns and cost structures on optimal standard level.

Expensive downtime, at times poses a problem to the management, particularly, in line paced operation. James[11] showed how to alleviate this problem by introducing incentive in the system.

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CHAPTER IV

APPLICATION OF OPERATIONS RESEARCH TECHNIQUES TO FORMULATE AND SOLVE AN INCENTIVE PROBLEM -- A CASE STUDY

4.1 Introduction

This chapter is primarily concerned with the application of some optimization techniques to solve some decision problems regarding wage incentive scheme. This is essentially a case study. In this section the management's problem and policy have been discussed. In section 4.2 and 4.3, the formulation and solution of the problems are provided. In this case study, a situation in a light engineering concern has been considered wherein the management is currently scheduling overtime hours to meet its production schedule. It wants to put a stop to giving overtime and get the same or more production without overtime through the installation of an incentive plan that will eventually improve operator's efficiency, increase machine utilization accompanied by less power consumption.

Management does not want the worker's weekly paycheck to be affected. By having the same output during normal working hours, it hopes to reduce the overhead expenses associated with having the firm work longer hours.

In this particular case, it is proposed that for twelve departments and for two groups of workers in each department, namely skilled and unskilled owrker, group incentive plan is suitable.

4.2 Problem Formulation

The following nomenclatures were used for the formulation of the problem:

N_k = Total number of workers in Group k in the department i.

J_i = Total number of operations done in department i.

x_1 = Proposed base level efficiency in % from which incentive should start.

C_{ki} = Current efficiency in % of Group k in the department i.

U_{kij} = Total number of units produced by Group k for jth operation in the department after the implimentation of incentive scheme.

H_{ki} = Input labor hours by kth group of workers in the department i.

t_{kij} = Standard time in hours per unit for jth operation done in the department i by kth group of worker.

x_{k+1} = Incentive rate per point rise in efficiency per hour (i.e. $/%/hr.) for kth group of workers.

d_k = Average overtime in $ paid per hour to Group k.

It is also desired that x_1, the proposed base level efficiency should be same for all groups of worker.

So the desired objective function is to :

Minimize $$\sum_k \sum_i N_{ki}(x_1 - C_{ki})^2$$

Constraints:

(1) Total incentive to be paid must not exceed the total overtime payment, i.e.

$$\sum_i \left[\sum_{j=1}^{J_i} \frac{U_{kij}\ t_{kij}}{H_{ki}} - x_1 \right] x_{k+1} \leqslant d_k \qquad \text{for } k = 1,2$$

Essentially $U_{kij}t_{kij}/H_{ki}$ gives the new efficiencies of two groups ($k = 1$ and 2). If they are defined as C_{1i} and C_{2i} then $(C_{1i} - x_1)$ and $(C_{2i} - x_1)$ are the total rise in efficiencies by two groups of workers for ith department after the implementation of the scheme.

(2) Again to have good motivation, the incentive rate should not be less than the 'per hour wage' evaluated on per point basis at the optimum base level efficiency, i.e.

$$x_{k+1} \geqslant \frac{w_k}{x_1}$$ where w_k is the average wage rate for kth group of workers.

This is quite clear from the relationship (line BC) shown in Figure 4.1.

The number of workmen for the two groups and for twelve departments are shown in Appendix 3. After time study, the current performance index (P.I.) i.e. C_{ik_1} and C_{ik_2} of the

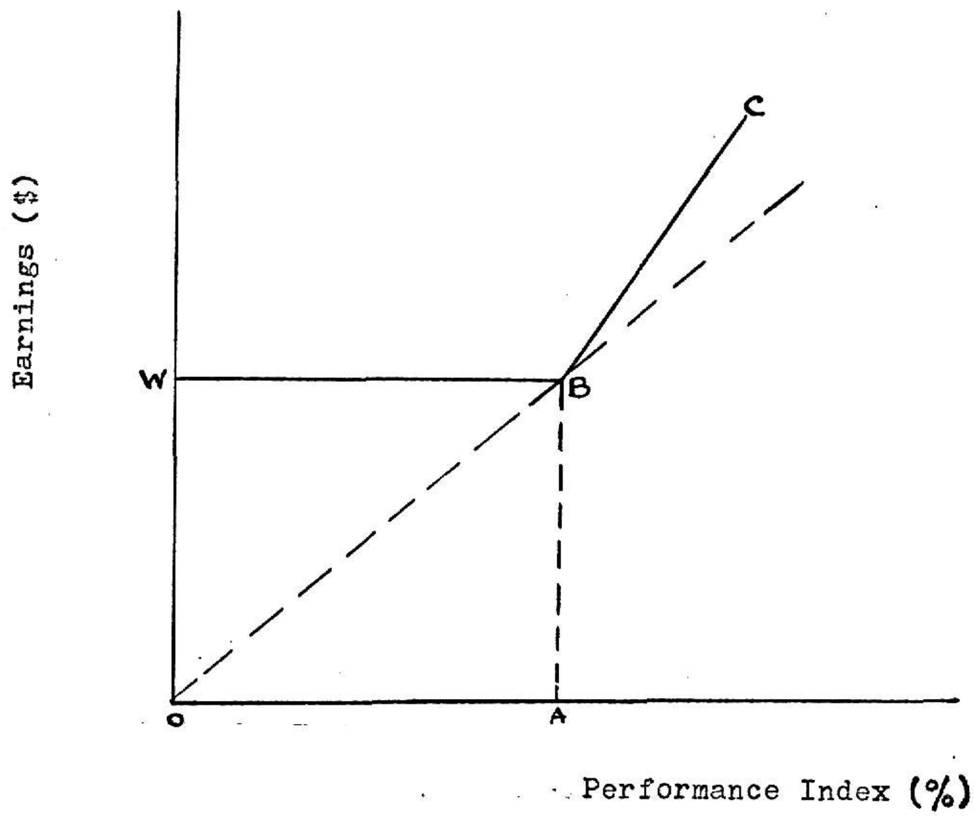


Fig. 4.1 Relationship between the earnings and performance index under the proposed incentive scheme.

Note: Guaranteed minimum wage is W even when the output falls below the base level efficiency A. The slope of BC (i.e. the incentive rate) is greater than the slope of OB. This provides greater motivation. Constraint 2 is essentially derived from this condition.

two groups for each department are evaluated and provided in the same Appendix 3. It also has been found that the management gets the same output if the two groups of workers work at 105% and 100% P.I. during normal working hours. The wage rate for skilled and unskilled groups of workers are assumed as \$4 and \$3 per hour respectively. From the past records in the account section, the overtime earning per hour per worker for the two groups are found to be \$2 and \$1.50 respectively.

So using those data the problem is rewritten as:

$$
\begin{aligned}
Z = {} & 10(x_1-78)^2 + 20(x_1-68)^2 + 18(x_1-82)^2 + 28(x_1-85)^2 + 5(x_1-83)^2 \\
& + 5(x_1-69)^2 + 40(x_1-95)^2 + 5(x_1-75)^2 + 80(x_1-80)^2 + 10(x_1-96)^2 \\
& + 20(x_1-91)^2 + 10(x_1-65)^2 + 30(x_1-71)^2 + 20(x_1-66)^2 + 40(x_1-76)^2 \\
& + 32(x_1-75)^2 + 10(x_1-78)^2 + 10(x_1-65)^2 + 60(x_1-89)^2 + 10(x_1-69)^2 \\
& + 60(x_1-72)^2 + 5(x_1-90)^2 + 10(x_1-85)^2 + 5(x_1-62)^2 \quad \ldots \text{ eqn.}(4.1)
\end{aligned}
$$

S.T. $(105-x_1)x_2 \leqslant 2$ . . . . eqn. (4.2)

$(100-x_1)x_3 \leqslant 1.5$ . . . eqn. (4.3)

$x_2 \geqslant \frac{4}{x_1}$ . . . . eqn. (4.4)

$x_3 \geqslant \frac{3}{x_1}$ . . . . eqn. (4.5)

$x_1 = C_p$ = Optimum base level efficiency in %

x_2 = Incentive rate per point rise per hour for skilled group (\$/%/hr.)

x_3 = Incentive rate per point rise per hour for unskilled group. (\$/%/hr.)

4.3 Generalized Reduced Gradient Formulation

To solve the above problem by the GRG method, the objective functions and the constraints may be represented by:

Maximize $f_o(\bar{x})$

Subject to the constraints

$$f(\bar{x}) = 0$$

$$\bar{a} \leqslant \bar{x} \leqslant \bar{b}$$

Any inequality constraints can be converted into equality constraints using the standard procedure of adding slack variables and changing the sign, if necessary.

The basic underlying principle of this technique is to change the constrained optimization problem into an unconstrained one. This is done by dividing the solution vector components into two groups, independent $(\bar{x})$ and dependent $(\bar{y})$. The dependent variables denoted by the vector $\bar{y}$ are solved in terms of independent vector $\bar{x}$, through the constrain functions.

Therefore on this basis the constraints may be rewritten as:

$$\bar{f}(\bar{x}) = \bar{F}(\bar{x}, \bar{y}) = 0$$

Solving $\bar{y} = \phi(\bar{x})$

The objective function also is rewritten in terms of $\bar{x}$ and $\bar{y}$ and substituting the value of $\bar{y}$ in that one gets

$$f_o(\bar{x}) = f_o(\bar{x}, \bar{y}) = f_o(\bar{x}, \phi(\bar{x})) = F(\bar{x})$$

Hence the problem is to maximize

$$F(\bar{x})$$

Subject to $\bar{a} \leqslant \bar{x} \leqslant \bar{b}$

Since $F(\bar{x}) = f_o(\bar{x}, \bar{y})$

$\therefore$ The reduced gradient can be evaluated as:

$$\frac{\partial F}{\partial \bar{x}} = \frac{\partial f_o}{\partial \bar{x}} + \frac{\partial f_o}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \bar{x}}$$

$\frac{\partial \bar{y}}{\partial \bar{x}}$ is determined indirectly from the constraints.

$$\because \bar{f}(\bar{x}) = \bar{f}(\bar{x}, \bar{y}) = 0$$

$$\therefore \frac{\partial \bar{f}}{\partial \bar{x}} + \frac{\partial \bar{f}}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \bar{x}} = 0$$

or $$\frac{\partial \bar{y}}{\partial \bar{x}} = -\left[\frac{\partial \bar{f}}{\partial \bar{y}}\right]^{-1} \left[\frac{\partial \bar{f}}{\partial \bar{x}}\right]$$

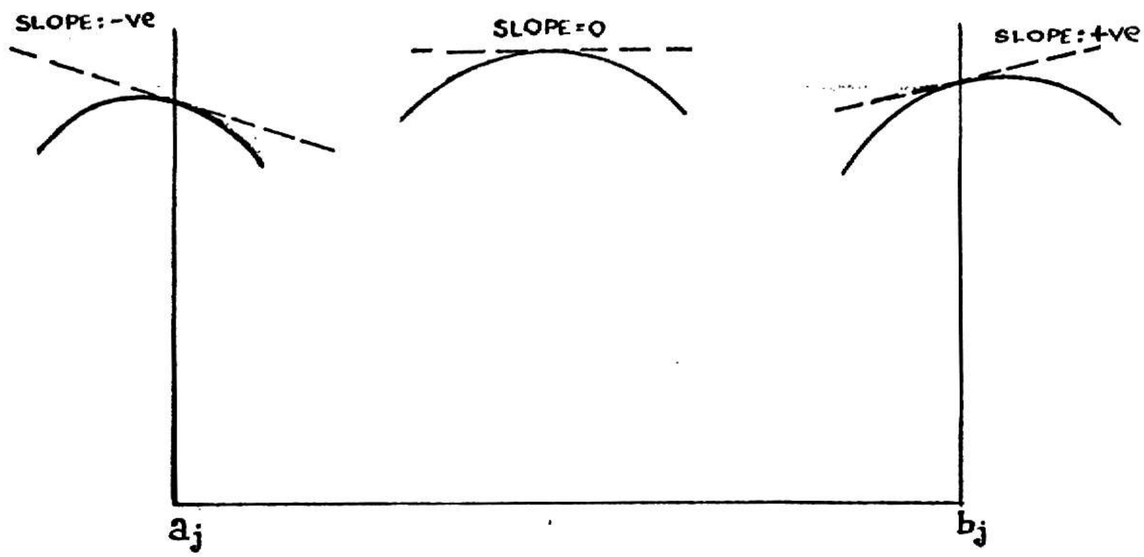
$$\therefore \bar{g}^T = \frac{\partial F}{\partial \bar{x}} = \frac{\partial f_o}{\partial \bar{x}} - \frac{\partial f_o}{\partial \bar{y}} \left[\frac{\partial \bar{f}}{\partial \bar{y}}\right]^{-1} \left[\frac{\partial \bar{f}}{\partial \bar{x}}\right]$$

The conditions that determine an optimum solution, $\bar{x}^*$ are as given below (for all j)

$$\frac{\partial F}{\partial x_j^*} = 0 \quad \text{if} \quad a_j < x_j^* < b_j$$

$$\frac{\partial F}{\partial x_j^*} \leqslant 0 \quad \text{if} \quad x_j^* = a_j$$

$$\frac{\partial F}{\partial x_j^*} \geqslant 0 \quad \text{if} \quad x_j^* = b_j$$



Slope: -ve satisfies condition $\frac{\partial F}{\partial x_j{}^*} = 0$ if $x_j{}^* = a_j$

Slope =0 satisfies condition $\frac{\partial F}{\partial x_j{}^*} = 0$ if $a_j < x_j{}^* < b_j$

Slope: +ve satisfies condition $\frac{\partial F}{\partial x_j{}^*} = 0$ if $x_j{}^* = b_j$

Fig. 4.2 Graphical representation of the optimum conditions used in GRG technique.

These conditions are graphically represented in Fig. 4.2.

The underlying assumptions for this alogrithim are that for a given iteration

1) There exists a set of dependent variables contained within the boundary conditions

2) The Jacobian $\frac{\partial \bar{f}}{\partial \bar{y}}$ is non-singular.

Using the above information, the basic GRG algorithims can be stated in five steps which are provided in the flow chart (Appendix 4).

Theoretically, the stopping condition is when the projected reduced gradient $P_i^{\,o} = 0$, $i = j, \ldots .$ N-M, where N is the number of variables in the original objective function and M is the number of constraints, N-M being the reduced dimension.

In practice, the following three stopping criteria are employed.

1) $\| \bar{P}^o \| = \sqrt{\sum_{i=1}^{N-M} (P_i^o)^2} < \epsilon_1$

2) $P_i^o < \epsilon_2 \qquad i = 1, 2, \cdots \cdot (N-M)$

3) $| f_o(\bar{x}^1) - f_o(\bar{x})^o | \leqslant \epsilon_3$

TABLE 4.1

RESULTS OBTAINED BY GRG METHOD USING THREE DIFFERENT STARTING POINTS

Run No.	Starting Values	Solution	Function Value	Norm of Reduced Gradient	$\Delta F1$	ETA	No. of Iterns.
1	$x_1=85\%$ $x_2=.03$ $x_3=.03$	Job Abandoned					8
2	$x_1=60\%$ $x_2=.05$ $x_3=.05$	$x_1=79.27$ $x_2=.078$ $x_3=.05$	$0.4276X10^5$	1.5	10	44	23
3	$x_1=75\%$ $x_2=.04$ $x_3=.04$	$x_1=79.27$ $x_2=.051$ $x_3=.04$	$0.4276X10^5$	0.0	0.0	0.0	7

Note: 1) In Run # 2, the termination occured since same function values are obtained in the last two iterations before it meets the other stopping criteria.

2) $\Delta F1 = \sum |P_i(a_i - x_i)|$ when $P_i < 0$

$= \sum |P_i(b_i - x_i)|$ when $P_i > 0$

ETA = Max. $|P_i(a_i - x_i)|$ for $P_i < 0$

or Max. $|P_i(b_i - x_i)|$ for $P_i > 0$

Where P_i is the gradient of the function with respect to the variable.

4.4 Solution of the Incentive Problem by G.R.G. Method

The incentive problem as formulated in section 4.2 was solved by GRG method using the GREG program which was developed by Abadie and his associates of Electricite de France. The program was run thrice using three different starting values (Table 4.1). The number of iterations required, the optimum value of the variables and the value of the objective function etc. are given in Table 4.1. Appendices 5-7 are the computer printout for the three runs which provide the other informations like the stopping criteris etc.

The variable x_3, i.e. the incentive rate for unskilled group, assumes the same optimal value as the starting value in both the feasible runs although the function values are same. Hence it may be concluded that the objective function is very flat near the optimum.

4.5 Separable Programming

Separable programming is a special case of non-linear programming. When the objective function and the constraints are constructed or can be constructed of separable functions, this method can be used effectively. The basic principle is to approximate the non-linear function to piecewise linear functions and thereby changing the problem into a restricted linear programming problem.

Thus the incentive problem given by eqns. 4.1 to 4.5 can be changed to seperable programming problem which may be defined as:

$$C(\bar{x}) = \sum_{i=1}^{m} f_i(x_i)$$

Subject to constraints:

$$\sum_{i=1}^{m} g_{ki}(x_i) \leqslant b_k \qquad k = 1, 2, \ldots, p$$

$$x_i \geqslant 0 \qquad i = 1, 2, \ldots, m$$

Partitioning each variable x_i into n_i divisions and approximating the functions $f_i(x_i)$ and $g_{ki}(x_i)$ one can write as:

$$x_i = x_i^{o} + \sum_{j=1}^{n_i} \Delta x_i^{j} D_i^{j}$$

$$f_i(x_i) \cong f_i(x_i^{o}) + \sum_{j=1}^{n_i} f_i^{j} D_i^{j}$$

$$g_{ki}(x_i) \cong g_{ki}(x_i^{o}) + \sum_{j=1}^{n_i} g_{ki}^{j} D_i^{j}$$

$k = 1, 2, \ldots p$, and $i = 1, 2, \ldots m$

x_i^{o} = lower boundary of variable x_i. $i = 1, 2, \ldots m$

x_i^{o} may or may not be equal to zero

$f_i(x_i)$ and $g_{ki}(x_i)$ are the corresponding values of the objective function and constraints at x_i^{o}. These values may or may not be equal to zero.

$D_i^{\,j}$ represents a variable created for the jth partition of variable x_i.

Thus the original problem can be written as:

Optimize (max. or min.)

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n_i} f_i^{\,j} D_i^{\,j} + \sum_{=1}^{m} f_i(x_i^{o})$$

Subject to: $$\sum_{i=1}^{m} \sum_{j=1}^{n_i} g_{ki}^{j} D_i^{\,j} \quad b_k - \sum_{L=i}^{m} g_{ki}(x_i^{o}),$$

$$k = 1, 2, \ldots p$$

Grid equation: $$x_i - \sum_{j=1}^{n_i} \Delta x_i^{\,j} D_i^{\,j} = x_i^{o}, \quad L = 1, 2, \ldots, m$$

$$0 \leqslant D_i^{\,j} \leqslant 1, \quad i = 1, 2, \ldots m, \; j = 1, 2, \ldots n_i$$

$$x_i \geqslant 0, \qquad i = 1, 2, \ldots m$$

4.6 Solution of the Incentive Problem by Separable Method

The incentive problem as formulated in section 4.2 and defined by the eqns.4.1 to 4.5 can be seperated as follows:

The objective function on expansion yields

$$\text{Min} = 543\ x_1^2 - 86082\ x_1 + 3454417 \ldots \quad \text{eqn.4.6}$$

The constraints are separated according to the principle of separable programming by taking logarithims on both sides.

Thus the constraints are:

$$\log (105 - x_i) + \log x_2 \leqslant \log 2 \quad . . . \text{eqn. } 4.7$$

$$\log (100 - x_i) + \log x_3 \leqslant \log 1.5 \quad . . . \text{eqn. } 4.8$$

$$\log x_2 \quad \log x_1 \geqslant \log 4 \quad . . . . \text{eqn. } 4.9$$

$$\log x_3 + \log x_1 \geqslant \log 3 \quad . . . . \text{eqn. } 4.10$$

The starting value of the variables x_1, x_2 and x_3 are 75%, \$.06, and \$.04 respectively. The upperbounds are 90%, \$.12 and \$.09 and the number of partitions required for linearisation are 20, 10 and 10 respectively.

The linearized form of the non-linear components of the objective function and of constraints are furnished in Appendix 8.

Since $f(x_i)$ and $g_{ki}(x_i)$ where x_i is the starting value of the ith variable and f and g stand for objective function and constraint respectively, are not zero so the right hand side of constraints and also the d_{ij} function are to be adjusted.

Thus the original incentive problem is represented as:

Maximize:

$$F = (-z) = \sum_{j=1}^{20} \Delta f (D_1)^j - 543 (x_1^\circ)^2 + 86082\, x_1 - 3454417$$

S.T.

$$\sum_{j=1}^{20} \Delta g_1^j D_1^j + \sum_{j=1}^{10} \Delta g_1 D_2^j \leqslant \log 2 - g_1(x_1^\circ, x_2^\circ)$$

$$\sum_{j=1}^{20} \Delta g_2^j D_1^j + \sum_{j=1}^{10} \Delta g_2 D_3^j \leqslant \log 1.5 - g_2(x_1^\circ, x_3^\circ)$$

$$\sum_{j=1}^{10} \Delta g_3^j D_2^j + \sum_{j=1}^{20} \Delta g_3 D_1^j \geqslant \log 4 - g_3(x_1^\circ, x_2^\circ)$$

$$\sum_{j=1}^{10} \Delta g_4^j D_3^j + \sum_{j=1}^{20} \Delta g_4^j D_1^j \geqslant \log 3 - g_4(x_1^\circ, x_3^\circ)$$

Grid equations:

$$x_1 - \sum_{j=1}^{20} \Delta x_1^j D_1^j = 75$$

$$x_2 - \sum_{j=1}^{10} \Delta x_2^j D_2^j = 0.06$$

$$x_3 - \sum_{j=1}^{10} \Delta x_3^j D_3^j = 0.04$$

$$x_1, x_2, x_3 \geqslant 0, \quad 0 \leqslant D_i^j \leqslant 1 \qquad \begin{array}{l} i = 1,2,3 \ \& \\ j = 1,2,3 \cdots\cdots 20 \end{array}$$

The linear equations, the grid equations are given in details in Appendix 8. The problem then eventually was solved by linear programming using MPS/360 programm , the results are given at the end of Appendix 8.

The results are summarized in Table 4.2 and the value of the objective function also was given in the same table after manipulating the constant terms using eqn. 4.6.

TABLE 4.2

Results Obtained Using Separable Programming

Variable	Lower bound	Upper bound	Starting Value	Number of partitions	Number of iterations for Linear programming solution	Solution	
						Local	Global
x_1	75%	90%	75%	20	9	79.5	79.5
x_2	.06	.12	.06	10		.06	.078
x_3	.04	.09	.04	10		.04	.074

$$\text{value of obj. function} = -\left[6466000 - 3454417 - 543(75)^2\right]$$

$$= -\left[-42792\right] \quad = .42792 \times 10^5$$

4.7 Conclusion

The GRG mehtod appears to be a very powerful tool for handling optimization of non-linear objective function subjected to non-linear constraints. As can be seen from the results (Table 4.1) the convergence rate is quite fast and only a very small amount of computer time and computer memory are needed to solve the problem.

Separable programming is also a powerful non-linear programming technique since it will yield, as with any other non-linear technique, at least a local optimum solution, if it exists. It's only pitfall is precision, but this is of little consequence since most engineering problems need only a good approximation.

As contrast to the GRG technique, separable programming requires more manipulation since a large number of new variables have to be introduced and it can solve only certain non-linear programming problems.

As far as our given incentive problem is concerned, separable programming and GRG yield nearly the (Table 4.3) same objective function value and the base efficiency but the incentive rates obtained by seperable programming are higher than those by GRG method. Since higher motivation will be generated by higher values of incentive rates, so results of separable programming may be recommended with a insignificant change in the value of objective function.

Table 4.3 Comparison of the results obtained by G.R.G. method and Seperable Programming

Method	Starting values	No of iterations	Optimum solns.	Optimum func. values
G. R.G. Method	x_1=75% x_2=.04 (\$) x_3=.04 (\$)	7	x_1=79.27% x_2=.05 (\$) x_3=.04 (\$)	$.4276 \times 10^5$
Seperable programming	x_1=75% x_2=.06 (\$) x_3=.04 (\$)	9	x_1=79.5% x_2=.078 (\$) x_3=.074 (\$)	$.4279 \times 10^5$

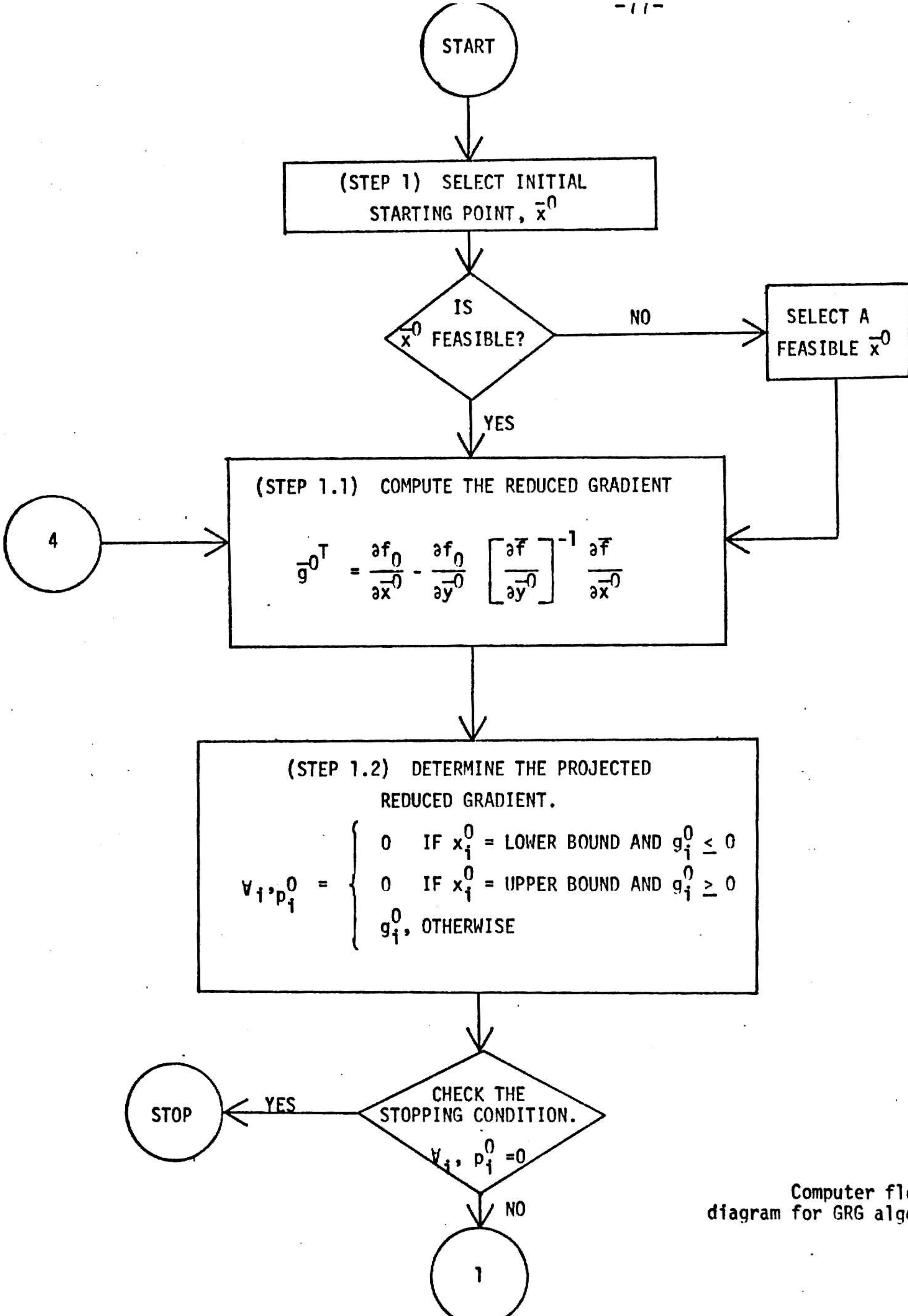
APPENDIX 2

CURRENT MANPOWER AND PERFORMANCE INDEX OF TWO GROUPS OF WORKERS FOR VARIOUS DEPARTMENTS

	SKILLED		UNSKILLED	
Dept.	No. of men	Efficiency (%)	No. of men	Efficiency (%)
Power Press	10	78	30	71
Auto Screw Cutting	20	68	20	66
Drilling	18	82	40	76
Milling	28	85	32	75
Plating	5	83	10	78
Painting	5	69	10	65
Sub. Assem.	40	95	60	89
Spring Mfg.	5	75	10	69
Assembly	80	80	60	72
Salvage	10	96	5	90
Grinding	20	91	10	85
Heat Treatment	10	65	5	62
Total	251		292	

A P P E N D I X : 3

COMPUTER FLOW CHART FOR G.R.G METHOD



Computer flow diagram for GRG algorithm

1

(STEP 1.3) COMPUTE THE DIRECTION OF MOVEMENT, $\bar{h}^0$, FOR $\bar{x}^0$

A SIMPLE EXAMPLE IS $\bar{h}^0 = \bar{p}^0$

(STEP 2) COMPUTE THE DIRECTION OF MOVEMENT $\bar{k}^0$, FOR $\bar{y}^0$

(STEP 2.1)

$$\bar{k}^0 = - \left[\frac{\partial \bar{f}}{\partial \bar{y}^0}\right]^{-1} \left[\frac{\partial \bar{f}}{\partial \bar{x}^0}\right] \bar{h}^0$$

(STEP 2.2) USE A ONE-DIMENSIONAL SEARCH TO $\max_{\Theta} f_0(\bar{x}^0 + \Theta\bar{h}^0, \bar{y}^0 + \Theta\bar{k}^0)$

3

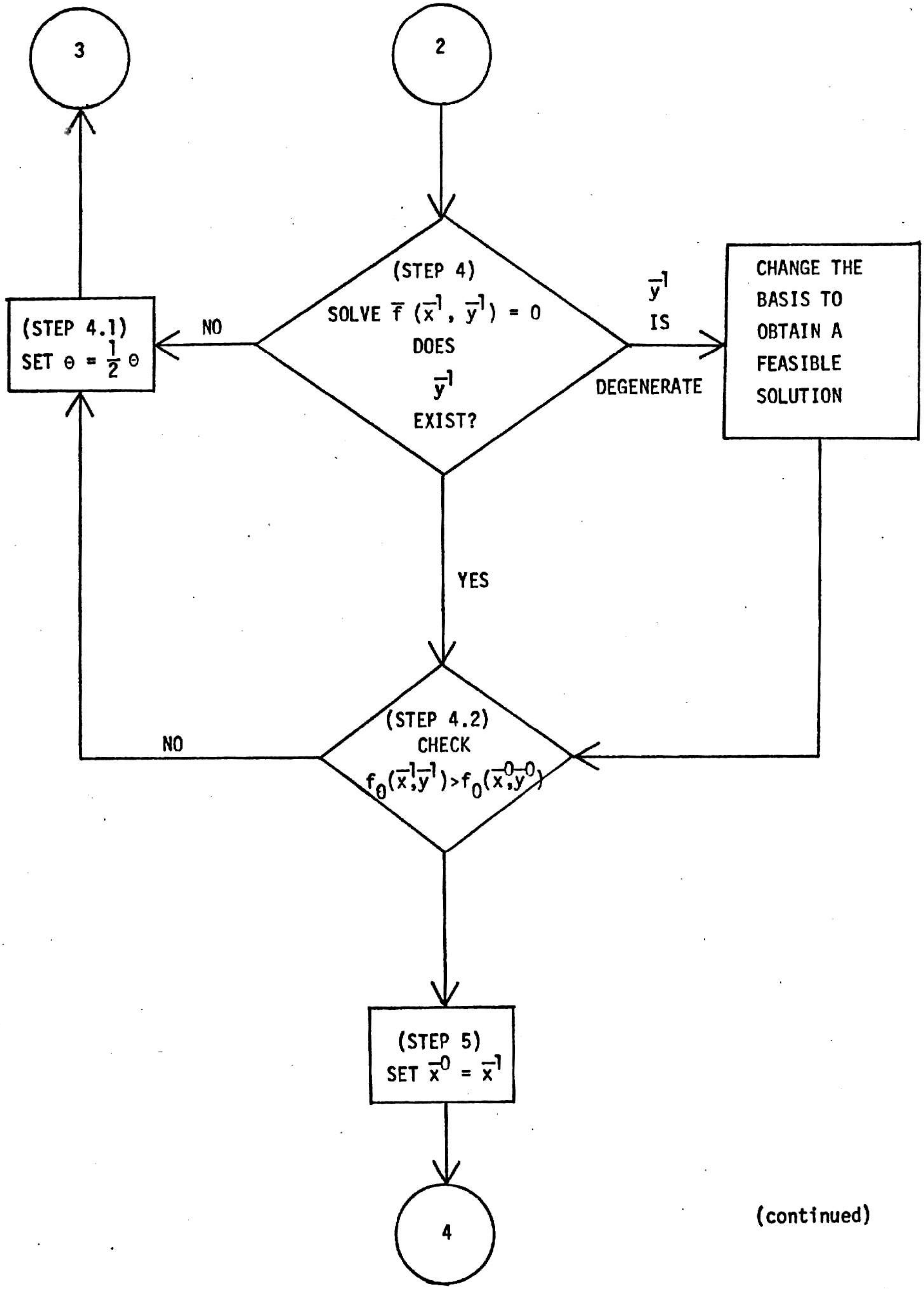
(STEP 3) CALCULATE $\tilde{x}^1 = \bar{x}^0 + \Theta\bar{h}^0$, $\bar{y}^1 = \bar{y} + \Theta\bar{k}^0$

PROJECT $\tilde{x}^1$ INTO P.

$$\forall_j, \quad x^1{}_j = \begin{cases} \text{UPPER BOUND IF } x_j^0 + \Theta h_j^0 \geq \text{UPPER BOUND} \\ \text{LOWER BOUND IF } \bar{x}_j^0 + \Theta h_j^0 \leq \text{LOWER BOUND} \\ x_j^0 + h_j^0 \text{ OTHERWISE} \end{cases}$$

2

(continued)


3
2
(STEP 4)
SOLVE $\bar{f}(\bar{x}^1, \bar{y}^1) = 0$
DOES
$\bar{y}^1$
EXIST?
NO
(STEP 4.1)
SET $\Theta = \frac{1}{2}\Theta$
$\bar{y}^1$
IS
DEGENERATE
CHANGE THE
BASIS TO
OBTAIN A
FEASIBLE
SOLUTION
YES
(STEP 4.2)
CHECK
$f_0(\bar{x}^1,\bar{y}^1) > f_0(\bar{x}^0,\bar{y}^0)$
NO
(STEP 5)
SET $\bar{x}^0 = \bar{x}^1$
4

(continued)

A P P E N D I X : 4

COMPUTER PRINTOUT OF THE RESULTS OBTAINED

BY G.R.G. METHOD

PARAMETRES .

NV	3
NIN	4
NEG	0
NEVL	20
NFC	6
ITET	20
ICCNJ	1
ICIAG	1
ITMAX	50
KEIL	0
KLIN	0
NCC	10
ITSOR	1
ISCLSP	10
EPSIL	C.1E-01
EPSILO	C.1E-C4
EPSIL1	C.1E-02
EPSIL2	C.1E-C2
PC	C.0

GRADIENT REDUIT GENERALISE

```
NCMBRE  DE  VARIABLES  NATURELLES     3
NCMBRE  TCTAL  DE  VARIABLES          9
NCMBRE  CE  CCNTRAINTES               4

EPSILCN  CE  NEWTCN           C.1CC0E-C4
EPSILCN  TEST  GRADIENT       0.1000E-02

FCNCTICN  ECCNCMICUE           -0.6062200CCOCCOOE 05

            BORNE  INFERIEURE                         VARIABLE  NATURELLE                        BORNE  SUPERIEURE

      XI(  1)      0.5C000CCCC0C000E 02        X(  1)      0.85000000000000E 02        XS(  1)      0.99000000000000E 02
      XI(  2)      0.99999979138374E-02        X(  2)      0.29999997466803E-01        XS(  2)      0.20000000000000E 01
      XI(  3)      0.99999979138374E-02        X(  3)      0.29999997466803E-01        XS(  3)      0.2000C000C0000E 01

          VALEUR  CES  CCNTRAINTES

     C(  1)      -C.13999996185303E 01
     C(  2)      -C.1C499992370605E 01
     C(  3)       C.145C0CC76293955E 01
     C(  4)       0.45000076293945E 00
```

```
IT  1   PHI-0.61922993750000E 06   DIR.GRAD.  NO  3   YN 0.17E 09   DELTAFI 0.34E 09   ETA 0.34E 09   NCDB  0   NCN  3   NITN  3
IT. 2   PHI-0.51766818750000E 06   DIR.CONJ.  LC  2
IT  3   PHI-0.51765250000000E 06   DIR.CONJ.  LC  2                                                      NCDB  0   NCN  2   NITN  2
IT  4   PHI-0.51764943750000E 06   DIR.GRAD.  NO  3   YN 0.17E 09   DELTAFI 0.34E 09   ETA 0.34E 09   NCDB  1   NCN  7   NITN  3
IT  5   PHI-0.51764850000000E 06   DIR.GRAD.  NO  4   YN 0.10E 07   DELTAFI 0.34E 09   ETA 0.10E 17   NCDB  0   NCN 26   NITN 26
IT  6   PHI-0.51764812500000E 06   DIR.GRAD.  NO  3   YN 0.86E 08   DELTAFI 0.17E 09   ETA 0.17E 09   NCDB  0   NCN 27   NITN 25
IT  7   PHI-0.51764800000000E 06   DIR.GRAD.  NO  3   YN 0.86E 08   DELTAFI 0.17E 09   ETA 0.17E 09   NCDB  0   NCN 26   NITN 24
IT  8   PHI-0.51764793750000E 06   DIR.GRAD.  NO  3   YN 0.86E 08   DELTAFI 0.17E 09   ETA 0.17E 09   NCDB  0   NCN 27   NITN 25
```

A P P E N D I X : 5

COMPUTER PRONTOUT OF THE RESULTS OBTAINED BY G.R.G. METHOD

PARAMETRES

NV	3
NIN	4
NEG	0
NEVL	20
NTD	6
ITET	20
ICONJ	1
IDIAG	1
ITMAX	50
KFIL	0
KLIN	0
NCC	10
ITSCE	1
ISCLSR	10
EPSIL	0.1E-01
EPSILO	0.1E-04
EPSIL1	0.1E-02
EPSIL2	0.1E-02
EC	0.0

GRADIENT REDUIT GENERALISE

```
NOMBRE DE VARIABLES NATURELLES     3
NOMBRE TOTAL DE VARIABLES         10
NOMBRE DE CONTRAINTES              4

EPSILON DE NEWTON          0.1000E-04
EPSILON TEST GRADIENT      0.1000E-02

FONCTION ECONOMIQUE        -0.244297000000000E 06
```

BORNE INFERIEURE		VARIABLE NATURELLE		BORNE SUPERIEURE	
XI(1)	0.500000000000000E 02	X(1)	0.600000000000000E 02	XS(1)	0.990000000000000E 02
XI(2)	0.999999791383174E-02	X(2)	0.459999970195768E-01	XS(2)	0.200000000000000E 01
XI(3)	0.999999791383174E-02	X(3)	0.459999970195768E-01	XS(3)	0.200000000000000E 01

VALEUR DES CONTRAINTES

C(1)	0.249999904632568E 00
C(2)	0.499999504632568E 00
C(3)	0.100000095367431E 01
C(4)	0.953674316406250E-06

```
IT  1   PHI-0.20215090000000E 07   ITERATION  SPECIALE  BASE  DEGENEREE                                   NCDB  3   NCN 20   NITN241
IT  2   PHI-0.11308700000000E 07   DIR.GRAD.  NO  3   YN 0.28E 07   DELTAFI 0.76E 07   ETA 0.12E 17   NCDB  1   NCN 75   NITN882
IT  3   PHI-0.10802070000000E 07   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.76E 07   ETA 0.10E 17   NCDB  0   NCN  3   NITN 12
IT  4   PHI-0.10800800000000E 07   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.76E 07   ETA 0.10E 17   NCDB  0   NCN  5   NITN 18
IT  5   PHI-0.10799710000000E 07   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.76E 07   ETA 0.10E 17   NCDB  0   NCN  5   NITN 18
IT  6   PHI-0.10799660000000E 07   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.76E 07   ETA 0.10E 17   NCDB  0   NCN  5   NITN 24
IT  7   PHI-0.10799580000000E 07   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.76E 07   ETA 0.10E 17   NCDB  0   NCN  4   NITN 18
IT  8   PHI-0.10219425000000E 06   DIR.GRAD.  NO  4   YN 0.41E 08   DELTAFI 0.76E 07   ETA 0.79E 08   NCDB  4   NCN  6   NITN 30
IT  9   PHI-0.10200181250000E 06   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  1   NCN  2   NITN  5
IT 10   PHI-0.10195400000000E 06   DIR.GRAD.  NO  4   YN 0.16E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  2   NCN  7   NITN 28

SOLUTION  OBTENUE  A  L ITERATION   10

X(  1) =  0.65546624755859E 02
X(  2) =  0.57055201381445E-01
X(  3) =  0.49999997019768E-01
X(  4) =  0.13062925972918E-04
X(  5) =  0.0
X(  6) =  0.0
X(  7) =  0.0
X(  8) = -0.26724343128107E-02
X(  9) = -0.51809593141079E-02
X( 10) =  0.49733012514658E 00
IT 11   PHI-0.97537000000000E 05   DIR.GRAD.  NO  4   YN 0.16E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  3   NCN 14   NITN 52
IT 12   PHI-0.97341750000000E 05   DIR.GRAD.  NO  3   YN 0.17E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  0   NCN  7   NITN 18
IT 13   PHI-0.97324125000000E 05   DIR.GRAD.  NO  3   YN 0.17E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  0   NCN  8   NITN 24
IT 14   PHI-0.97319812500000E 05   DIR.GRAD.  NO  3   YN 0.17E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  0   NCN  6   NITN 24
IT 15   PHI-0.97317687500000E 05   DIR.GRAD.  NO  3   YN 0.17E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  0   NCN  5   NITN 24
IT 16   PHI-0.97317437500000E 05   DIR.GRAD.  NO  3   YN 0.17E 07   DELTAFI 0.76E 07   ETA 0.11E 17   NCDB  0   NCN  6   NITN 30
IT 17   PHI-0.93915437500000E 05   DIR.GRAD.  NO  4   YN 0.41E 08   DELTAFI 0.76E 07   ETA 0.79E 08   NCDB  3   NCN  7   NITN 34
IT 18   PHI-0.93850375000000E 05   DIR.GRAD.  NO  4   YN 0.10E 07   DELTAFI 0.53E 07   ETA 0.53E 07   NCDB  1   NCN  3   NITN  2
IT 19   PHI-0.91000250000000E 05   DIR.GRAD.  NO  5   YN 0.19E 06   DELTAFI 0.53E 07   ETA 0.54E 07   NCDB  5   NCN 10   NITN 13
IT 20   PHI-0.89728062500000E 05   DIR.GRAD.  NO  3   YN 0.15E 07   DELTAFI 0.53E 07   ETA 0.10E 17   NCDB  3   NCN  3   NITN 10

SOLUTION  OBTENUE  A  L ITERATION   20

X(  1) =  0.70000259359414E 02
X(  2) =  0.57138379663229E-01
X(  3) =  0.49999997019768E-01
X(  4) =  0.16209458408412E-03
X(  5) =  0.0
X(  6) =  0.50498114433140E-04
X(  7) =  0.0
X(  8) =  0.0
X(  9) = -0.34841150045395E-03
X( 10) =  0.50001186132431E 00
IT 21   PHI-0.42814792968750E 05   DIR.GRAD.  NO  4   YN 0.13E 08   DELTAFI 0.53E 07   ETA 0.98E 17   NCDB  1   NCN  6   NITN 42
LES  VARIABLES  ARTIFICIELLES  SONT  TOUTES  ANNULEES
IT 22   PHI-0.42763781250000E 05   DIR.GRAD.  NO  5   YN 0.34E 04   DELTAFI 0.23E 04   ETA 0.23E 04   NCDB  0   NCN  1   NITN  1
IT 23   PHI-0.42763769531250E 05   DIR.CONJ.  LC  1                                                      NCDB  0   NCN  1   NITN  1
        PHI-0.42763769531250E 05                      YN 0.15E 01   DELTAFI 0.10E 02   ETA 0.44E 02
```

VARIABLE NATURELLE	VARIABLE DUALE ASSOCIEE
X(1) = C.752665863C3711E 02	V(1) = -C.15COCCOOCCOOOOE 01
X(2) = C.7771331071E536E-C1	V(2) = 0.0
X(3) = C.4599995 7C15768E-01	V(3) = C.0

CCNTRAINTE		VARIABLE DUALE ASSOCIEE
CCNTPAINTE 1	-C.1716613765 5313E-C3	U(1) = 0.0
CCNTRAINTE 2	-0.4633302688598 6E 00	U(2) = C.0
CCNTPAINTE 3	-C.216CC6851196295 01	U(3) = 0.0
CCNTFAINTE 4	-C.563328361511 23E CO	U(4) = 0.0

A P P E N D I X : 6

COMPUTER PRINTOUT OF THE RESULTS OBTAINED

BY G.R.G. METHOD

PARAMETRES

NV	3
NIN	4
NEG	0
NEVL	20
NTD	6
ITET	20
ICONJ	1
IDIAG	1
ITMAX	50
KFIL	0
KLIN	0
NCO	10
ITSOR	1
ISOLSR	10
EPSIL	0.1E-01
EPSILO	0.1E-04
EPSIL1	0.1E-02
EPSIL2	0.1E-02
PC	0.0

GRADIENT REDUIT GENERALISE

NOMBRE DE VARIABLES NATURELLES	3
NOMBRE TOTAL DE VARIABLES	8
NOMBRE DE CONTRAINTES	4

EPSILON DE NEWTON	0.1000E-04
EPSILON TEST GRADIENT	0.1000E-02

FONCTION ECONOMIQUE -0.52642000000000E 05

BORNE INFERIEURE		VARIABLE NATURELLE		BORNE SUPERIEURE	
XI(1)	0.50000000000000E 02	X(1)	0.75000000000000E 02	XS(1)	0.99000000000000E 02
XI(2)	0.99999979138374E-02	X(2)	0.39999999105930E-01	XS(2)	0.20000000000000E 01
XI(3)	0.99999979138374E-02	X(3)	0.39999999105930E-01	XS(3)	0.20000000000000E 01

VALEUR DES CONTRAINTES

C(1)	-0.80000019073486E 00
C(2)	-0.50000005960464E 00
C(3)	0.10000009536743E 01
C(4)	0.95367431640625E-06

```
IT 1   PHI-0.96671262500000E 06   DIR.GRAD.  NO  2   YN 0.76E 08 FI 0.15E 09   ETA 0.15E 09   NCDB  6   NCN 23   NITN 31
IT 2   PHI-0.34527900000000E 06   DIR.GRAD.  NO  2   YN 0.11E 09 FI 0.15E 09   ETA 0.11E 17   NCDB  3   NCN  9   NITN 17
IT 3   PHI-0.34527743750000E 06   DIR.GRAD.  NO  2   YN 0.11E 07 FI C.15E 09   ETA 0.93E 15   NCDB  1   NCN  5   NITN 24
IT 4   PHI-0.42964253906250E 05   DIR.GRAD.  NO  3   YN 0.14E 07 FI 0.15E 09   ETA 0.10E 17   NCDB  3   NCN  2   NITN  4
LES VARIAFLES ARTIFICIELLES SONT TOUTES ANNULEES
IT 5   PHI-0.42964003906250E 05   DIR.GRAD.  NO  3   YN 0.66E 03 FI 0.13E 05   ETA 0.13E 05   NCDB  1   NCN  5   NITN 18
IT 6   PHI-0.42763773437500E 05   DIR.GRAD.  NO  3   YN 0.86E 05 FI 0.13E 05   ETA 0.61E 15   NCDB  1   NCN 12   NITN 21
IT 7   PHI-0.42763769531250E 05   DIR.CONJ.  LC  1                                             NCDB  0   NCN  1   NITN  C
       PHI-0.42763769531250E 05                      YN 0.0       FI C.0        ETA 0.0

CUREE CU CALCUL     12  CENTISECONDES
```

VARIABLE NATURELLE	VARIABLE DUALE ASSOCIEE
X(1) = 0.79265228271484E 02	V(1) = 0.0
X(2) = 0.50546824932098E-01	V(2) = 0.
X(3) = 0.39999999105930E-01	V(3) = 0.0

CONTRAINTE		VARIABLE DUALE ASSOCIEE
CONTRAINTE 1	-C.69916918609619E 00	U(1) = 0.0
CONTRAINTE 2	-C.67C60917615891E 00	U(2) = 0.0
CONTRAINTE 3	-C.66051483154297E-02	U(3) = 0.0
CONTRAINTE 4	-C.17060852050781E 00	U(4) = 0.0

A P P E N D I X : 7

COMPUTER PRINTOUT OF THE RESULTS OBTAINED

BY SEPARABLE PROGRAMMING

```
0001   PROGRAM
0002   INITIALZ
0065   MOVE(XDATA,'DATA-SET')
0066   MOVE(XPBNAME,'EXAMPLE')
0067   CONVERT
0068   BCDOUT
0069   MOVE(XOBJ,'RONTOI')
0070   MOVE(XRHS,'LIMITS')
0071   TITLE('PRIMAL-LOCAL OPTIMUM')
0072   XSEFLB=-1     $* BOUNDED VARIABLES AT LOWER LIMIT
0074   SETUP('BOUND','SEPBOUND','MAX')
0075   PRIMAL
0076   SOLUTION      $* LOCAL OPTIMUM
0078   TITLE('PRIMAL-GLOBAL OPTIMUM')
0079   XSEFLB=+1     $* BOUNDED VARIABLES AT UPPER LIMIT
0081   SETUP('BOUND','SEPBOUND','MAX')
0082   PRIMAL
0083   SOLUTION      $* GLOBAL OPTIMUM
0085   TITLE('DUAL-GLOBAL OPTIMUM')
0086   SETUP('BOUND','SEPBOUND','MAX')
0087   DUAL
0088   SOLUTION      $* GLOBAL OPTIMUM DUAL
0090   EXIT
0091   PEND
```

THIS IS RECYCLED PAPER "UNMARK"™ BOND" THIS FORM IS RECYCLABLE "UNMARK"™ BOND" IS A TRADEMARK OF SHADE INFORMATION SYSTEMS

```
NAME          DATA-SET
ROWS
 N  ROW101
 L  ROW102
 L  ROW103
 G  ROW104
 G  ROW105
 E  GRID101
 E  GRID102
 E  GRID103
COLUMNS
    X1        ROW101      86082.00000    GRID101          1.00000
    X2        GRID102         1.00000
    X3        GRID103         1.00000
    SETI01    'MARKER'                   'SEPORG'
    P1001     ROW101     -61392.00000    ROW102     -      .02500
    P1001     ROW103     -     .03000    ROW104            .00900
    P1001     ROW105           .00995    GRID101    -      .75000
    P1002     ROW101     -62004.00000    ROW102     -      .02500
    P1002     ROW103     -     .03100    ROW104            .00900
    P1002     ROW105           .00985    GRID101    -      .75000
    P1003     ROW101     -62615.00000    ROW102     -      .02600
    P1003     ROW103     -     .03200    ROW104            .00900
    P1003     ROW105           .00976    GRID101    -      .75000
    P1004     ROW101     -63226.00000    ROW102     -      .02700
    P1004     ROW103     -     .03300    ROW104            .00900
    P1004     ROW105           .00965    GRID101    -      .75000
    P1005     ROW101     -63836.00000    ROW102     -      .02800
    P1005     ROW103     -     .03400    ROW104            .00900
    P1005     ROW105           .00957    GRID101    -      .75000
    P1006     ROW101     -64447.00000    ROW102     -      .02800
    P1006     ROW103     -     .03500    ROW104            .00900
    P1006     ROW105           .00948    GRID101    -      .75000
    P1007     ROW101     -65058.00000    ROW102     -      .02900
    P1007     ROW103     -     .03700    ROW104            .00900
    P1007     ROW105           .00939    GRID101    -      .75000
    P1008     ROW101     -65670.00000    ROW102     -      .03000
    P1008     ROW103     -     .03800    ROW104            .00900
    P1008     ROW105           .00930    GRID101    -      .75000
    P1009     ROW101     -66279.00000    ROW102     -      .03100
    P1009     ROW103     -     .04000    ROW104            .00900
    P1009     ROW105           .00922    GRID101    -      .75000
    P1010     ROW101     -66891.00000    ROW102     -      .03200
    P1010     ROW103     -     .04100    ROW104            .00900
    P1010     ROW105           .00913    GRID101    -      .75000
    P1011     ROW101     -67502.00000    ROW102     -      .03300
    P1011     ROW103     -     .04300    ROW104            .00900
    P1011     ROW105           .00905    GRID101    -      .75000
    P1012     ROW101     -68113.00000    ROW102     -      .03500
    P1012     ROW103     -     .04500    ROW104            .00800
    P1012     ROW105           .00897    GRID101    -      .75000
    P1013     ROW101     -68723.00000    ROW102     -      .03600
    P1013     ROW103     -     .04800    ROW104            .00800
    P1013     ROW105           .00889    GRID101    -      .75000
```

THIS IS RECYCLED PAPER "UNIMARK" BOND" THIS FORM IS RECYCLABLE "UNIMARK" BOND" IS A TRADEMARK OF TRADE INFORMATION SYSTEMS

P1014	ROW101	-69334.00000	ROW102	- .03700
P1014	ROW103	- .05000	ROW104	- .00800
P1014	ROW105	.00881	GRID101	- .75000
P1015	ROW101	-69945.00000	ROW102	- .03900
P1015	ROW103	- .05300	ROW104	- .00800
P1015	ROW105	.00873	GRID101	- .75000
P1016	ROW101	-70557.00000	ROW102	- .04000
P1016	ROW103	- .05600	ROW104	- .00800
P1016	ROW105	.00865	GRID101	- .75000
P1017	ROW101	-71166.00000	ROW102	- .04200
P1017	ROW103	- .05900	ROW104	- .00800
P1017	ROW105	.00858	GRID101	- .75000
P1018	ROW101	-71778.00000	ROW102	- .04400
P1018	ROW103	- .06300	ROW104	- .00800
P1018	ROW105	.00851	GRID101	- .75000
P1019	ROW101	-72389.00000	ROW102	- .04600
P1019	ROW103	- .06700	ROW104	- .00800
P1019	ROW105	.00844	GRID101	- .75000
P1020	ROW101	-73000.00000	ROW102	- .04800
P1020	ROW103	- .07200	ROW104	- .00800
P1020	ROW105	.00837	GRID101	- .75000
SET102	'MARKER'		'SEPORG'	
P1021	ROW102	.09500	ROW104	.09500
P1021	GRID102	- .00600		
P1022	ROW102	.08700	ROW104	.08700
P1022	GRID102	- .00600		
P1023	ROW102	.08000	ROW104	.08000
P1023	GRID102	- .00600		
P1024	ROW102	.07400	ROW104	.07400
P1024	GRID102	- .00600		
P1025	ROW102	.06800	ROW104	.06800
P1025	GRID102	- .00600		
P1026	ROW102	.06400	ROW104	.06400
P1026	GRID102	- .00600		
P1027	ROW102	.06000	ROW104	.06000
P1027	GRID102	- .00600		
P1028	ROW102	.05700	ROW104	.05700
P1028	GRID102	- .00600		
P1029	ROW102	.05400	ROW104	.05400
P1029	GRID102	- .00600		
P1030	ROW102	.05100	ROW104	.05100
P1030	GRID102	- .00600		
SET103	'MARKER'		'SEPORG'	
P1031	ROW103	.11200	ROW105	.11200
P1031	GRID103	- .00490		
P1032	ROW103	.10100	ROW105	.10100
P1032	GRID103	- .00490		
P1033	ROW103	.09200	ROW105	.09200
P1033	GRID103	- .00490		
P1034	ROW103	.08400	ROW105	.08400
P1034	GRID103	- .00490		
P1035	ROW103	.07700	ROW105	.07700
P1035	GRID103	- .00490		
P1036	ROW103	.07200	ROW105	.07200

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    P1036     GRID103   -   .00490
    P1037     ROW103        .06700   ROW105        .06700
    P1037     GRID103   -   .00490
    P1038     ROW103        .06300   ROW105        .06300
    P1038     GRID103   -   .00490
    P1039     ROW103        .05900   ROW105        .05900
    P1039     GRID103   -   .00490
    P1040     ROW103        .05500   ROW105        .05500
    P1040     GRID103   -   .00490
    ENDSET    'MARKER'               'SEPEND'
RHS
    LIMITS    ROW102        .10536   ROW103        .40546
    LIMITS    ROW104    -   .11778   GRID101     75.00000
    LIMITS    GRID102       .06000   GRID103       .04000
BOUNDS
 UP SEPBOUND  P1001        1.00000
 UP SEPBOUND  P1002        1.00000
 UP SEPBOUND  P1003        1.00000
 UP SEPBOUND  P1004        1.00000
 UP SEPBOUND  P1005        1.00000
 UP SEPBOUND  P1006        1.00000
 UP SEPBOUND  P1007        1.00000
 UP SEPBOUND  P1008        1.00000
 UP SEPBOUND  P1009        1.00000
 UP SEPBOUND  P1010        1.00000
 UP SEPBOUND  P1011        1.00000
 UP SEPBOUND  P1012        1.00000
 UP SEPBOUND  P1013        1.00000
 UP SEPBOUND  P1014        1.00000
 UP SEPBOUND  P1015        1.00000
 UP SEPBOUND  P1016        1.00000
 UP SEPBOUND  P1017        1.00000
 UP SEPBOUND  P1018        1.00000
 UP SEPBOUND  P1019        1.00000
 UP SEPBOUND  P1020        1.00000
 UP SEPBOUND  P1021        1.00000
 UP SEPBOUND  P1022        1.00000
 UP SEPBOUND  P1023        1.00000
 UP SEPBOUND  P1024        1.00000
 UP SEPBOUND  P1025        1.00000
 UP SEPBOUND  P1026        1.00000
 UP SEPBOUND  P1027        1.00000
 UP SEPBOUND  P1028        1.00000
 UP SEPBOUND  P1029        1.00000
 UP SEPBOUND  P1030        1.00000
 UP SEPBOUND  P1031        1.00000
 UP SEPBOUND  P1032        1.00000
 UP SEPBOUND  P1033        1.00000
 UP SEPBOUND  P1034        1.00000
 UP SEPBOUND  P1035        1.00000
 UP SEPBOUND  P1036        1.00000
 UP SEPBOUND  P1037        1.00000
 UP SEPBOUND  P1038        1.00000
 UP SEPBOUND  P1039        1.00000
```

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```
 UP SEPBOUND  P1040          1.00000
ENDATA
```


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SOLUTION (OPTIMAL)

TIME = 0.19 MINS. ITERATION NUMBER = 9

...NAME...	...ACTIVITY...	DEFINED AS
FUNCTIONAL	6465999.00000	ROW101
RESTRAINTS		LIMITS
BOUNDS....		SEPBOUND

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SECTION 1 - ROWS

	NUMBER	...ROW..	AT	...ACTIVITY...	SLACK ACTIVITY	..LOWER LIMIT.	..UPPER LIMIT.	.DUAL ACTIVITY
	1	ROW101	BS	6465999.00000	6465999.00000-	NONE	NONE	1.00000
	2	ROW102	BS	.15900-	.26436	NONE	.10536	.
	3	ROW103	BS	.19500-	.60046	NONE	.40546	.
	4	ROW104	BS	.05400	.17178-	.11778-	NONE	.
	5	ROW105	BS	.05827	.05827-	.	NONE	.
	6	GRID101	EQ	75.00000	.	75.00000	75.00000	86082.00000-
Δ	7	GRID102	EQ	.06000	.	.06000	.06000	.
Δ	8	GRID103	EQ	.04000	.	.04000	.04000	.

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SECTION 1 - ROWS

	NUMBER	...ROW..	AT	...ACTIVITY...	SLACK ACTIVITY	..LOWER LIMIT.	..UPPER LIMIT.	.DUAL ACTIVITY
	1	ROW101	BS	6465999.00000	6465999.00000-	NONE	NONE	1.00000
A	2	ROW102	UL	.10535	.	NONE	.10536	.
A	3	ROW103	UL	.40546	.	NONE	.40546	.
	4	ROW104	BS	.31836	.43614-	.11778-	NONE	.
	5	ROW105	BS	.65873	.65873-	.	NONE	.
	6	GRID101	EQ	75.00000	.	75.00000	75.00000	86082.00000-
A	7	GRID102	EQ	.06000	.	.06000	.06000	.
A	8	GRID103	EQ	.04000	.	.04000	.04000	.

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SECTION 2 - COLUMNS

	NUMBER	.COLUMN.	AT	...ACTIVITY...	..INPUT COST..	..LOWER LIMIT.	..UPPER LIMIT.	.REDUCED COST.
	9	X1	BS	79.50000	86082.00000	.	NONE	.
	10	X2	BS	.06000	.	.	NONE	.
	11	X3	BS	.04000	.	.	NONE	.
	12	P1001	UL	1.00000	61392.00000-	.	1.00000	3169.50000
	13	P1002	UL	1.00000	62004.00000-	.	1.00000	2557.50000
	14	P1003	UL	1.00000	62615.00000-	.	1.00000	1946.50000
	15	P1004	UL	1.00000	63226.00000-	.	1.00000	1335.50000
	16	P1005	UL	1.00000	63836.00000-	.	1.00000	725.50000
	17	P1006	UL	1.00000	64447.00000-	.	1.00000	114.50000
	18	P1007	LL	.	65058.00000-	.	1.00000	496.50000-
	19	P1008	LL	.	65670.00000-	.	1.00000	1108.50000-
	20	P1009	LL	.	66279.00000-	.	1.00000	1717.50000-
	21	P1010	LL	.	66891.00000-	.	1.00000	2329.50000-
	22	P1011	LL	.	67502.00000-	.	1.00000	2940.50000-
	23	P1012	LL	.	68113.00000-	.	1.00000	3551.50000-
	24	P1013	LL	.	68723.00000-	.	1.00000	4161.50000-
	25	P1014	LL	.	69334.00000-	.	1.00000	4772.50000-
	26	P1015	LL	.	69945.00000-	.	1.00000	5383.50000-
	27	P1016	LL	.	70557.00000-	.	1.00000	5995.50000-
	28	P1017	LL	.	71166.00000-	.	1.00000	6604.50000-
	29	P1018	LL	.	71778.00000-	.	1.00000	7216.50000-
	30	P1019	LL	.	72389.00000-	.	1.00000	7827.50000-
	31	P1020	LL	.	73000.00000-	.	1.00000	8438.50000-
Δ	32	P1021	LL	.	.	.	1.00000	.
Δ	33	P1022	LL	.	.	.	1.00000	.
Δ	34	P1023	LL	.	.	.	1.00000	.
Δ	35	P1024	LL	.	.	.	1.00000	.
Δ	36	P1025	LL	.	.	.	1.00000	.
Δ	37	P1026	LL	.	.	.	1.00000	.
Δ	38	P1027	LL	.	.	.	1.00000	.
Δ	39	P1028	LL	.	.	.	1.00000	.
Δ	40	P1029	LL	.	.	.	1.00000	.
Δ	41	P1030	LL	.	.	.	1.00000	.
Δ	42	P1031	LL	.	.	.	1.00000	.
Δ	43	P1032	LL	.	.	.	1.00000	.
Δ	44	P1033	LL	.	.	.	1.00000	.
Δ	45	P1034	LL	.	.	.	1.00000	.
Δ	46	P1035	LL	.	.	.	1.00000	.
Δ	47	P1036	LL	.	.	.	1.00000	.
Δ	48	P1037	LL	.	.	.	1.00000	.
Δ	49	P1038	LL	.	.	.	1.00000	.
Δ	50	P1039	LL	.	.	.	1.00000	.
Δ	51	P1040	LL	.	.	.	1.00000	.

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SECTION 2 - COLUMNS

	NUMBER	.COLUMN.	AT	...ACTIVITY...	..INPUT COST..	..LOWER LIMIT.	..UPPER LIMIT.	.REDUCED COST.
	9	X1	BS	79.50000	86082.00000	.	NONE	.
	10	X2	BS	.07819	.	.	NONE	.
	11	X3	BS	.07397	.	.	NONE	.
	12	P1001	UL	1.00000	61392.00000-	.	1.00000	3169.50000
	13	P1002	UL	1.00000	62004.00000-	.	1.00000	2557.50000
	14	P1003	UL	1.00000	62615.00000-	.	1.00000	1946.50000
	15	P1004	UL	1.00000	63226.00000-	.	1.00000	1335.50000
	16	P1005	UL	1.00000	63836.00000-	.	1.00000	725.50000
	17	P1006	UL	1.00000	64447.00000-	.	1.00000	114.50000
	18	P1007	LL	.	65058.00000-	.	1.00000	496.50000-
	19	P1008	LL	.	65670.00000-	.	1.00000	1108.50000-
	20	P1009	LL	.	66279.00000-	.	1.00000	1717.50000-
	21	P1010	LL	.	66891.00000-	.	1.00000	2329.50000-
	22	P1011	LL	.	67502.00000-	.	1.00000	2940.50000-
	23	P1012	LL	.	68113.00000-	.	1.00000	3551.50000-
	24	P1013	LL	.	68723.00000-	.	1.00000	4161.50000-
	25	P1014	LL	.	69334.00000-	.	1.00000	4772.50000-
	26	P1015	LL	.	69945.00000-	.	1.00000	5383.50000-
	27	P1016	LL	.	70557.00000-	.	1.00000	5995.50000-
	28	P1017	LL	.	71166.00000-	.	1.00000	6604.50000-
	29	P1018	LL	.	71778.00000-	.	1.00000	7216.50000-
	30	P1019	LL	.	72389.00000-	.	1.00000	7827.50000-
	31	P1020	LL	.	73000.00000-	.	1.00000	8438.50000-
A	32	P1021	UL	1.00000	.	.	1.00000	.
A	33	P1022	UL	1.00000	.	.	1.00000	.
A	34	P1023	UL	1.00000	.	.	1.00000	.
	35	P1024	BS	.03189	.	.	1.00000	.
A	36	P1025	LL	.	.	.	1.00000	.
A	37	P1026	LL	.	.	.	1.00000	.
A	38	P1027	LL	.	.	.	1.00000	.
A	39	P1028	LL	.	.	.	1.00000	.
A	40	P1029	LL	.	.	.	1.00000	.
A	41	P1030	LL	.	.	.	1.00000	.
A	42	P1031	UL	1.00000	.	.	1.00000	.
A	43	P1032	UL	1.00000	.	.	1.00000	.
A	44	P1033	UL	1.00000	.	.	1.00000	.
A	45	P1034	UL	1.00000	.	.	1.00000	.
A	46	P1035	UL	1.00000	.	.	1.00000	.
A	47	P1036	UL	1.00000	.	.	1.00000	.
	48	P1037	BS	.93231	.	.	1.00000	.
A	49	P1038	LL	.	.	.	1.00000	.
A	50	P1039	LL	.	.	.	1.00000	.
A	51	P1040	LL	.	.	.	1.00000	.

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SEARCH TECHNIQUES
AND WAGE INCENTIVE PLANS

by

ROBINDRA N. PAL

B.M.E. JADAVPUR UNIVERSITY,CALCUTTA,INDIA.1965

AN ABSTRACT OF THE MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY
Manhattan,Kansas

1976

ABSTRACT

This thesis has tow sections. Part 1 deals with the literature survey and the development of new techniques to handle search problems. Since the effectiveness of the search procedure is characterized by its rate of convergence, much of research work has been and are still being done to reduce the computation time. An attempt was made to solve one-dimensional search problems for convex functions by bisecting the enveloping cone of the function and then rotating it till the bisector becomes vertical. The generalization of this new method for any unimodal function by coupling with Fibonacci search was also discussed. This approach essentially cuts down the total number of experiments required to reach at optimum. A new method for multi-dimensional search problems based on the intersection of quadratics passing through the line-optimums in co-ordinate directions was developed and exemplified along with the comparison with other standard methods to show its efficiency.

In the second section, a case study was made with a view to show how operations research technique can be applied to formulate and solve certain wage incentive problems. Since the basic problem in an incentive scheme is to define the base level efficiency from which the incentive should start and also the incentive rates, the problem was formulated with the objective as to minimize the variance between the optimum base

level efficiency and the current different efficiencies of various departments. A constraint was that the total incentive to be paid to the workers must not exceed the current overtime expenses. This problem was solved by Generalized Reduced Gradient method and separable programming.