TOPOLOGY OF FIBER BUNDLES

by

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Abstract

This report introduces the fiber bundles. It includes the definitions of fiber bundles such as vector bundles and principal bundles, with some interesting examples. Reduction of the structure groups, and covering homotopy theorem and some specific computation using obstruction classes, Cech cohomology, Stiefel-Whitney classes, and first Chern classes are included.

Table of Contents

List of Figures

Chapter 1

Introduction to Fiber Bundles

A fiber bundle is a space that is locally a product space, but globally may have a different topological structure. We can describe it as a continuous surjective map $\pi : E \to B$ that in small neighborhood U behaves like the projection of $U \times F$ to U. Usually, we call the map π the projection of the bundle. The space E, B, F are known as total space, base space, and fiber space respectively.

In general, there are two kinds of fiber bundles. The trivial bundle is $B \times F$, the map π is the canonical projection from the product space to the first factor. In the non-trivial case, bundles will have totally different topological structures globally, such as the Mӧbius strip and the Klein bottle, as well as non-trivial covering spaces.

Definition 1. *A fiber bundle is a structure (π, E, B, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition, which is called the bundle projection. The space E is the total space, B is called the base space of the bundle, and F is the fiber space.*

The locally triviality condition is that for any point $x \in B$, there exists an open neighborhood

$U_x \subset B$ such that $\pi^{-1}(U_x)$ is homeomorphic to the product space $U_x \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_x) \arrow[r, "\phi"] \arrow[d, "\pi"] & U_x \times F \arrow[dl, "\mathbf{proj}"] \\ U_x & \end{tikzcd} \tag{1.1}$$

where $\mathbf{proj} : U_x \times F \to U_x$ *is the natural projection. The set of all $\{(U_j, \phi_j)\}$ is called a local trivialization of the bundle.*

Thus for any p in B, the preimage $\pi^{-1}(p)$ is homeomorphic to $F$and is called the fiber over p. Every fiber bundle $\pi : E \to B$ is an open map since the projection of products are open maps. Therefore B carries the quotient topology determined by the map π.

Transition Functions

If let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a open cover of the base space, then one can construct many different local homeomorphisms by taking different open sets. Then, the nonempty interscetion of two local homeomorphisms will give two different systems for the same point in the overlap. Transition functions give a invertible transformation of the fiber over the point in two different systems.

Given a vector bundle $E \to B$, and a pair of neighborhood U_α and U_β over which the bundle trivializes via two homeomorphisms,

$$h_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha) \tag{1.2}$$

$$h_\beta : U_\beta \times F \to \pi^{-1}(U_\beta) \tag{1.3}$$

the composite function $\overline{\psi}_{\alpha\beta}$ is defined to be

$$h_\alpha^{-1} \circ h_\beta : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F \tag{1.4}$$

where $U_{\alpha\beta}$ is $U_\alpha \cap U_\beta$, which is well defined on the overlap by

$$h_\alpha^{-1} \circ h_\beta(x, v) := \overline{\psi}_{\alpha\beta}(x, v) = (x, \psi_{\alpha\beta}(x)(v)) \tag{1.5}$$

where the map

$$\psi_{\alpha\beta} : U_{\alpha\beta} \to \text{Homeo(F)} \tag{1.6}$$

sends x to $\psi(x)$ such that

$$\psi_{\alpha\beta}(x) : F \to F \tag{1.7}$$

is the automorphism of the fiber F with $f \to \psi_{\alpha\beta}(x)(f)$ for all $f \in F$.

Lemma 1. $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$.

Proof. Note $\overline{\psi}_{\alpha\beta}$ is defined to be $h_\alpha^{-1} \circ h_\beta : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$, therefore, $\overline{\psi}_{\alpha\beta} \circ \overline{\psi}_{\beta\gamma} = h_\alpha^{-1} \circ h_\beta \circ h_\beta^{-1} \circ h_\gamma = h_\alpha^{-1} \circ h_\gamma = \overline{\psi}_{\alpha\gamma}$. Thus, by this fact, for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $f \in F$ we have $\overline{\psi}_{\alpha\beta} \circ \overline{\psi}_{\beta\gamma}(x, f) = \overline{\psi}_{\alpha\gamma}(x, f) \Rightarrow \overline{\psi}_{\alpha\beta}(x, \psi_{\beta\gamma}(x)(f)) = (x, \psi_{\alpha\gamma}(x)(f)) \Rightarrow (x, \psi_{\alpha\beta}(x)(\psi_{\beta\gamma}(x)(f))) = (x, \psi_{\alpha\gamma}(x)(f)) \Rightarrow (x, \psi_{\alpha\beta} \circ \psi_{\beta\gamma}(x)(f)) = (x, \psi_{\alpha\gamma}(x)(f)) \Rightarrow \psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$ □

1.1 Trivial Bundles

Let $E = B \times F$ and let $\pi : E \to B$ be the projection onto the first factor. Then E is a fiber bundle over B. The bundle E is not just a locally product but globally one. Thus, a trivial bundle is homeomorphic to the Cartesian product.

Lemma 2. *Any fiber bundle over a contractible CW-complex is trivial.*

Proof. Let B be a contractible CW-complex. Since B is contractible, there is a point $x \in B$ and a homotopy between id_B and the constant map $f : B \to \{x\}$. Then, we can define the *pullback* with $id_B^*(E) \backsimeq f^*(E)$. Therefore, $E = Id_B^*$ and $f^* = B \times E_x$ imply that $E \backsimeq B \times E_x$, which is a trivial bundle.[1] $\square$

1.2 Vector Bundles

Definition 2. *A vector bundle is a fiber bundle in which every fiber is a vector space. For every point in the base space B, there is an open neighborhood U and a homeomorphism $h : U \times \mathbb{R}^k \to \pi^{-1}(U)$ such that for all $x \in U$,*

1. *$(\pi \circ \phi)(x, v) = x$ for all vectors v in $\mathbb{R}^k$*
2. *$v \to \phi(x, v)$ is an isomorphism between the vector space $\mathbb{R}^k$ and $\pi^{-1}(x)$,*

and the vector bundle over the fields are defined similarly.

1.3 Principal Bundles

A principal bundle P is a mathematical object which formalizes some essential features of the Cartesian product $X \times G$ of a topological space with a structure group G equipped with:

1. An action of G on P, analogous to $(x, g)h = (x, gh)$ for a product space

2. A natural projection onto X, this is just the projection onto the first factor, $(x, g) \to x$

Definition 3. *A principle G-bundle P, where G denotes any topological group, is a fiber bundle $\pi : P \to X$ together with a continuous right action $P \times G \to P$ such that G preserves the fibers of P and acts freely and transitively on them i.e. let P_x be a fiber over any point x, then $\forall$ x_0 and $x_1 \in P_x$, $\exists$ some $g \in G$ such that $x_0 g = x_1$*

The definition implies that each fiber of the bundle is homeomorphic to the group G itself. Unlike the product space, principal bundles P lack a preferred choice of identity section. Likewise, there is no general projection onto G generalizing the projection onto the second factor, $X \times G \to G$ which exists for the Cartesian product.

Chapter 2

Example: The Bundle of Frames $(\pi, \mathbf{SO}_3, \mathbf{S}^2, \mathbf{SO}_2)$

Every rotation maps an orthonormal basis of $\mathbf{R}^3$ to another orthonormal basis. Like any linear transformation of a finite-dimensional vector space, a rotation can always be represented by a matrix. The orthonormality condition can be expressed as $R^T R = I$, where R is any rotation matrix in $\mathbf{R}^3$ and I is the 3×3 identity matrix. Matrices for which this property holds are called orthogonal matrices. The group of the orthogonal matrices is called orthogonal group, and $\mathbf{SO}_3 := \{A \in \mathbf{Mat}_{3\times3} | A^T A = I, \mathbf{det}(A) = 1\}$ is the subgroup of the orthogonal group with determinant +1. It is called the special orthogonal group, and the $det(A) = 1$

Similiarly, $\mathbf{SO}_2$ is the orientation preserving orthogonal group in $\mathbf{R}^2$. In order to carry out a rotation using the point (x, y) to be rotated is written as a vector, then multiplied by a matrix from the rotating angle θ:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.1}$$

where (x', y') are the coordinates of the point after rotation, and the formulae for x' and y' can be seen to be

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta.$$

Note that unlike higher dimension, the group of vector rotations in $\mathbb{R}^2$ is commutative.

Before showing the bundle structure of $(\pi, \mathbf{SO}_3, \mathbf{S}^2, \mathbf{SO}_2)$, let us consider the map $\widetilde{\pi} : \mathbf{SO}_3/\mathbf{SO}_2 \to \mathbf{S}^2$ defined by $\widetilde{\pi}([A]) = A \cdot e_3$ where $A \in \mathbf{SO}_3$ and $e_3 = [0, 0, 1]^T$. Here, the equivalence class of [A] is defined via the equivalence

$$A \sim A \cdot \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$$

where $B \in \mathbf{SO}_2$. We are trying to show that this map is a bijective continuous closed map, therefore, a homeomorphism.

Proof. We first show that this map is well-defined. Let $[A_1] = [A_2]$, so $A_1 = A_2 \cdot \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\widetilde{\pi}[A_1] = [A_1] \cdot e_3 = [A_2] \cdot \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \cdot e_3 = [A_2] \cdot e_3 = \widetilde{\pi}[A_2]$.

Assume $\widetilde{\pi}[A_1] = \widetilde{\pi}[A_2]$ so that $A_1 \cdot e_3 = A_2 \cdot e_3$, where $A_1, A_2 \in \mathbf{SO}_3$. Now $A_2^T A_1 \cdot e_3 = A_2^T A_2 \cdot e_3 = I_3 \cdot e_3 = e_3$, thus, there exists one element $B \in \mathbf{SO}_2$ such that $A_2^T A_1 = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$,

which implies that $A_1 = A_2 \cdot \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, $[A_1] = [A_2]$ implies that $\widetilde{\pi}$ is injective.

For each vector $\overrightarrow{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{S}^2$, there exists an $A \in \mathbf{SO}_3/\mathbf{SO}_2$ with

$$A = \begin{bmatrix} \frac{xz}{\sqrt{1-z^2}} & \frac{-y}{\sqrt{1-z^2}} & x \\ \frac{yz}{\sqrt{1-z^2}} & \frac{x}{\sqrt{1-z^2}} & y \\ -\sqrt{1-z^2} & 0 & z \end{bmatrix}$$

such that

$$\widetilde{\pi}([A]) = A \cdot e_3 = \begin{bmatrix} \frac{xz}{\sqrt{1-z^2}} & \frac{-y}{\sqrt{1-z^2}} & x \\ \frac{yz}{\sqrt{1-z^2}} & \frac{x}{\sqrt{1-z^2}} & y \\ -\sqrt{1-z^2} & 0 & z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where the way to get A is applying the spherical system by composing the rotation of θ and the rotation of ϕ. Since rotation of θ is on the xy-plane, then

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

and the rotation of ϕ can be fixed on the xz-plane without lost generality, then

$$R(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \tag{2.3}$$

therefore, their composition is

$$R(\theta) \circ R(\phi) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \tag{2.4}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\phi) & -\sin(\theta) & \cos(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) & \cos(\theta) & \sin(\theta)\sin(\phi) \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \tag{2.5}$$

Thus, set

$$\cos(\theta)\sin(\phi) = x\,,\ \sin(\theta)\sin(\phi) = y\,,\ \cos(\phi) = z\,,$$

we can solve each entry of the matrix which is exactly where we get A. Therefore, $\widetilde{\pi}$ is surjective.

By the construction of $\widetilde{\pi}$, the map actually is the projection of the third column by multiplying A and the third base element. If we just consider the map $\pi : \mathbf{SO}_3 \to \mathbf{S}^2$ defined by

$$\pi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

Note that $\mathbf{SO}_3 \subset \mathbf{Mat}_{3\times 3}(\mathbb{R})$ is a subspace of $\mathbb{R}^9$, and $\mathbb{R}^9$ is homeomorphic to $\mathbb{R}^6 \times \mathbb{R}^3$ which is the product topological space. Thus, π is just the projection from $\mathbb{R}^9$ to $\mathbb{R}^3$. Then, applying the result that the projection from the product topology is continuous will lead that $\widetilde{\pi}$ is also continuous.

$\mathbf{SO}_3$ is closed as it is the inverse image of a closed set $\mathbf{S}^2$, and bounded by computing its norm, therefore, it is compact. It is clear that $\mathbf{S}^2$ is a Hausdorff Space. By the closed map lemma, every continuous map from a compact space to a Hausdorff space is closed and proper, we conclude that the continuous map $\widetilde{\pi}$ is closed.

$\square$

Hence, we just proved that $\widetilde{\pi}$ is a homeomorphism by the fact that a bijective continuous map is homeomorphism if and only if it is a closed map.

In fact, when H is a closed subgroup of a Lie group G, the projection $G \to G/H$ will be a principal H-bundle. The example of $\mathbb{Q} \in \mathbb{R}$ shows that one must make some assumption on H.

We will now show directly the $\mathbf{SO}_3 \to \mathbf{S}^2$ is a principal $\mathbf{SO}_2$-bundle by replacing H to $\mathbf{SO}_2$ and G to $\mathbf{SO}_3$. Define a map $\pi : \mathbf{SO}_3 \to \mathbf{S}^2$ by $\pi(A) = A \cdot e_3$. The similarly way as we did for $\widetilde{\pi}$ shows that π is a surjective projection.

Now we check the local triviality condition. By doing the stereographic projection, let $U_- = \mathbf{S}^2 - [0, 0, 1]^T$, define a map $\phi : \mathbb{R}^2 \to \mathbf{S}^2$. Pick a point $(x, y, 0) \in \mathbb{R}^2$, then the line equation

$$\overrightarrow{r}(t) = t(x, y, 0) + (1 - t)(0, 0, -1) = (tx, ty, t - 1).$$

Since this point is on the unit sphere $\mathbf{S}^2$, we get

$$t = \frac{2}{x^2 + y^2 + 1}$$

by solving $t^2x^2 + t^2y^2 + t^2 - 2t + 1 = 1$. So the corresponding point on the sphere is

$$(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1}).$$

Thus, we can find the new basis for every element of $\mathbb{R}^2$ by the differentiation respect to x and y. Denote the new basis by (e_1, e_2, e_3),

$$e_1 = \frac{\phi_* \partial_x}{||\phi_* \partial_x||} = (\frac{2-2x^2+2y^2}{(x^2+y^2+1)^2}, \frac{-4xy}{(x^2+y^2+1)^2}, \frac{-4x}{(x^2+y^2+1)^2})$$

$$e_2 = \frac{\phi_* \partial_y}{||\phi_* \partial_y||} = (\frac{-4xy}{(x^2+y^2+1)^2}, \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}, \frac{-4y}{(x^2+y^2+1)^2})$$

$$e_3 = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1})$$

Now define $h_- : U_- \times \mathbf{SO}_2 \to \pi^{-1}(U_-)$ by

$$h_-((x,y), \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}) = (\cos\theta e_1(x,y) + \sin\theta e_2(x,y), -\sin\theta e_1(x,y) + \cos\theta e_2(x,y), e_3(x,y)) \tag{2.6}$$

$$= (e_1(x,y), e_2(x,y), e_3(x,y)) \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{SO}_3. \tag{2.7}$$

Then, define $g_- : \pi^{-1}(U_-) \to U_- \times \mathbf{SO}_2$ by $g_-(A) = (\phi^{-1}(A \cdot e_3), (e_1(x,y), e_2(x,y), e_3(x,y)^{-1} A)$ as a well-defined continuous map. It is easy to check that $h_-^{-1} \circ g_- = id_{\pi^{-1}(U_-)}$ and $g_-^{-1} \circ h_- = id_{U_- \times \mathbf{SO}_2}$. Therefore, h_- is a homeomorphism.

Similarly, we can constrcut a map $h_+ : U_+ \times \mathbf{SO_2} \to \pi^{-1}(U_+)$ by

$$h_+((x,y), \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}) = (\cos\theta f_1(x,y) + \sin\theta f_2(x,y), -\sin\theta f_1(x,y) + \cos\theta f_2(x,y), f_3(x,y)) \tag{2.8}$$

$$= (f_1(x,y), f_2(x,y), f_3(x,y)) \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{SO_3} \tag{2.9}$$

for $U_+ = \mathbf{S}^2 - [0,0,-1]^T$, where the basis (f_1, f_2, f_3) is

$$f_1 = \frac{\phi_* \partial_x}{||\phi_* \partial_x||} = (\frac{2 - 2x^2 + 2y^2}{(x^2+y^2+1)^2}, \frac{-4xy}{(x^2+y^2+1)^2}, \frac{4x}{(x^2+y^2+1)^2})$$

$$f_2 = \frac{\phi_* \partial_y}{||\phi_* \partial_y||} = (\frac{-4xy}{(x^2+y^2+1)^2}, \frac{2x^2 - 2y^2 + 2}{(x^2+y^2+1)^2}, \frac{4y}{(x^2+y^2+1)^2})$$

$$f_3 = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$$

and define $g_+ : \pi^{-1}(U_+) \to U_+ \times \mathbf{SO_2}$ by $g_+(A) = (\phi^{-1}(A \cdot e_3), (f_1(x,y), f_2(x,y), f_3(x,y)^{-1}A)$ as a well-defined continuous map. It is easy to check that $h_+^{-1} \circ g_+ = id_{\pi^{-1}(U_+)}$ and $g_+^{-1} \circ h_+ = id_{U_+ \times \mathbf{SO_2}}$. Therefore, h_+ is a homeomorphism.

Hence, combining the fact of the closed subgroup of the Lie group, we claim that this structure is a princiapl $\mathbf{SO_2}$-bundle. It can also be verified that $\mathbf{SO_3}$ is not homeomorphic to $\mathbf{S}^2 \times \mathbf{SO_2}$ by computing fundamental groups. The fundamental group $\pi_1(\mathbf{SO_2} \times \mathbf{S}^2)$ is equal to $\pi_1(\mathbf{SO_2}) \times \pi_1(\mathbf{S}^2)$, which is homeomorphic to $\mathbb{Z} \times \{1\} \cong \mathbb{Z}$. While the fundamental group $\pi_1(\mathbf{SO}_3)$ is homeomorphic to $\mathbb{Z}_2$, so $(\pi, \mathbf{SO}_3, \mathbf{S}^2, \mathbf{SO}_2)$ is not a trivial bundle.

Chapter 3

Trivialization and Section

Definition 4. *A section of a fiber bundle is a continuous right inverse of the projection* π*. If* $\pi : E \to B$ *is a fiber bundle, then a section is a continuous map* $s : B \to E$ *such that* $\pi(s(x)) = x$ *for all* $x \in B$

The most common question regarding any fiber bundle is whether or not it is trivial. There is a nice lemma to answer it, but it is not true for the other fiber bundles.

Lemma 3. *A principal bundle is trivial if and only if it admits a global section.*

Proof. If a principal G-bundle P is trivial (where G denotes any topological group), then, $P = B \times G$, where G and B are fiber space and base space, is just the Cartesian product. Define $\pi : P \to B$ as a fiber projection, and $\mathbf{proj} : B \times G \to B$ with $\mathbf{proj}(b, g) = b$ as a Cartesian projection, and we will be able to find a section $s : B \to P$ such that s is just the inclusion map $i : B \to B \times G$ with $i(b) = (b, 1)$. Thus, $\pi(s(b)) = \mathbf{proj}(i(b)) = \mathbf{proj}((b, 1)) = b \forall b \in B$, which implies that the inclusion map i is a global section.

Now if a principal G-bundle P admits a global section. A fiber bundle $\pi : P \to B$ with a continuous right action $\rho : P \times G \to P$ defined by $\rho(p, g) = pg$ for all $g \in G$, which actually defines an equivalence class of any element p in P such that $p \sim pg$. Now let $B \times G$ be a Cartesian product and define a map $\phi : B \times G \to P$ by $\phi(b, g) = \rho(s(b), g) = s(b)g$, we want to show that ϕ is an homeomorphism, thus, P is a trivial bundle.

Check injective: Let $\phi(b_1, g_1) = \phi(b_2, g_2)$, so $\rho(s(b_1), g_1) = \rho(s(b_2), g_2)$, which implies $s(b_1)g_1 = s(b_2)g_2$. Then, by the definition of section, we have $b_1 = \pi(s(b_1))$ and $b_2 = \pi(s(b_2))$, where $s(b_1), s(b_2) \in P$. Since ρ defines the equivalence class such that $p \sim pg \forall g \in G$, $s(b_1) \sim s(b_1)g_1$ and $s(b_2) \sim s(b_2)g_2$ for some $g_1, g_2 \in G$. Thus, $b_1 = \pi(s(b_1)) = \pi([s(b_1)g_1]) = \pi(s(b_1)g_1)$ and $b_2 = \pi(s(b_2)) = \pi([s(b_2)g_2]) = \pi(s(b_2)g_2)$. So, $b_1 = b_2$ and $g_1 = g_2$ since $s(b_1)g_1 = s(b_2)g_2 = s(b_2)g_1$. Hence, $(b_1, g_1) = (b_2, g_2)$.

Check surjective: Given any $p \in P$, set $\pi(p) = b$ for some $b \in B$ by the projection map $\pi : P \to B$. By the map $s : B \to P$, we know that $s(b) \in P$. Then, there exists some $g \in G$ such that $p = s(b)g$. Then, $p = s(b)g = \rho(s(b), g) = \phi(b, g) = \phi(\pi(p), g) = \rho(s(\pi(p)), g) = s(p)g$. Note that $[p] \sim [s(\pi(p))]$ since $\pi(p) = \pi(s(\pi(p)))$. Therefore, there exists $b = \pi(p) \in B$ and $g \in G$ with $p = s(b)g$ such that $p = \phi(b, g)$.

Check continuous: By the definition of the map $\phi : B \times G \to P$ defined by $\phi(b, g) = \rho(s(b), g) = s(b)g$. We know that ϕ depends on two continuous map ρ and s, thus, ϕ is a continuous map by the fact that the composition of the continuous maps is still continuous.

Check open: Now we pick any open set $W \in B \times G$ to show that it maps to an open set in P. Pick U_b, where U_b is the open neighborhood of the arbitrary points b in B. Consider the local triviality $h_b : U_b \times G \to \pi^{-1}(U_b)$ which is a homeomorphism, therefore, $h_b^{-1} \circ \phi : U_b \times G \to U_b \times G$ is defined by $h_b^{-1} \circ \phi(x, g) = h_b^{-1}(s(x)g) = (x, \mathbf{Pr}((h_b^{-1}(s(x)g)))) = (x, \mathbf{Pr}((h_b^{-1}(s(x))))g)$, where $\mathbf{Pr}$ is just the natural projection from $U_b \times G$ to G, therefore, continuous. $h_b^{-1} \circ \phi$ is continuous since the composition of two continuous map is continuous. Let $\Psi : U_b \times G \to U_b \times G$ defined by $\Psi(x, g) = (x, \mathbf{Pr}((h_b^{-1}(s(x))))^{-1} g)$ and $\Theta : U_b \times G \to U_b \times G$ defined by $\Theta(x, g) = (x, \mathbf{Pr}((h_b^{-1}(s(x))))g)$, where Ψ and Θ are both continuous by definition. Now, $\Psi \circ \Theta = 1_\Theta$ and $\Theta \circ \Psi = 1_\Psi$. Thus, Θ is a homeomorphism and $\phi|_{U_b \times G} : U_b \times G \to \pi^{-1}(U_b)$ is a homeomorphism as $\phi = h_b \circ \Theta$. Thus, $\phi|_{U_b \times G}$ is open and $\phi(U_b \cap W)$ is open. Therefore, $\phi(W) = \phi(\cup(U_b \cap W)) = \cup(\phi(U_b \cap W))$ is open. Hence, ϕ is open.

$\square$

Chapter 4

Reduction of the structure group

Definition 5. *Given a principal G-bundle P with $\pi : P \to B$ and a monomorphism $H \to G$, where G is the structure group, then a reduction of the structure group to H is an principal H-bundle Q such that $Q \times_H G \cong P$.*

If W is a right G-space and X is a left G-space, the balanced product $W \times_G X$ is the quotient space $W \times X/\sim$, where $(wg, x) \sim (w, gx)$. Equivalently, we can simply convert X to a right G-space, and take the orbit space of the diagonal action $(w, x)g = (wg, g^{-1}x)$; thus $W \times_G X = (W \times X)/G$. The following special cases should be noted:

1. If $X = *$ is a point, $W \times_G * = W/G$
2. If $X = G$ with the left translation action, the right action of G on itself makes $W \times_G G$ into a right G-space, and the action map $W \times G \to W$ induces a G-equivalent homeomorphism $W \times_G G \cong W$.

Let G and H be topological groups. A (G, H)-space is a space Y equipped with a left G-action and right H-action, such that the two actions commute: $(gy)h = g(yh)$.Note that if Y is a (G, H)-space and X is a right G-space, $X \times_G Y$ has a right H-action defined by

$[x, y]h = [x, yh]$; similarly $Y \times_H Z$ has a left G-action defined by $g[y, z] = [gy, z]$if Z is a left H-space.[2]

Proposition 1. The balanced product is associative up to a natural isomorphism: Let X be a right G-space, Y a (G, H)-space, and Z a left H-space. Then there is a natural homeomorphism

$$(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z).$$

Proof. By the definition, the balanced product $X \times_G Y$ is the quotient group $X \times Y/\sim$ where $(xg, y) \sim (x, gy)$. Thus, $(X \times_G Y) \times_H Z$ is the quotient group $((X \times_G Y) \times Z)/\sim$ where $([x, y]h, z) \sim ([x, y], hz)$ and $[x, y]$ is the equivalence class defined as above. Therefore, $((xg, y)h, z) \sim ((x, gy)h, z) \sim ((x, gy), hz) \sim (x, gy, hz)$. Similarly, the balanced product $Y \times_H Z$ is the quotient group $Y \times Z/\sim$ where $(yh, z) \sim (y, hz)$. Thus, $X \times_G (Y \times_H Z)$ is the quotient group $(X \times (Y \times_H Z))/\sim$ where $(xg, [y, z]) \sim (x, g[y, z])$ and $[y, z]$ is the equivalence class. Therefore, $(xg, (yh, z)) \sim (x, g(yh, z)) \sim (x, g(y, hz)) \sim (x, (gy, hz)) \sim (x, gy, hz)$. □

Proposition 2. The bundle $E = P \times_G (G/H)$ associated with P with standard fiber G/H can be identified with P/H. An element $(p, gH) \in P \times_G (G/H)$ maps into the element $pg \in P/H$. Consequently, $P(E, H)$ is a principal bundle over the base $E = P/H$ with structure group H. The projection $P \to E$ maps $p \in P$ into $pg \in E$.

Proof. The proof is straightforward by the definition of the fiber bundle, except the local triviality of the bundle $P(E, H)$, which follows from local triviality of $E(B, G/H, G, P)$ and $G(G/H, H)$. Let U be an open set of B such that $\pi_E^{-1}(U) \cong U \times G/H$ and let V be an open set of G/H such that $p^{-1}(V) \cong V \times H$, where $p : G \to G/H$ is the projection. Let W be the open set of $\pi_E^{-1}(U)$ which corresponds to $U \times V$ under the identification $\pi_E^{-1}(U) \cong U \times G/H$. If $\mu : P \to E = P/H$ is the projection, then $\mu^{-1}(W) \cong W \times H$.[3] □

Theorem 1. *The structure group of a principal G-bundle P can be reduced to H if and only if $E = P \times_G (G/H) \to B$ admits a section, where H is the closed subgroup of the Lie group G.*

Proof. Suppose that the structure group of P can be reduced to a principal H-bundle Q such that $Q \times_H G \cong P$. Then, $E = P \times_G (G/H) = Q \times_H G \times_G (G/H) = Q \times_H G \times_G G \times_H * = Q \times_H G \times_H * = Q \times_H (G/H)$ by the proposition of the balanced product, where $*$ corresponds to any point in H. Since the identity coset in G/H is an H-fixed point, then we can define a map $s : * \to G/H$. Applying the functor $Q \times_H (-)$ to the map s, there is a morphism, $B \to Q \times_H (G/H)$ which is equivalent to $B \to P \times_G (G/H)$, a section.[2]

Suppose that $P \times_G (G/H) \to B$ admits a section $s : B \to P \times_G (G/H)$. By the proposition above, $E = P \times_G (G/H) \cong P/H$. Define $\mu : P \to P/H$ to be the projection, and let Q be the set of points $x \in P$ such that $\mu(x) = s(\pi(x))$. In other words, Q is the inverse image of $s(B)$ by the projection μ. For every $b \in B$, there is $x \in Q$ such that $\pi(x) = b$ since $\mu^{-1}(s(b))$ is non-empty. Given x and y in the same fiber of P, if $x \in Q$ then $y \in Q$ if and only if $x = yh$ for some $h \in H$. Thus, the restriction $\mu|_Q : Q \to Q/H$ is induced by μ, so Q is a closed sub-manifold of P.[3] By the fact that G/H is a smooth manifold and $G \to G/H$ is a submersion since H is the closed subgroup of the Lie group G, thus Q is a principal bundle imbedded in P with $Q \times_H G \cong P$.[4] $\square$

The balanced product let one considers many fiber bundles from one principal bundle. Consequently, given any fiber bundle $E \to B$, one constructs a principal Homeo(F)-bundle such that

$$P^{\psi} = \coprod_{\alpha} U_\alpha \times \mathrm{Homeo(F)}/ \sim,$$

where the equivalence class is $(x, f) = (x, \psi_{\alpha\beta}(x)(f))$, and ψ is the transition function.

Example 1. As a concrete example, every even-dimension real vector space is the underlying real space of a complex vector space: it admits a linear complex structure. A real vector bundle admits an complex structure if and only if it is the underlying real bundle of a complex vector bundle. This is a reduction along the inclusion $GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$.

Example 2. Let $G = GL(n)$ and $H = GL^{+}(n)$. A reduction to $GL_{+}(n)$-bundle is a fiber orientation, and we can define a map $GL(n)/GL^{+}(n) \to \mathbf{Z}_2$ by sending the equivalence class $[A]$ to the sign of $det(A)$.

Example 3. Let $G = GL(n)$ and $H = O(n)$. Then G/H is homeomorphic to the group of upper triangular matrices with positive diagonal entries, and so is contractible. This implies that the associated bundle $P \times_{GL(n)} GL(n)/O(n)$ admits a section. Hence, any $GL(n)$-bundle is induced from an $O(n)$-bundle.

Example 4. Let $G = GL(n, \mathbb{R})$ and $H = SL(n, \mathbb{R})$, where $H \subset G$. Then $G/H = GL(n, \mathbb{R})/SL(n, \mathbb{R}) \cong \mathbb{R}^{\times}$, therefore, a reduction to $SL(n)$ cooresponds to an oriented fiber volume form.

These examples demonstrate the fact that the structure group of a real vector bundle of rank n can always be reduced to $O(n)$, which can be reduced to $SO(n)$ if and only if the vector bundle is orientable. And the vector bundle is orientable if its structure group may be reduced to $GL^{+}(n)$.

Chapter 5

Covering Homotopy Theorem

Definition 6. *Given a map $\pi : E \to B$, and any topological space X. Then, (X, π) has the* **homotopy lifting property** *if:*

1. *For any homotopy $f : X \times I \to B$*
2. *For any $\widetilde{f} : X \to E$ lifting $f_0 = f|_{X \times \{0\}}$ such that $f_0 = \pi \circ \widetilde{f}$,*

there exists a homotopy $\widetilde{f} : X \times I \to E$ lifting f such that $f = \pi \circ \widetilde{f}$ with $\widetilde{f_0} = \widetilde{f}|_{X \times \{0\}}$ as the following diagram.

$$\begin{tikzcd} X \arrow[r, "\widetilde{f_0}"] \arrow[d, "i_0"] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "f"] \arrow[ur, "\widetilde{f}"] & B \end{tikzcd} \tag{5.1}$$

Definition 7. *A Hurewicz-fibration is a continuous map $\pi : E \to B$ satisfies he homotopy lifting property with respect to any space,and a Serre-fibration is a continuous map which has the homotopy lifting property with respect to any finite CW-complex.*

Example 5. The projection map $B \times F \to B$ is a trivial fibration, and the fiber over every point is homeomorphic to F. Indeed, given f and $\widetilde{f_0}$, define $\widetilde{f} : X \times I \to B \times F$ by $\widetilde{f}(x,t) = (f(x,t), \mathbf{Pr}(\widetilde{f_0}(x)))$.

Example 6. Any covering space $E \to B$ is fibration. And, in fact, for a covering space the lifting $\widetilde{f}$ is uniquely determined.

Example 7. Any fiber bundle $E \to B$ over paracompact base space B is a fibration, *e.g.* any CW-complex.

Example 8. In general,any fiber bundle in which the base space is para-compact is a fibration. A map $\pi : E \to B$ is a fiber bundle with fiber F if B has a cover $\mathcal{U}$ and $h : U \times F \to \pi^{-1}(U)$ is a homeomorphism for each $U \in \mathcal{U}$. But, without the para-compact hypothesis on the base, any fiber bundle is at least a Serre-fibration.

Theorem 2. *Any fiber bundle $\pi : E \to B$ is a Serre-fibration.*

Proof. Let $\pi : E \to B$ be a fiber bundle, and suppose given a lifting diagram

$$
\begin{tikzcd}
I^n \arrow[r, "f"] \arrow[d, "i_0"] & E \arrow[d, "\pi"] \\
I^n \times I \arrow[r, "h"] & B
\end{tikzcd}
$$

Let $\mathcal{U}$ be a covering for B on which we have the local trivialization. I can divide I^n into sub-cubes C and I into subintervals J,such that each $C \times J$ is in a single $h^{-1}(U)$. For each C, I can build a lift on each $C \times J$, starting with the J containing 0, so I have a lift along the initial point of the interval. Moreover, the cube C borders other cubes, so I suppose that some union D of faces of ∂C has a lift along $D \times I$. For the fixed C and J, the fiber bundle is the trivial fiber bundle $U \times F \to U$. A lift in the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "i_0"'] & U \times F \arrow[d, "\pi"] \\ C \times I \arrow[r, "h"] \arrow[ur, "\exists\widetilde{h}"] & U \end{tikzcd}$$

must be given in the first coordinate by the map h, so it remains to describe the second coordinate of the lift. By assumption, we already have a lift $C \cup_D D \times J \to U \times F \to F$. Since the space $C \cup_D D \times J$ is a retract of $C \times J$, so I can compose the lift with a choice of a retraction $C \times J \to C \cup_D D \times J$.[5] □

Corollary 1. If $F \to E \to B$ is a fiber bundle and B is paracompact and contractible, then E is trivial.

Proof. Let $P \to B$ be the associated principal bundle, we will show that it has a section. Since the base B is contractible, it is homotopy equivalent to a single point $*$. Let $h : B \times I \to B$ be a contraction such that $h(b, 0) = b_0$ and $h(b, 1) = b$. Now pushforward of $\pi^{-1}(b_0)$ to get the commutative square diagram, then there is a map $\widetilde{h} : B \times I \to P$ such that

$$\begin{tikzcd} B \arrow[r, "\widetilde{h}_0"] \arrow[d, "i_0"'] & P \arrow[d, "\pi"] \\ B \times I \arrow[r, "h"] \arrow[ur, "\widetilde{h}"] & B \end{tikzcd}$$

So we define $\sigma : B \to P$ by $\sigma(b) = \widetilde{h}(b, 1)$. Since P has a section, P is trivial, then $P = B \times \text{Homeo(F)}$. Thus, $E = P \times_{\text{Homeo(F)}} F = B \times F$. □

5.1 Hopf-fibration: $(S^1 \to S^3 \to S^2)$

We can identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$ by letting

1. (x_1, x_2, x_3, x_4)as$(z_0 = x_1 + ix_2, z_1 = x_3 + ix_4)$
2. (x_1, x_2, x_3) as $(z = x_1 + ix_2, x = x_3)$.

Thus $\mathbb{S}^3$ is identified with the subset of all $(z_0, z_1) \in \mathbb{C}^2$ such that $|z_0|^2 + |z_1|^2 = 1$, and $\mathbf{S}^2$ is identified with the subset of all $(z, x) \in \mathbb{C} \times \mathbb{R}$ such that $|z|^2 + x^2 = 1$. Then the Hopf-fibration π is defined by

$$\pi(z_0, z_1) = (2z_0\overline{z_1}, |z_0|^2 - |z_1|^2).$$

The first component is a complex number and the second one is real. Then, we need to check that the image of π maps into $\mathbf{S}^2$, which can be verified by squaring two parts of the image

$$(2z_0\overline{z_1})^2 + (|z_0|^2 - |z_1|^2)^2 = 4|z_0|^2|z_1|^2 + |z_0|^4 - 2|z_0|^2|z_1|^2 + |z_1|^4 \tag{5.2}$$

$$= (|z_0|^2 + |z_1|^2)^2 = 1 \tag{5.3}$$

It is easy to see that this is a surjective map since $\mathbf{S}^2 \to \mathbf{S}^3$ is an inclusion map. Then, if two points on $\mathbf{S}^3$ map to the same point on the $\mathbf{S}^2$ such that $\pi(z_0, z_1) = \pi(w_0, w_1)$, then (w_0, w_1) must equal to $(\lambda z_0, \lambda z_1)$ for some complex number λ with $|\lambda|^2 = 1$. Since the set of the complex numbers λ with $|\lambda|^2 = 1$ forms the unit circle in the complex plane, it follows that for each point $x \in \mathbf{S}^2$, the inverse image $\pi^{-1}(x)$ is isomorphic to $\mathbf{S}^1$. This structure admits a local trivialization which implies that it is a fiber bundle, hence, a fibration since the base space $\mathbf{S}^2$ is a paracompact CW-complex.

Chapter 6

Obstruction Theory

Suppose we want to construct a section from a CW-complex X into a bundle E with fiber F. We do this by induction: given a section $\sigma : X^{(k)} \to E$ on the kth-skeleton $X^{(k)}$ and a (k+1)-cell $i : D^{k+1} \to X$, we want to extend σ over the i. The obstruction to extend over a (k+1)-cell is an element of $\pi_k(F)$, the k-th homotopy group of the fiber.[6]

Define the pullback $i^*(E) := \{(p, q) \in D^{k+1} \times E | i(p) = \pi(q)\}$, where $q := \sigma(i(p))$ and the fiber of $i^*(E)$ over $x \in X$ is the fiber of E over $i(p) \in X$. Therefore, it admits the bundle structure

$$\begin{tikzcd} & i^*(E) \arrow[d] \\ \mathbf{S}^k \arrow[ur] \arrow[r] & D^{k+1} \end{tikzcd}$$

Thus, $i^*(E)$ admits a trivialization ϕ such that

$$\phi : i^*(E) \to D^{k+1} \times F$$

then, there exists a continuous projection $\mathbf{P}_2$ maps to the fiber space F. Now we can define the obstruction class $\mathcal{O}^\sigma(i) = [\mathbf{P}_2 \circ \phi(b, \sigma \circ i)] \in \pi_k(F)$, where $\mathcal{O}^\sigma \in C^{k+1}_{CW}(X, \pi_k(F))$. These obstructions fit together to give a cellular cochain $\mathcal{O}$ on X with coefficients in this π_k. In fact, this cochain is a cocycle, so it defines an obstruction class $\mathcal{O}(E)$ in $H^{k+1}(X, \pi_k(E))$. Then there exists a cross-section over the (k+1)-skeleton if and only if a certain well defined obstruction class is zero. If the cochain is 0, then there exists a map $\mu : D^{k+1} \to F$. Then the section extending to (k+1)-skeleton $\widetilde{\sigma} : X^{k+1} \to E$ is defined to be $\mathbf{P}_2(\phi^{-1}(v, \mu(v)))$, where $v \in D^{k+1}$.

Example 9. (1st Stiefel-Whitney class)

Let $E \to X$ be a real vector bundle with structure group $GL(n, \mathbb{R})$ as example 2 in section 2, where X is a CW-complex space. Then, E is orientable if and only if its structure group can be reduced to the subgroup $GL^+(n, \mathbb{R})$. Therefore, we have the associated bundle $Z(E) = GL(n, E) \times_{GL(n,\mathbb{R})} (GL(n, \mathbb{R}/GL^+(n.\mathbb{R})))$ with fiber $\mathbb{Z}_2$ as the following diagram,

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r, "i"] & Z(E) \arrow[d, "\pi"] \\
X \arrow[ur, "\sigma"] \arrow[r, "Id_x"] & X
\end{tikzcd} \tag{6.1}$$

where the orientation is a section σ.

So, we build σ_k inductively on the k-th skeleton $X^{(k)}$. Define $\sigma_0 : X^{(0)} \to E$ as a section such that $\pi \circ \sigma_k = Id$ since $X^{(0)} \hookrightarrow X$ is an inclusion map. Note that the fiber $\mathbb{Z}_2$ is a group so that its 0th-homotopy group $\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$. In this case, the obstruction cocycle is the 1st Stiefel-Whitney class $w_1^\sigma(E) \in C^1_{CW}(X, \mathbb{Z}_2)$. Hence, a section defined on 0-cells is extendable if and only if the 1st Stiefel-Whitney class $w_1^{\sigma}(E) \in \mathbf{H}^1(X, \mathbb{Z}_2)$ is 0.

(**1**). It is a cocycle since its coboundary is 0. Indeed, let $i : D^2 \to X$ be 1-cells and $(\delta w_1^{\sigma_0}(E))(i) = w_1^{\sigma_0}(E)(\partial i) = (\sigma_0(1, e_2) - \sigma_0(1, e_1)) + (\sigma_0(1, e_3) - \sigma_0(1, e_2)) + ... + (\sigma_0(1, e_1) - \sigma_0(1, e_n)) = 0$ where e_i is the 0-cells of D^2 and e_n need not be distinct.

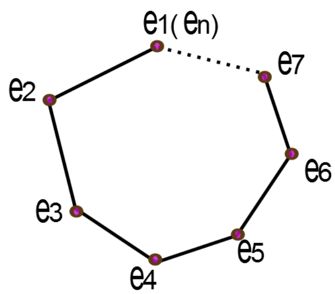

Figure 6.1: *cell decomposition*

(**2**). The 1st Stiefel-Whitney class is independent of the choice of sections. Let's pick two distinct sections σ and τ. Then, define $\triangle^{\sigma,\tau} \in C^1_{CW}(X, \pi_0(\mathbb{Z}_2))$ by $\triangle^{\tau,\sigma}(x) = \sigma(x) - \tau(x) \in \pi_0(\mathbb{Z}_2)$. Thus,

$$
\begin{aligned}
(\delta\triangle^{\tau,\sigma})(e) &= \triangle^{\tau,\sigma}(\partial e) && (6.2)\\
&= \triangle^{\tau,\sigma}(1,e) - \triangle^{\tau,\sigma}(-1,e) && (6.3)\\
&= (\sigma(1,e) - \tau(1,e)) - (\sigma(-1,e) - \tau(-1,e)) && (6.4)\\
&= w_1^{\sigma} - w_1^{\tau} = 0 && (6.5)
\end{aligned}
$$

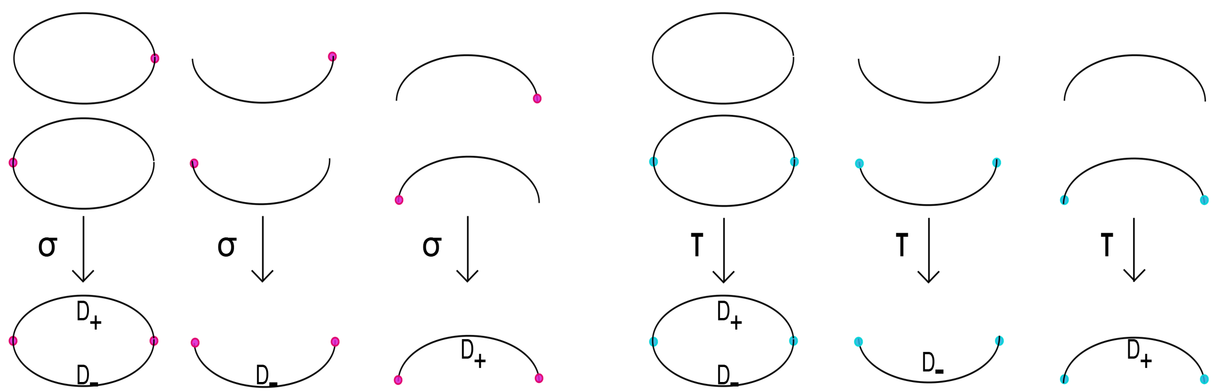

Figure 6.2: *1st Stiefel Class of total space E*

Exercise 1. Compute $w_1^{\sigma}(E)$ and $w_1^{\tau}(E)$, where $\sigma, \tau : S^{(0)} \to E$. Let us first consider the function σ where the chosen two 0-cells are mapping into different sides of the total space E, and we can define an orientation of $\mathbf{S}^1$ as the graph shows by separating $\mathbf{S}^1$ into two arcs denoted D_+ and D_-. By drawing two different arcs of the projections from E to $\mathbf{S}^1$, there is no continuous mapping from the boundary of D_- to the bundle space since the two points on E is arc-wise disconnected, so does D_+. Therefore, I conclude that $w_1^{\sigma}(E)(D_-) = w_1^{\sigma}(E)(D_+) = 1$, which implies that $w_1^{\sigma}(E)(S^1) = 1 + 1 = 2 = 0$ in $\mathbb{Z}_2$. Let another function τ maps two 0-cells into the same side of the total space as the graph shows. Then, $w_1^{\tau}(E)(D_-) = w_1^{\tau}(E)(D_+) = 0$, so $w_1^{\tau}(E)(S^1) = 0$. It actually shows that the obstruction class is independent with the choice of the functions. Therefore, the zero obstruction class implies that we can extend the function from 0-cell to 1-cell.

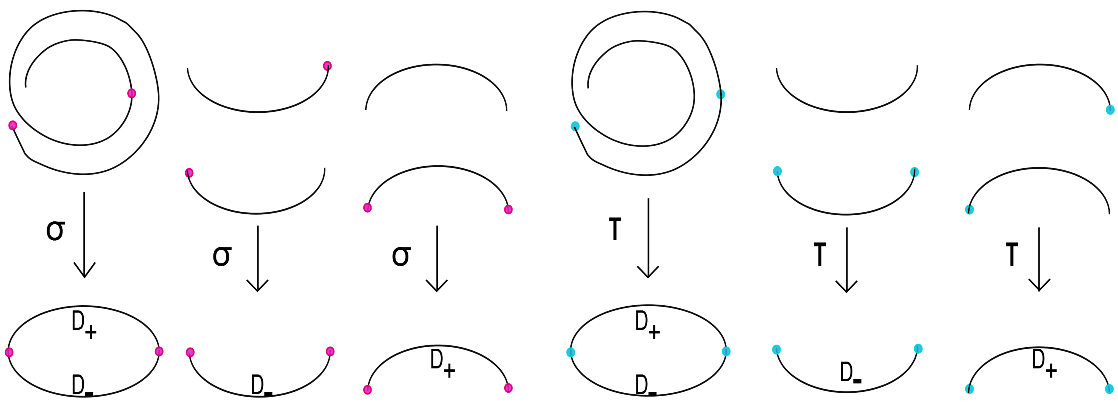

Figure 6.3: *1st Stiefel Class of total space E'*

Exercise 2. Compute $w_1^\sigma(E')$ and $w_1^\tau(E')$ of the different bundles E', where σ and τ are the same as part B. By separating the base space S^1 into two parts D_+ and D_- as we did above. Then, $w_1^\sigma(E')(D_-) = 1$ and $w_1^\sigma(E')(D_+) = 0$, which implies that $w_1^\sigma(E')(S^1) = 1 + 0 = 1$ in $\mathbb{Z}_2$. We know that the obstruction class is independent with the choice of the functions, thus, $w_1^\tau(E')(D_-) = 0$ and $w_1^\tau(E')(D_+) = 1$ which can also be proved by the graph. So, $w_1^\tau(E')(S^1) = 1 + 0 = 1$. Hence, we can not extend any function on 0-cells over 1-celsl on the vector bundle E'.

Example 10. (1st Chern class) Recall the definition of the complex vector bundle: a complex vector bundle E of complex dimension n over B and projection map $\pi : E \to B$, together with the structure of a complex vector space in each fiber $\pi^{-1}(b)$ with local triviality such that $h : U \times \mathbb{C}^n \to \pi^{-1}(U)$ is a homeomorphism which maps each fiber $\pi^{-1}(b)$ complex linearly onto $b \times \mathbb{C}^n$

Just as the structure group of a real vector bundle can be reduced to the orthogonal group

$O(n)$, the structure group of a rank n complex vector bundle can be reduced to the unitary group $U(n)$. Every complex vector bundle E of rank n has an underlying real vector bundle $E_{\mathbb{R}}$ of rank 2n, obtained by discarding the complex structure on each fiber.

Construct the composition $GL(n, \mathbb{C}) \to \mathbb{C}^{\times} \to \mathbf{S}^1$, where the map from the complex linear group $GL(n.\mathbb{C})$ to the multiplicative group of complex number $\mathbb{C}^{\times}$ is a determinant function, and the map from the total space to the base space $\mathbf{S}^1$ is an argument function. Then, there exists a reduced bundle structure of the complex vector bundle P with the base space X denoted as $E := P \times_{GL(n,\mathbb{C})} (GL(n, \mathbb{C})/\mathbf{Ker}(arg \circ det))$, where $GL(n, \mathbb{C})/\mathbf{Ker}(arg \circ det)$ is homeomorphic to $\mathbf{S}^1$. Since $\pi_0(\mathbf{S}^1) = 0$ and $\pi_1(\mathbf{S}^1) = \mathbb{Z}$, the obstruction class $\mathcal{O}(E)$ is an element in $C^2_{CW}(X, \mathbb{Z})$. In this case, the 1st Chern class $c_1(E)$ is defined to be the obstruction class $\mathcal{O}(E)$, an element in 2nd cohomology group $\mathbf{H}^2(X, \mathbb{Z})$.

Exercise 3. Consider the complex vector bundle $\pi : E \to \mathbf{S}^2$ with fiber $\mathbb{C}$, where the total bundle E is the tangent space of $\mathbf{S}^2$. To compute the first Chern class $c_1(E)(\mathbf{S}^2)$.
Cut the 2-sphere into two halves with one labeled D^2_+ and another one D^2_- as the graph shows, which are homeomorphic to D^2, a one dimensional disk over $\mathbb{C}$. By pointing out the vectors on the equator on both halves, I take the projection of the vectors from the equator to the boundary of the disk as graphs. Define $\psi_* : D^1_+ \to \mathbf{R}^3$ by $\psi_*(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. Then, taking the patrial derivative to x,

$$\partial_x \psi_*(x, y) = (1, 0, -x(1 - x^2 - y^2)^{-\frac{1}{2}}), \tag{6.6}$$

normalizing this vector, we get

$$\frac{\partial_x \psi_*(x, y)}{||\partial_x \psi_*(x, y)||} = \frac{((1 - x^2 - y^2)^{\frac{1}{2}}, 0, -x)}{(1 - y^2)^{\frac{1}{2}}} \tag{6.7}$$

Now, plugging in some values for (x, y), we can tell that the orientation of the vectors on D^1_+ does not change, while the orientation of the vectors on D^1_- changes twice by defining the similar parametrization function. Therefore, the first Chern class is

$c_1(E)(\mathbf{S}^2) = c_1(E)(D^1_+) + c_1(E)(D^1_+) = 0 + 2 = 2,$

which can be verified by our definition related to the Euler class of $\mathbf{S}^2$, $c_1(E)(\mathbf{S}^2) = e(E_{\mathbf{R}^2})(\mathbf{S}^2) = \sum (-1)^k(\sharp \text{ of the k-cell}) = (-1)^0(1) + (-1)^1(0) + (-1)^2(1) = 1 + 0 + 1 = 2$

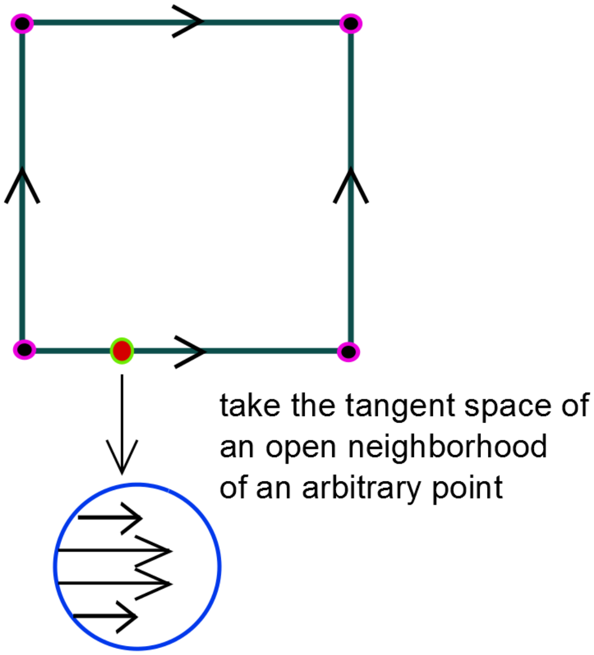

Figure 6.4: *Cell decomposition of torus*

Exercise 4.

Let us see another example denoted as $\mathbf{C} \to T(T^2) \to T^2$, where the total space is the tangent space of the 2-torus. By definition, $T^2 = \mathbf{C}/(\mathbf{Z} \oplus i\mathbf{Z})$. Thus, we have the graph of

the square representing the 2-torus, which is homeomorphic to the unit circle. Since A_1 and A_2 are equivalent, then we can define a continuous function $a : A_1 \to A_2$ by $a(z) = z + i$, where z is a complex number. So this map preserves the sign of the vector, therefore, the orientation of the vectors on A_1 and A_2 is the same. We can define another continuous function $b : B_1 \to B_2$ by $b(z) = 1 + z$, which also preserves the orientation of the vectors on B_1 and B_2. Note that the four vertices are identified by gluing together, so the orientation of the vectors on four sides preserves. Hence, the orientation of the vectors on the tangent bundle preserves, which implies that the first Chern class $c_1(TT^2)(T^2) = 0$.
It can also be verified by computing the Euler class of T^2, $e(T^2) = (-1)^0(1) + (-1)^1(2) + (-1)^2(1) = 0$

6.1 Stiefel-Whitney Classes

The Stiefel-Whitney classes are a set of topological invariants of a real vector bundle that describe the obstructions to constructing independent sections. Let $H^i(B;\mathbb{Z}_2)$ denote the i-th singular cohomology group of B with coefficients in $\mathbb{Z}_2$, here are four axioms which characterize the Stiefel-Whitney cohomology.[7]

Axiom 1. For each vector bundle E there corresponds a sequence of cohomology classes

$$w_i(E) \in H^i(B(E);\mathbb{Z}_2), i = 0, 1, 2, ...,$$

called the Stiefel-Whitney classes of E. The class $w_0(E)$ is the unit element

$$1 \in H^0(B(E);\mathbb{Z}_2),$$

and $w_i(E) = 0 \ \forall \ i \geq n$ if E is an n-dimensional bundle.

Axiom 2. If $f : Y \to X$ is covered by a bundle map from E to E', then

$$w_i(E) = f^* w_i(E'),$$

where f^* is the pullback.

Axiom 3. If E and E' are vector bundles over the same base space, then

$$w_n(E \oplus E') = \sum_{i=0}^{n} w_i(E) \cup w_{n-i}(E'),$$

where $\cup$ denotes the cap product. For example, $w_1(E \oplus E') = w_1(E) + w_1(E')$, and $w_2(E \oplus E') = w_2(E) + w_1(E)w_1(E') + w_2(E')$.

Axiom 4. For the line bundle E_1^1 over the circle P^1(real projective plane), the Stiefel-Whitney class $w_1(E_1^1)$ is non-zero.

Proposition 3. If Ei is isomorphic to E' then $w_i(E) = w_i(E')$.

Proposition 4. If E is a trivial vector bundle then $w_i(E) = 0$ for $i > 0$.

Proposition 5. If E is trivial then $w_i(E \oplus E') = w_i(E')$

6.2 Chern Class

The Chern classes are a set of topological invariants of a complex vector bundle that describe the obstructions to constructing independent sections. Let $H^{2i}(B; \mathbb{Z})$ denote the 2i-th singular cohomology group of B with coefficients in $\mathbb{Z}$, here are four axioms which characterize the Chern cohomology.[7]

Axiom 5. For each complex vector bundle E there corresponds a sequence of cohomology classes

$$c_i(E) \in H^{2i}(B(E); \mathbb{Z}), i = 0, 1, 2, ...,$$

called the Chern classes of E. The class $c_0(E)$ is the unit element

$$1 \in H^0(B(E); \mathbb{Z}),$$

and $c_i(E) = 0 \ \forall \ i \geq n$ if E is an n-dimensional bundle.

Axiom 6. If $f : Y \to X$ is covered by a bundle map from E to E', then

$$c_i(E) = f^* c_i(E'),$$

where f^* is the pullback.

Axiom 7. If E and E' are vector bundles over the same base space, then

$$c_n(E \oplus E') = \sum_{i=0}^{n} c_i(E) \cup c_{n-i}(E'),$$

where $\cup$ denotes the cap product. For example, $c_1(E \oplus E') = c_1(E) + c_1(E')$, and $c_2(E \oplus E') = c_2(E) + c_1(E)c_1(E') + c_2(E')$.

Axiom 8. For the line bundle E_1^1 over the circle CP^1(complex projective plane), the Chern class $c_1(E_1^1)$ is -1.

Proposition 6. If Ei is isomorphic to E' then $c_i(E) = c_i(E')$.

Proposition 7. If E is a trivial vector bundle then $c_i(E) = 0$ for $i > 0$.

Proposition 8. If E is trivial then $c_i(E \oplus E') = c_i(E')$

Chapter 7

Čech Cohomology

In general, to define homotopy groups, one must pick a base point. We did not need the base point in our discussion of the first Stiefel-Whitney classes as one does not need a base point to define an element of $\pi_0(\mathbb{Z}_2)$. Similarly, one does not need a base point to discuss an element of $\pi_1(\mathbf{S}^1) = \mathbb{Z} = \mathbf{H}_1(\mathbf{S}^1)$.

In order to work with different base points, one needs to use the cohomology group with twisted coefficients. Then, Čech cohomology is a tool applies abelian sheaf cohomology by using coverings and systems of coefficients on the covering and all non-empty finite intersections. More generally, it applies to non-abelian cohomology, therefore, can be used to compute classes of fiber bundles.

Čech cohomology is obtained using an open cover of a topological space and it arise using purely combinatorial data. The idea being that if one has information about the open sets that make up a space as well as how those sets are glued together one can deduce global properties of the space from the local data.

Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a connected manifold M. For $\alpha_0, \cdots, \alpha_n \in A$, we denote

$$U_{\alpha_0 \cdots \alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$$

or, equivalently, in multi-index notation, if $a = \{\alpha_0, \cdots, \alpha_k\}$

$$U_a = \bigcap_{\alpha_i \in a} U_{\alpha_i}$$

Let X be a topological space, and let $\mathcal{U}$ be an open cover of X. Define a simplicial complex $\mathcal{N}(\mathcal{U})$, called the nerve of the covering as follows:
1.There is one vertex for each element of $\mathcal{U}$ 2.There is one edge for each pair U_1, $U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \varnothing$ 3.There is one k-simplex for each k+1-element subset $\{U_0, \cdots, U_k\}$ of $\mathfrak{U}$ for which $U_0 \cap \cdots \cap U_k \neq \varnothing$

The ideal of Čech cohomology is that, if we choose a nice cover $\mathcal{U}$ containing of sufficiently small open sets, the resulting simplicial complex $\mathcal{N}(\mathcal{U})$ should be a good conmbinatorial model for the space X. For such a cover, the Čech cohomology of X is defined to be the simplicial cohomology of the nerve.

Now let X be a topological space, and Let $\mathcal{F}$ be the abelian group of coefficients. Let $\mathcal{U}$ be an open cover of X. An q-simplex σ of $\mathcal{N}(\mathcal{U})$ is an ordered collection of q+1 sets chosen from $\mathcal{U}$ such that the intersection of all these sets is non-empty. This intersection is called the *support* of σ.

Now let $\sigma = (U_i)_{i \in \{0,\ldots,q\}}$ be such a q-simplex. The j-th partial boundary of σ is defined to be the (q-1)-simplex obtained by removing the j-th set from σ, that is

$$\partial_j\sigma = (U_i)_{i\in\{0,\dots\hat{j},\dots,q\}}$$

the boundary of σ is defined as the alternating sum of the partial boundaries

$$\partial\sigma = \sum_{j=0}^{q}(-1)^j\partial_j\sigma$$

A q-cochain of $\mathcal{U}$ with coefficient in $\mathcal{F}$ is a map which associates to each q-simplex σ and we denote the set of all q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$ by $C^q(\mathcal{U},\mathcal{F})$.

In fact, all one needs is a way to associate an abelian group to an open say $\mathcal{F}(U)$(which functions $U \to \mathcal{F}$) and a homeomorphism $i_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ for the subset $V \hookrightarrow U$ such that

1. $i_U^U = id$ and $i_V^U \circ i_W^V = i_W^U$.
2. Let $U = \cup U_\alpha$, if $i_{U_\alpha}^U \cdot f = i_{U_\alpha}^U \cdot g \ \forall\ \alpha$, then $f = g$.
3. For any $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $i_{U_{\alpha\beta}}^{U_\alpha} \cdot f_\alpha = i_{U_{\beta\alpha}}^{U_\beta} \cdot f_\beta$. Then, $f \in U$ such that $f_\alpha = i_{U_\alpha}^U \cdot f$.

Such a structure is called a Sheaf.

The cochain groups can be made into a cochain complex $(C^k(\mathcal{U},\mathcal{F}),\delta)$ by defining the coboundary operator $\delta_q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F})$ by $(\delta_q\omega) = \sum_{j=0}^{q}(-1)^j res_{|\sigma|}^{|\partial_j\sigma|}\omega(\partial_j\sigma)$, where $res_{|\sigma|}^{|\partial_j\sigma|}\omega(\partial_j\sigma)$ is the restriction morphism on the intersection. It also satisfies the composition that $\delta_{q+1}\circ\delta_q = 0$.

A q-cochain is called a q-cocycle if it is in the kernel of δ, hence $Z^q(\mathcal{U},\mathcal{F}) := \mathbf{ker}(\delta_q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F}))$ is the set of all q-cocycles. Thus a (q-1)-cochain f is a cocycle if for all q-simplices σ the cocycle condition $\sum_{j=0}^{q}(-1)^j res_{|\sigma|}^{|\partial_j\sigma|} f(\partial_j\sigma) = 0$ holds. In particular, a 1-cochain f is a 1-cocycle if $f(B\cap C)|_U - f(A\cap C)|_U + f(A\cap B)|_U = 0\ \forall\ U = A\cap B\cap C$, where $\{A, B, C\} \in U$.

A q-cochain is called a q-coboundary if it is in the image of δ and $B^q(\mathcal{U}, \mathcal{F})\colon = \mathbf{im}(\delta_{q-1} : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F}))$ is the set of all q-coboundaries. For instance, a 1-cochain f is a 1-coboundary if there exists a 0-cochain h such that $f(U) = (\delta h)(U) = h(A)|_U$ - $h(B)|_U$ $\forall$ $U = A\cap B$, where $\{A, B \in \mathcal{U}\}$.

Then, the Čech cohomology of $\mathcal{U}$ with values in $\mathcal{F}$ is defined to be the cohomology of the cochain complex $(C^k(\mathcal{U}, \mathcal{F}), \delta)$. Thus the q-th Čech cohomology is given by

$$\check{\mathbf{H}}^q(\mathcal{U}; \mathcal{F}) = \mathbf{H}^q((C^q(\mathcal{U}, \mathcal{F}), \delta)) = Z^q(\mathcal{U}, \mathcal{F})/B^q(\mathcal{U}, \mathcal{F})$$

The Čech cohomology of X is defined by considering refinements of open covers. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ then there is a map in cohomology $\check{\mathbf{H}}^*(\mathcal{U}, \mathcal{F}) \to \check{\mathbf{H}}^*(\mathcal{V}, \mathcal{F})$. The open covers of $\mathbf{X}$ form a directed set under refinement, so the above map leads to a direct system of abelian groups. The **Cech cohomology** of X with values in F is defined as the direct limit $\check{\mathbf{H}}^*(\mathbf{X}, \mathcal{F}) = \lim_{\to\mathcal{U}} \check{\mathbf{H}}^*(\mathcal{U}, \mathcal{F})$.

Related to other cohomology

If $\mathbf{X}$ is homotopy equivalent to a CW-complex, then the Čech cohomology $\check{\mathbf{H}}^*(\mathcal{U}; \mathcal{F})$ is naturally isomorphic to the singular cohomology $\mathbf{H}^*(\mathcal{U}; \mathcal{F})$. If $\mathbf{X}$ is a differential manifold, then $\check{\mathbf{H}}^*(\mathcal{U}; \mathcal{F})$ is naturally isomorphic to the de Rham cohomology of $\mathbf{X}$. For some less well-behaved spaces that fails for the closed topologist's sine curve, its Čech cohomology $\check{\mathbf{H}}^1(\mathbf{X}; \mathbb{Z}) = \mathbb{Z}$, whereas $\mathbf{H}^1(\mathbf{X}; \mathbb{Z}) = 0$.

Example 11. Compute the $\check{\mathbf{H}}^*(\mathbf{S}^1;\mathbb{Z})$ if the open cover $\mathfrak{U} = U_1 \cup U_2 = \{(x,y)|x^2+y^2=1, y < \frac{1}{2}\} \cup \{(x,y)|x^2+y^2=1, y > -\frac{1}{2}\}$. $U_1 \cap U_2 = \{(x,y)|x^2+y^2=1, -\frac{1}{2} < y < \frac{1}{2}\}$, which is the disconnected two components. Note, for any continuous map $f \in C^0(U_1 \to \mathbb{Z})$, its image $f(U_1)$ is connected since U_1 is connected, but the only connected subspaces of $\mathbb{Z}$ are $\varnothing$ and singleton points which implies that $f(U_1) =$ singleton point, therefore, isomorphic to $\mathbb{Z}$ since the U_1 is not empty. Also, the disconnection of $U_1 \cap U_2$ will give $C^0(U_{12} \to \mathbb{Z})$ two distinct components. Then, $C^0(\mathcal{N}(U)) = C^0(U_1 \to \mathbb{Z}) \oplus C^0(U_2 \to \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $C^1(\mathcal{N}(U)) = C^0(U_{12} \to \mathbb{Z}) = C^0(U_{12}^+ \to \mathbb{Z}) \oplus C^0(U_{12}^- \to \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Define $\delta_{-1} : C^{-1}(\mathcal{N}(U)) \to C^0(\mathcal{N}(U))$, $\delta_0 : C^0(\mathcal{N}(U)) \to C^1(\mathcal{N}(U))$, and $\delta_1 : C^1(\mathcal{N}(U)) \to C^2(\mathcal{N}(U))$. The maps δ_q will all be 0 map since $C^q(\mathcal{N}(U)) = 0 \forall q \geq 2$. Note, for any continuous map $f \in C^0(\mathcal{N}(U))$ has two distinct components with $f = (f_1, f_2)$. Thus, $\delta_0 f = \delta_0(f_1, f_2) = f_1|_{U_{12}} - f_2|_{U_{12}}$, so
$\mathbf{Ker}\delta_0 = \{(f_1,f_2) \in C^0(\mathcal{N}(U))|\delta_0(f_1,f_2) = f_1|_{U_{12}} - f_2|_{U_{12}} = 0\} = \{(f_1,f_2) \in C^0(\mathcal{N}(U))|f_1|_{U_{12}} = f_2|_{U_{12}}\} = \{(f_1,f_2) \in C^0(\mathcal{N}(U))|f_1 = f_2\} \cong (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}$ since f_1 and f_2 are both entirely constant map. And, $\mathbf{Im}\delta_{-1} = 0$ since $C^{-1}(\mathcal{N}(U)) = 0$. Thus, $\check{\mathbf{H}}^0(\mathbf{S}^1;\mathbb{Z}) = \mathbf{Ker}\delta_0/\mathbf{Im}\delta_{-1} = \{f_1, f_2|f_1 = f_2\} \cong \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}$. $\mathbf{Ker}\delta_1 = \{g = (g_1,g_2) \in C^1(\mathcal{N}(U))|\delta_1(g_1,g_2) = 0\} = 0$ since $C^2(\mathcal{N}(U))$ is 0. So, $\mathbf{Ker}\delta_1 = C^1(\mathcal{N}(U))$ which generates by (g_1,g_2), thus, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, where g_1 generates $C^0(U_{12}^+$ and g_2 generates $C^0(U_{12}^-$. And $\mathbf{Im}(\delta_0) = \{g \in C^1(\mathcal{N}(U))|g = \delta_0 f = (f_1,f_2) = f_1|_{U_{12}} - f_2|_{U_{12}} = g_1 + g_2\} \cong \mathbb{Z}$. Thus, $\check{\mathbf{H}}^1(\mathbf{S}^1;\mathbb{Z}) = \mathbf{Ker}\delta_1/\mathbf{Im}\delta_0 = \{g_1,g_2\}/(g_1+g_2) \cong \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}$.

Therefore, $\check{\mathbf{H}}^q(\mathbf{S}^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for} \quad q = 0,1 \\ 0 & \text{for} \quad q \geq 2 \end{cases}$.

As the 1-dimensional sphere can be constructed using CW-complex, its Čech cohomology is the same as its singular cohomology.

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