

OPTIMIZATION OF SOME MULTISTAGE STOCHASTIC MANAGEMENT SYSTEMS

by 45

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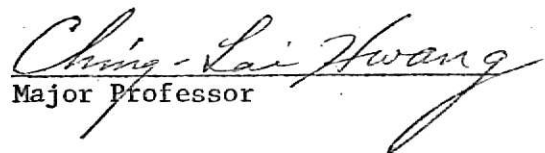
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1. INTRODUCTION

The scope of the field of Operations Research in its present status, is stated as the application of scientific methods, techniques and tools to problems involving the operation of a system so as to provide those in control of the system the optimum solution to the problem. One of the most important phase of operation in conducting an operations research study of a management decision problem is constructing a mathematical model to represent the system under study and deriving a solution from the model.

The mathematical models are generally classified into two categories: 1) Deterministic models and 2) Stochastic models. All the mathematical models are artificial, although some are more realistic than others. In deterministic models, the cause and effect relationships are unequivocally set forth. The advantage of the deterministic model is that it stresses the principle aspects of the problem. As a result, this model often yields equations simple enough to be handled analytically. On the other hand, the stochastic models try to describe the problem more realistically, yet with no attempt to examine each possible outcome of uncertainty. The stochastic system analysis is based on the concepts of the probability theory which is defined as the study of mathematical models of random phenomena (1). A random phenomenon is defined as an empirical phenomenon that obeys probabilistic rather than deterministic laws.

Stochastic problems appear in practical situations for many reasons (2). Some of these are stated below:

- (1) The equations describing the process are only approximations, since the "true" equations are not known. The transformations from stage to stage are probabilistic;
- (2) The state of the system can not be clearly identified;
- (3) The physical, chemical and economic data in the models are estimates of varying degrees of certainty;
- (4) The stochastic variables can not be clearly identified;
- (5) The probability density function is not known;
- (6) The probability density function is known, but the parameters vary;
- (7) The objective of the problems change from time to time;
- (8) A great deal of uncertainty exists in the measurements taken in the process.

The various reasons indicate the occurrence of stochastic processes in the different fields of science and technology. Stochastic systems exist in industrial and management problems involving production scheduling, inventory control, forecasting, the analysis of economic fluctuations and the design of control systems for industrial processes. Also the concepts of quality control and the newly developed science, Queuing Theory, which play an important role in Operations Research are based upon the probability theory. The products which engineers design and produce are used by various segments of the population. Since large numbers of people differing in many ways are always involved, the description can only be probabilistic in nature. Thus the techniques of queuing theory and quality control deal with the stochastic systems in all types of industries. In statistical physics stochastic processes provide models for physical phenomena such as

thermal noise in electrical circuits and the Brownian motion of a particle immersed in a liquid or gas. In addition, randomly varying time dependent functions enter many engineering problems, such as signal detections in the presence of noise. Thus a study and analysis of stochastic systems is seen to be important.

PURPOSE AND SCOPE

The purpose of this report is to formulate stochastic models for some industrial management systems and to obtain a solution to these models by the application of the discrete stochastic maximum principle. The mathematical models considered include different types of models such as a linear model, a non-linear model, with known probability distribution functions and known parameters (the parameters being the mean and variance of the random variables in the model). The models solved here are taken from original deterministic cases and are converted to stochastic models by properly introducing random variables in their structure, thus making their solution more realistic. The main purpose of the report is an attempt to apply the Pontryagin's maximum principle to the optimization of the discrete multi-stage stochastic processes. For all models, a purely probabilistic mathematical model will be formulated and a solution will be sought. In chapter 2 a weakened form of the local maximum principle for stochastic problems is obtained by following the derivation presented in chapter 10 of Fan, et al. (3). In the other chapters the following stochastic models are solved by applying the maximum principle algorithm.

- (1) A three-stage production scheduling problem;
- (2) A five-stage resource allocation problem;

- (3) A hydro-electric water storage system;
- (4) A production and inventory control problem with discrete probability distribution.

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2. BASIC CONCEPTS AND THE STOCHASTIC MAXIMUM PRINCIPLE

RANDOM VARIABLES AND PROBABILITY THEORY (1)

From the mathematical theory of probability, a stochastic process is best defined as a collection $(X(t), t \in T)$ of random variables. The set T is called the index set of the process and no restriction is placed on the nature of T . However, the two important cases arise when $T = (0, 1, 2, \dots)$, in which case the stochastic process is said to be a discrete parameter process, or when $T = (t; -\infty < t < \infty)$ or $T = (t; t \geq 0)$, in which case the stochastic process is said to be a continuous process. A random variable X is a real valued quantity, which has the property that for every set B of real numbers, there exists a probability, denoted by $P(X \in B)$, that X is a member of B . Thus X is a random variable whose values are taken randomly (that is, in accord with a probability distribution).

The expectation, or mean, μ , of a random variable X , denoted by $E(X)$, is defined (when it exists) by

$$\mu = E(X) = \begin{cases} \int_{-\infty}^{\infty} xf(x)dx & \text{for a continuously distributed random variable,} \\ \sum_x xf(x) & \text{for a discrete type random variable,} \end{cases}$$

where $f(x)$ is the probability density function of X . The variance of the random variable X is defined by

$$\text{Var}(X) = \sigma^2 = E(X^2) - [E(X)]^2$$

The concepts of expectation (or mean) and the variance are very useful in the stochastic models. In these models to smooth out the probabilistic behavior of the variables, the kind of averaging procedure that is used is

the problem of expectation. As much as certainty is essential to the deterministic model, expectation is to the stochastic model.

One more concept that is used in the solution of stochastic models from the probability theory is the definition of independent random variables. A set of random variables X_1, X_2, \dots, X_n which are defined as functions on the same sample space are said to be independent if and only if the following statement is true

For all sets B_1, B_2, \dots, B_n of real numbers,

$$\begin{aligned} P(X_1 \text{ is in } B_1, X_2 \text{ is in } B_2, \dots, X_n \text{ is in } B_n) \\ = P(X_1 \text{ is in } B_1)P(X_2 \text{ is in } B_2)\dots P(X_n \text{ is in } B_n) \end{aligned}$$

Hence it follows that a set of n random variables X_1, X_2, \dots, X_n are independent if and only if their joint density is equal to the product of their marginal densities, that is,

$$f(X_1, X_2, \dots, X_n) = f_1(X_1)f_2(X_2)\dots f_n(X_n)$$

For a set of independent random variables, the expectation of the joint density is equal to the product of their marginal densities, that is,

$$E[f(X_1, X_2, \dots, X_n)] = E[f_1(X_1)]E[f_2(X_2)]\dots E[f_n(X_n)]$$

In the following section a weakened form of the local maximum principle for stochastic problems is presented by following the derivation presented in reference (5).

THE STOCHASTIC DISCRETE MAXIMUM PRINCIPLE FOR SIMPLE PROCESSES (3,4,5)

A schematical representation of a simple discrete process is shown in Fig. 1. The process consists of N stages connected in series. The transformation of the process stream at the n th stage is described by a set of performance equations.

$$x^n = f^n(x^{n-1}, \theta^n, \xi^n), \quad n = 1, 2, \dots, N, \quad (1)$$

$x^0 = C$, specified input vector, which may be a random variable with known probability density function,

where

x^n is a s -dimensional state vector at stage n ;

θ^n is a r -dimensional decision (or control vector) at stage n ;

ξ^n is a q -dimensional random vector with known probability density function at stage n .

The observation of the state vector at stage n is made and, in general, described by

$$y^n = G^n(x^n, \eta^n), \quad n = 1, 2, \dots, N \quad (2)$$

where

y^n is a s -dimensional observation vector at stage n ,

η^n is a $\sum_{i=1}^s p_i$ dimensional random vector (disturbance or error)

where p_i is the sources of error for each x_i^n , the number of

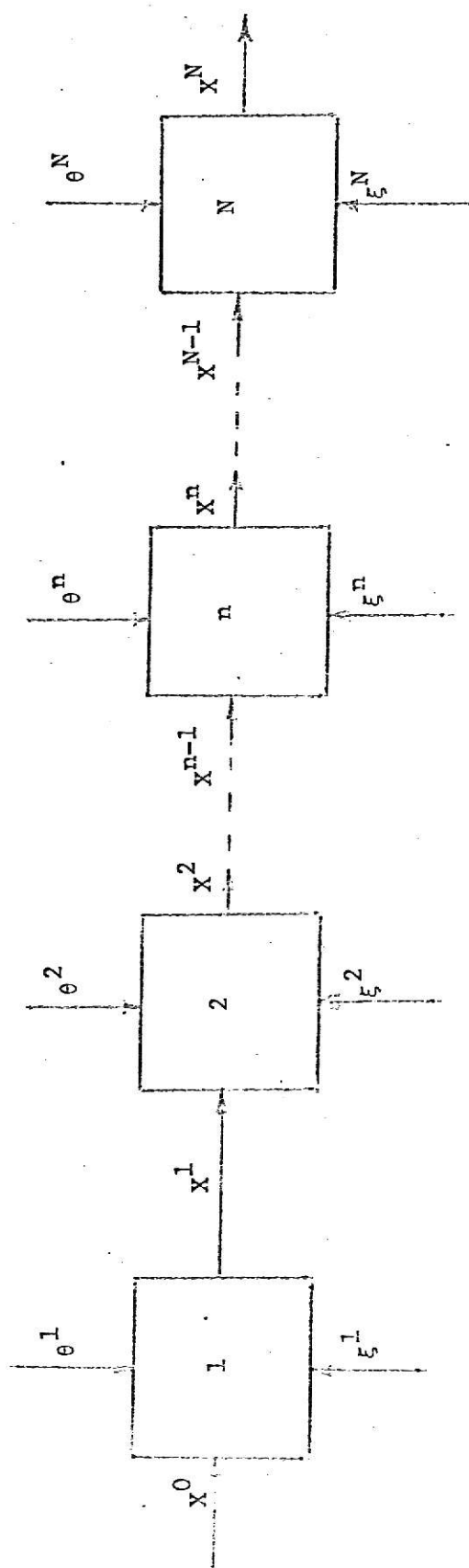


FIG. 1. A SIMPLE MULTISTAGE STOCHASTIC PROCESS

which may be different for each x_i^n .

If we assume that a perfect measurement on x^n is made and no disturbance presents, equation (2) reduces to

$$y^n = G^n(x^n), \quad n = 1, 2, \dots, N \quad (3)$$

In the following development, we assume

$$y^n = x^n, \quad n = 1, 2, \dots, N \quad (4)$$

The optimization problem associated with such a process is to choose a set of θ^n , $n = 1, 2, \dots, N$, satisfying certain conditions, such that the expected value of the scalar function (i.e., objective function)

$$E(S) = E((c)^T x^N) \quad (5)$$

attains its maximum value. Here c is a given column vector of constants and the superscript T denotes the transpose of the column vector.

The conditions that the decision variables must satisfy are

- (1) $\theta^n = \theta^n(x^{n-1})$, i.e., a decision variable is a function of the observed data; (6)
- (2) The values that θ^n can take on are subject to constraints in the form of

$$\psi^n(\theta^n) \geq 0, \quad n = 1, 2, \dots, N \quad (7)$$

A set of decisions θ^n , $n = 1, 2, \dots, N$, which satisfies conditions (1) and (2), is called a control policy.

The method of deriving the discrete stochastic maximum principle is

much like that used in deterministic case. It is first assumed that an optimal solution exists. This optimal solution is then perturbed slightly at stage n , and the resulting change in the objective function is observed. From an analysis of this variation, local conditions on the optimal decision are obtained.

To derive the optimization algorithm for the problem, we assume that

(1) $f^n(x^{n-1}, \theta^n, \xi^n)$, $n = 1, 2, \dots, N$ have continuous second order partial derivatives with respect to x^{n-1} and θ^n and the expected values of these partials are assumed to be uniformly bounded.

(2) The random vectors ξ^n , $n = 1, 2, \dots, N$ are assumed to be independent from stage to stage and the q -dimensional joint distribution function $P(\xi^n)$ is known.

(3) There exists a set of optimal decisions, denoted by $\bar{\theta}^n$, $n = 1, 2, \dots, N$, satisfying the conditions (6) and (7), for which the objective function (5) is maximized.

Corresponding to these optimal $\bar{\theta}^n$, the optimal trajectory is described by

$$\bar{x}^n = f^n(\bar{x}^{n-1}, \bar{\theta}^n(\bar{x}^{n-1}), \xi^n), \quad n = 1, 2, \dots, N \quad (8)$$

and

$$\bar{\theta}^n = \bar{\theta}^n(y^{n-1}) = \bar{\theta}^n(\bar{x}^{n-1}), \quad n = 1, 2, \dots, N \quad (9)$$

The objective function then attains its maximum value, i.e.,

$$E((c)^T x^N) = \max_{(\bar{\theta}^n)} E((c)^T x^N) \quad (10)$$

Suppose now that equation (1) represents a system perturbed away from its optimal state given by equation (8), due to a small perturbation in the state vectors. The variational equation of the perturbed system about the optimal system is then given by

$$\delta x^n = \left(\frac{\partial f^n}{\partial x^{n-1}} + \frac{\partial f^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x^{n-1}} \right) \delta x^{n-1} + 0(x^{n-1} - \bar{x}^{n-1}) \quad (11)$$

$$\left. \begin{aligned} x^{n-1} &= \bar{x}^{n-1} \\ \theta^n &= \bar{\theta}^n(\bar{x}^{n-1}) \end{aligned} \right\}$$

where $0(x^{n-1} - \bar{x}^{n-1})$ represents terms of higher order derivatives, and

$$\delta x^n = x^n - \bar{x}^n$$

The matrices $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial \theta}$ and $\frac{\partial \theta}{\partial x}$ are the Jacobian matrices of the functions f and θ . We now define a set of adjoint vectors by

$$(z^{n-1})^T = (z^n)^T F^n, \quad n = 1, 2, \dots, N \quad (12)$$

where

$$F^n = \left(\frac{\partial f^n}{\partial x^{n-1}} + \frac{\partial f^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x^{n-1}} \right) \left. \begin{aligned} x^{n-1} &= \bar{x}^{n-1}, \\ \theta^n &= \bar{\theta}^n(\bar{x}^{n-1}) \end{aligned} \right\} \quad (13)$$

$$z^N = c \quad (14)$$

Let the scalar product of z^n and $f^n(x^{n-1}, \theta^n, \xi^n)$ be denoted by H^n ; i.e.,

$$H^n(z^n, x^{n-1}, \theta^n) = (z^n)^T f^n(x^{n-1}, \theta^n, \xi^n) \quad (15)$$

Then we have the following necessary condition for optimality.

Theorem 1

Let the optimal decision $\bar{\theta}^n$, $n = 1, 2, \dots, N$ exist, and have continuous second derivative almost surely. Then there exists a solution to equations (12), (13) and (14) with the following properties:

$$(1) \quad E(H^N(z^N, \bar{x}^{N-1}, \bar{\theta}^N) \mid \bar{x}^{N-1}) \geq E(H^N(z^N, \bar{x}^{N-1}, \theta^N) \mid \bar{x}^{N-1}) \text{ a.s.} \quad (16)^*$$

(2) If $1 \leq n \leq N$ and if $\theta^n = \bar{\theta}^n + \delta \theta^n$ is allowed to lie on boundary of the constraint, $\psi^n(\theta^n) \geq 0$, then

$$E(H^n(z^n, \bar{x}^{n-1}, \bar{\theta}^n) \mid \bar{x}^{n-1}) \geq E(H^n(z^n, \bar{x}^{n-1}, \theta^n) \mid \bar{x}^{n-1}) + O(\epsilon^2) \text{ a.s.} \quad (17)$$

Note that the almost surely relation in the above theorem and in the sequel are stated with respect to the probability distribution induced by $\bar{\theta}^n$. It may then happen that the function $\bar{\theta}^n(\bar{x}^{n-1})$ have points at which continuity of second derivative fail to exist but the probability associated with these points is zero.

* a.s. is the abbreviation for almost surely.

The second assertion (2) above is not a local maximum condition but rather a stationary condition because of the appearance of higher order terms, $O(\epsilon^2)$, if the optimal solution lies within the constraints.

If the system is linear and the optimal solution is also linear, then the necessary condition can be formulated as a global maximum condition on H^n , i.e., if equation (1) can be written as

$$\dot{x}^n = A(\xi^n) x^{n-1} + B(\xi^n) \theta^n + C(\xi^n), \quad n = 1, 2, \dots, N \quad (18)$$

and if the optimal solution exists and is almost surely linear, then

$$E(H^n(z^n, \bar{x}^{n-1}, \bar{\theta}^n | \bar{x}^{n-1})) \geq E(H^n(z^n, \bar{x}^{n-1}, \theta^n | \bar{x}^{n-1})) \text{ a.s.} \quad (19)$$

The equations which described the adjoint vectors by equations (12), and (13) are somewhat more complicated than was the case for the analogous quantities in a deterministic system. This is due to the effect of perturbations in a decision on a future decision. However in many situations, equation (13) takes on a form identical to that of the deterministic case. This is stated in the following theorem.

Theorem 2.

Let the optimal decision θ^n , $n = 1, 2, \dots, N$ exist and have continuous second derivatives a.s.

At stage m if θ^m is not constrained or θ^m lies within the constraint, then, equation (12) of the adjoint vector can be replaced by

$$(\tilde{z}^{m-1})^T = (z^m)^T \tilde{F}^m \quad (20)$$

where

$$\tilde{F}^m = \frac{\partial f^m}{\partial x^{m-1}} \bigg|_{\substack{\bar{x}^{m-1} \\ \theta^m}} \quad (21)$$

with the following property,

$$E(\tilde{H}^{m-1}(\tilde{z}^{m-1}, \bar{x}^{m-2}, \bar{\theta}^{m-1}) \mid \bar{x}^{m-2}) \geq E(\tilde{H}^{m-1}(\tilde{z}^{m-1}, \bar{x}^{m-2}, \theta^{m-1}) \mid \bar{x}^{m-2}) + o(\epsilon^2) \quad (22)$$

Theorem 1 and 2 are very useful in the solution of stochastic models. In the following chapters, the above stated discrete stochastic maximum principle algorithm is used to obtain the solution for some stochastic management models.

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3. PRODUCTION SCHEDULING MODEL

INTRODUCTION

This study deals with the problem of optimal production planning. In any industry production is the basic operation performed to fulfill the objectives of the company. The manufactured products are sold out according to the demand for the product from the customer population. By production scheduling it is meant here that the rate of production of a commodity is determined over a period of time so that a desired criterion is optimized in the system. The criteria that are usually considered in the system analysis are

- (1) Maximization of the total profit derived from the system performance.
- (2) Minimization of the total cost of expenditure incurred in the operation of the system.

Here in the present study a mathematical model is formulated to represent a multi-stage production planning process considering the factors of production level, change in production level from time to time and the demand for the commodity. This model has been originally formulated and solved in detail using the discrete maximum principle by Hwang, Tillman, and Fan (1). Here in the present investigation the demand parameter (which was treated as a constant fixed quantity during each period in the original deterministic version) is considered as a random variable with known distribution and with known parameters (i.e., with known values of the mean demand during each period). The consideration of demand factor as a random variable fits the real situation more suitably since the demand for any

commodity from a large consumer population is always unpredictable and is expected to vary instantaneously from time to time. A general solution is first obtained as an n-stage process and then a numerical solution is presented for a four-stage process.

A PLANNING OF OPTIMUM PRODUCTION LEVEL

The problem is to determine the production level for a perishable commodity. The excess production over the actual demand is wasted at a cost of \$16 per unit. The cost of changing the production level is four times the square of the difference between two production levels. The demand for the product is unpredictable. It varies from time to time during each quarter. In statistical terms we say the demand is an independent random variable for each period. From the past history of the demand variations during each period a demand distribution can be obtained and from the distribution a mean demand level (or expected value of the demand) during each period can be estimated by the standard statistical methods (2). Hence for the following known values of the expectation of the demand during each period the problem is to determine the production level at each period which minimizes the cost. The last quarter production was 140 units.

Quarter	1	2	3	4
Expectation of Demand	115	125	100	95

SOLUTION BY THE STOCHASTIC MAXIMUM PRINCIPLE

In order to apply the discrete stochastic maximum principle each quarter is defined as a stage as shown in Fig. 1, and let

x_1^n = production level at nth stage (quarter),

θ^n = the change in the production level from the (n-1)th stage to the nth stage,

ξ^n = the demand at the nth stage (an independent random variable with specified mean).

Then the production level at the nth stage can be specified as

$$x_1^n = x_1^{n-1} + \theta^n, \quad n = 1, 2, \dots, N \quad (1)$$

$$\text{where } x_1^n \geq \xi^n \text{ and } x_1^0 = 140 \quad (2)$$

Now let

x_2^n = the sum of costs upto and including the nth stage (quarter).

Case 1 Linear Cost Function

Now let us consider the case when the cost function is linear with respect to the random variable, i.e., we consider the case when the unit cost of wastage of excess production over the actual demand is directly proportional to the quantity of excess production. Then,

$$x_2^n = x_2^{n-1} + 4(\theta^n)^2 + 16(x_1^n - \xi^n) \quad (3)$$

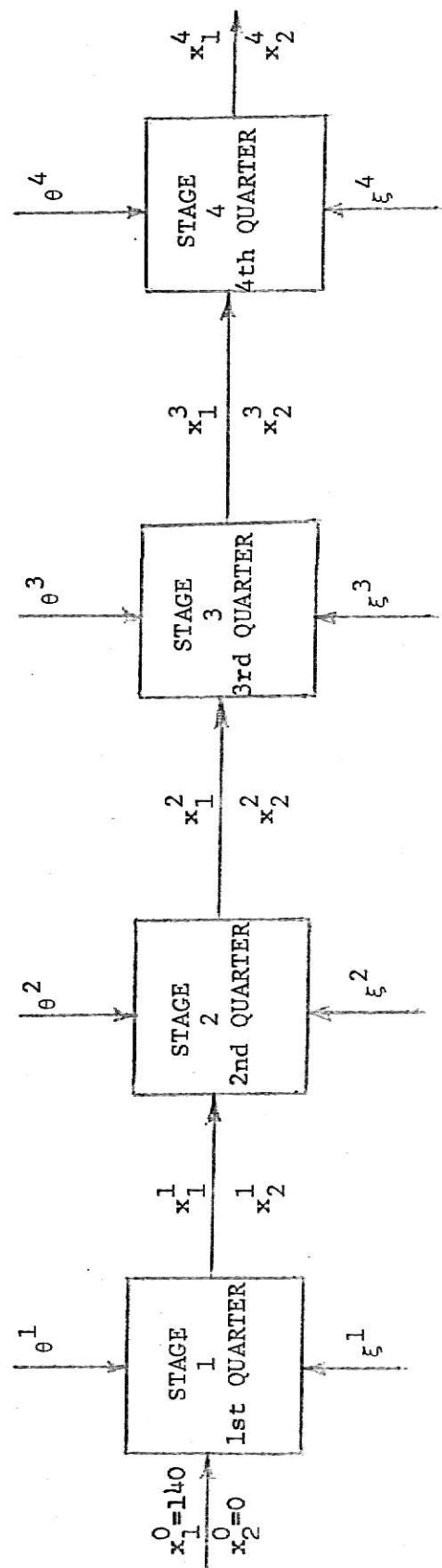


FIG. 1. MULTISTAGE PRODUCTION PLANNING PROCESS

$$x_2^0 = 0$$

Substitution of equation (1) into equation (3) yields

$$x_2^n = x_2^{n-1} + 4(\theta^n)^2 + 16(x_1^{n-1} + \theta^n - \xi^n) \quad n = 1, 2, \dots, N \quad (4)$$

The objective function to be minimized can be written as follows

$$E(S) = E(x_2^N) \quad (5)$$

with $c_1 = 0$ and $c_2 = 1$.

According to equation (2-15), we define the Hamiltonian function as

$$\begin{aligned} H^n &= z_1^n f_1^n + z_2^n f_2^n \\ &= z_1^n(x_1^{n-1} + \theta^n) + z_2^n[x_1^{n-1} + 4(\theta^n)^2 \\ &\quad + 16(x_1^{n-1} + \theta^n - \xi^n)], \quad n = 1, 2, \dots, N \end{aligned} \quad (6)$$

Since θ^n , $n = 1, 2, \dots, N$, are not constrained, we apply Theorem 2.

Thus,

$$(z_1^{n-1}, z_2^{n-1}) = (z_1^n, z_2^n) \begin{pmatrix} \frac{\partial f_1^n}{\partial x_1^{n-1}}, \frac{\partial f_1^n}{\partial x_2^{n-1}} \\ \frac{\partial f_2^n}{\partial x_1^{n-1}}, \frac{\partial f_2^n}{\partial x_2^{n-1}} \end{pmatrix}$$

$$= (z_1^n, z_2^n) \begin{pmatrix} 1, & 0 \\ 16, & 1 \end{pmatrix} \quad (7)$$

$$z_1^{n-1} = z_1^n + 16z_2^n \quad n = 1, 2, \dots, N$$

$$z_1^N = c_1 = 0 \quad (8)$$

$$z_2^{n-1} = z_2^n \quad n = 1, 2, \dots, N$$

$$z_2^N = c_2 = 1$$

Equation (8) can be further simplified to

$$z_1^{n-1} = z_1^n + 16, \quad n = 1, 2, \dots, N$$

$$z_1^N = 0 \quad (9)$$

$$z_2^n = 1, \quad n = 1, 2, \dots, N$$

Hence the Hamiltonian becomes

$$\begin{aligned} H^n &= z_1^n (x_1^{n-1} + \theta^n) + x_2^{n-1} + 16(x_1^{n-1} + \theta^n - \xi^n) \\ &\quad + 4(\theta^n)^2 \end{aligned} \quad (10)$$

Then the expecting value of the Hamiltonian becomes

$$\begin{aligned}
E(H^n | \bar{x}^{n-1}) &= E[z_1^n(x_1^{n-1} + \theta^n) + x_2^{n-1} + 16(x_1^{n-1} + \theta^n \\
&\quad - \xi^n) + 4(\theta^n)^2] \\
&= E(z_1^n)(x_1^{n-1} + \theta^n) + x_2^{n-1} + E[16(x_1^{n-1} + \theta^n - \xi^n)] + 4(\theta^n)^2 \\
&= x_2^{n-1} + x_1^{n-1} [E(z_1^n) + 16] + 4(\theta^n)^2 + \theta^n [E(z_1^n) + 16] - 16\mu^n
\end{aligned}$$

The stationary necessary condition for optimality is

$$E\left[\frac{\partial H^n}{\partial \theta^n} \mid \bar{x}^{n-1}\right] = 0, \quad n = 1, 2, \dots, N \quad (11)$$

It follows that

$$E\left[\frac{\partial H^n}{\partial \theta^n} \mid \bar{x}^{n-1}\right] = 8\theta^n + E(z_1^n) + 16 = 0$$

$$\text{or} \quad \bar{\theta}^n = \frac{-E(z_1^n)}{8} - 2 \quad (12)$$

Equation (12) gives the optimal decision at stage n .

From equation (9), we obtain

$$E(z_1^4) = 0$$

$$E(z_1^3) = 16$$

$$E(z_1^2) = 32$$

$$E(z_1^1) = 48$$

Substituting these values in (12), the optimal decisions are

$$\bar{\theta}^1 = -8$$

$$\bar{\theta}^2 = -6$$

$$\bar{\theta}^3 = -4$$

$$\bar{\theta}^4 = -2$$

From this, along with equation (1), the production level at each stage, where $x_1^n \geq \xi^n$, is determined as follows:

$$x_1^1 = \text{Max} \begin{cases} x_1^0 + \theta^1 = 140 - 8 = 132 \\ \mu^1 = 115 \end{cases}$$

$$= \underline{132}$$

$$x_1^2 = \text{Max} \begin{cases} x_1^1 + \theta^2 = 132 - 6 = 126 \\ \mu^2 = 125 \end{cases}$$

$$= \underline{126}$$

$$x_1^3 = \text{Max} \begin{cases} x_1^2 + \theta^3 = 126 - 4 = 122 \\ \mu^3 = 100 \end{cases}$$

$$= \underline{122}$$

$$x_1^4 = \text{Max} \begin{cases} x_1^3 + \theta^4 = 122 - 2 = 120 \\ \mu^4 = 95 \end{cases}$$

$$= \underline{120}$$

It is noted above that the determination of the optimum production level at each stage was not effected by the restriction to meet actual demand (as in the deterministic case), and thus this is the optimal solution. Using this optimal policy and according to equation (3), the minimum cost is

$$x_2^N = x_2^4 = \$1520$$

Case 2 Quadratic Cost Function

Here the situation when the unit cost of wastage of excess production over the actual demand or shortage cost is proportional to the square of the quantity of production is considered. In this case, the cost function is quadratic with respect to the random variable and is defined by

$$x_2^n = x_2^{n-1} + 4(\theta^n)^2 + 16(x_1^n - \xi^n)^2 \quad (13)$$

Then substituting equation (1) into equation (13) and simplifying, we get

$$x_2^n = x_2^{n-1} + 16(x_1^{n-1})^2 - 32x_1^{n-1}\xi^n + 20(\theta^n)^2 + 32\theta^n(x_1^{n-1} - \xi^n) + 16(\xi^n)^2, \quad n = 1, 2, \dots, N$$

The objective function to be minimized is

$$E(S) = E(x_2^N)$$

with $c_1 = 0$ and $c_2 = 1$

The Hamiltonian function is defined as

$$\begin{aligned} H^n &= z_1^n f_1^n + z_2^n f_2^n \\ &= z_1^n (x_1^{n-1} + \theta^n) + z_2^n [x_2^{n-1} + 16(x_1^{n-1})^2 - 32x_1^{n-1}\xi^n \\ &\quad + 20(\theta^n)^2 + 32\theta^n(x_1^{n-1} - \xi^n) + 16(\xi^n)^2] \end{aligned}$$

Since θ^n , $n = 1, 2, \dots, N$, are not constrained, we apply Theorem 2. The adjoint vectors become

$$\begin{aligned} (z_1^{n-1}, z_2^{n-1}) &= (z_1^n, z_2^n) \begin{pmatrix} \frac{\partial f_1^n}{\partial x_1^{n-1}}, & \frac{\partial f_1^n}{\partial x_2^{n-1}} \\ \frac{\partial f_2^n}{\partial x_1^{n-1}}, & \frac{\partial f_2^n}{\partial x_2^{n-1}} \end{pmatrix} \\ &= (z_1^n, z_2^n) \begin{pmatrix} 1, & 0 \\ 32(x_1^{n-1} - \xi^n + \theta^n), & 1 \end{pmatrix} \end{aligned} \quad (14)$$

$$z_1^{n-1} = z_1^n + 32z_2^n(x_1^{n-1} - \xi^n + \theta^n), \quad n = 1, 2, \dots, N$$

$$z_1^N = c_1 = 0$$

(15)

$$z_2^{n-1} = z_2^n, \quad n = 1, 2, \dots, N$$

$$z_2^N = c_2 = 1$$

Equation (15) can be further simplified to

$$z_1^{n-1} = z_1^n + 32(x_1^{n-1} - \xi^n + \theta^n), \quad n = 1, 2, \dots, N$$

$$z_1^N = 0$$

(16)

$$z_2^n = 1, \quad n = 1, 2, \dots, N$$

Taking expectation, from equation (16) we have

$$E(z_1^{n-1}) = E(z_1^n) + 32(x_1^{n-1} - \xi^n + \theta^n), \quad n = 1, 2, \dots, N$$

(17)

$$E(z_1^N) = 0$$

The Hamiltonian function becomes

$$\begin{aligned}
H^n = & z_1^n (x_1^{n-1} + \theta^n) + x_2^{n-1} + 16(x_1^{n-1})^2 - 32x_1^{n-1}\xi^n \\
& + 20(\theta^n)^2 + 32\theta^n(x_1^{n-1} - \xi^n) + 16(\xi^n)^2
\end{aligned} \tag{18}$$

The expecting value of the Hamiltonian becomes

$$\begin{aligned}
E(H^n \mid \bar{x}^{n-1}) &= E[z_1^n (x_1^{n-1} + \theta^n) + x_2^{n-1} + 16(x_1^{n-1})^2 \\
&\quad - 32x_1^{n-1}\xi^n + 20(\theta^n)^2 + 32\theta^n(x_1^{n-1} - \xi^n) \\
&\quad + 16(\xi^n)^2] \\
&= E(z_1^n)(x_1^{n-1} + \theta^n) + x_2^{n-1} + 16(x_1^{n-1})^2 \\
&\quad - 32x_1^{n-1}\mu_1^n + 20(\theta^n)^2 + 32\theta^n(x_1^{n-1} - \mu^n) + 16E[(\xi^n)^2] \\
&= x_2^{n-1} + 16(x_1^{n-1})^2 + x_1^{n-1}[E(z_1^n) - 32\mu^n] + 20(\theta^n)^2 \\
&\quad + \theta^n[E(z_1^n) + 32(x_1^{n-1} - \mu^n)] + 16E[(\xi^n)^2]
\end{aligned} \tag{19}$$

The stationary necessary condition for optimality is

$$E\left[\frac{\partial H^n}{\partial \theta^n} \mid \bar{x}^{n-1}\right] = 40\theta^n + E(z_1^n) + 32(x_1^{n-1} - \mu^n) = 0$$

Hence it follows that

$$\bar{\theta}^n = -\frac{E(z_1^n)}{40} - \frac{4}{5} (x_1^{n-1} - \mu^n) \quad (20)$$

or

Equation (20) gives the optimal decision at stage n . From equation (17), we obtain

$$E(z_1^4) = 0$$

$$E(z_1^3) = 32 (\bar{x}_1^3 + \bar{\theta}^4 - \mu^4)$$

$$E(z_1^2) = 32 (\bar{x}_1^3 + \bar{\theta}^4 - \mu^4 + \bar{x}_1^2 + \bar{\theta}^3 - \mu^3)$$

$$E(z_1^1) = 32 (\bar{x}_1^3 + \bar{\theta}^4 - \mu^4 + \bar{x}_1^2 + \bar{\theta}^4 - \mu^3 + \bar{x}_1^1 + \bar{\theta}^2 - \mu^2)$$

Substituting these values into equation (20) and simplifying, we get the following set of equations for the optimal decisions.

$$.8\bar{\theta}^1 + .8\bar{\theta}^2 + .8\bar{\theta}^3 + \bar{\theta}^4 = -36$$

$$.8\bar{\theta}^1 + .8\bar{\theta}^2 + \bar{\theta}^3 + .2\bar{\theta}^4 = -32$$

$$.8\bar{\theta}^1 + \bar{\theta}^2 - .2\bar{\theta}^3 = -12$$

$$\bar{\theta}^1 - .2\bar{\theta}^2 = -20$$

Solving these equations, the optimal decisions are

$$\bar{\theta}^1 = -20$$

$$\bar{\theta}^2 = 0$$

$$\bar{\theta}^3 = -17$$

$$\bar{\theta}^4 = -6$$

From this, along with equation (1), the optimal production level at each stage, where $x_1^n \geq \xi^n$, is determined as follows:

$$x_1^1 = \text{Max} \begin{cases} x_1^0 + \theta^1 = 140 - 20 = 120 \\ \mu^1 = 115 \end{cases}$$

$$= \underline{120}$$

$$x_1^2 = \text{Max} \begin{cases} x_1^1 + \theta^2 = 120 + 0 = 120 \\ \mu^2 = 125 \end{cases}$$

$$= \underline{125}$$

$$x_1^3 = \text{Max} \begin{cases} x_1^2 + \theta^3 = 125 - 17 = 108 \\ \mu^3 = 100 \end{cases}$$

$$= \underline{108}$$

$$x_1^4 = \text{Max} \begin{cases} x_1^3 + \theta^4 = 108 - 6 = 102 \\ \mu^4 = 95 \end{cases}$$

$$= \underline{102}$$

Using this optimal policy and according to equation (13), the minimum cost is

$$x_2^N = x_2^4 = \$5108$$

DISCUSSION OF THE RESULTS

It is found that the optimal control policy obtained with the linear cost function by the stochastic maximum principle is exactly the same as that obtained by using the deterministic maximum principle (1). This relationship is known as the certainty equivalence principle (3), (4), which is stated as: "The procedure to obtain control policies for stochastic systems is by considering the optimal control policies for the related deterministic systems where the random variables are replaced by their expected values." However, this principle is true only in certain classes of control systems, e.g., linear system equations with additive random variables (as in equation (1)) and objective functions of quadratic

form. For example a system like

$$x^n = ax^{n-1} + \xi_{\theta}^n,$$

where ξ^n is a random variable, this principle does not apply. Another point worth noting is that equations (12) and (20) for the optimal policies are obtained from the stationary necessary condition given by equation (11). This means that a policy or decision determined by the use of equation (11) is not necessarily an optimal policy. Equation (11) only provides a "candidate or candidates" for the optimal policy. In general, the second order variation of the objective function around the candidate policy must be examined in order to determine if it is indeed the optimal policy. It is very difficult, if not impossible, to do so for any sort of a complex discrete system and we often have to resort to other procedures.

In the case where the cost function is linear with respect to the random variable, it is observed that the determination of the optimum production level is independent of the expected value of the demand during any quarter (stage); however, for the case where the cost function is quadratic with respect to the random variable, the mean demand fixes the optimum production level in the second quarter and thus influences the other production levels also. The optimum production level as determined by the optimum control policy, when it is greater than the expected value of the demand during a stage, indicates that there is always an excess production available over the mean demand to meet small fluctuations in the demand over the mean value during that stage.

It is also found that for both linear and quadratic cost functions the optimal policy and the corresponding optimal production levels are not influenced by the demand variable when the expected values are considered. Another difference noted is that for linear cost function, the optimal decisions do not depend upon the previous quarter optimal decision but depend only on the expected demand during that quarter whereas for the quadratic cost function, the optimal policy for any quarter is dependent on both the previous quarter optimal decision and the expected demand for the present quarter. However the validity of the assumption for the cost function as linear or quadratic should be first analyzed carefully in any practical situation before any application of the above shown procedures.

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4. RESOURCE ALLOCATION MODEL

INTRODUCTION

This study deals with the problem of allocating certain amount of capital available for investment into two different stocks. The criterion considered is the total return or the profit earned from the money invested in the two stocks. Hence the problem is to find the proportion of the total money that is to be invested in each stock so that the total profit is maximized. In any capital market, the dividends from the different stocks fluctuate much through the years due to many different reasons like the nature of the industry, the type of demand, the economic conditions of the industry, etc. Hence here the rate of dividend from each stock is considered as a random variable in agreement to the realistic conditions. Also because of the above stated reasons, the net worth of a stock through different periods is subjected to change. Hence accordingly the rate of appreciation (or depreciation) of any particular stock with time is taken as an independently distributed random variable.

The mathematical model considered here has been originally formulated and solved using the discrete maximum principle by Hwang and Fan (1) in the deterministic way (i.e., considering the dividend rate and rate of appreciation for any period as fixed known values). In the present study a stochastic version of the above allocation system is formulated introducing properly defined random variables and is solved by the stochastic maximum principle for a five-stage process.

AN N-STAGE ALLOCATION PROCESS

Let us consider an N-stage allocation process for which a stage represents one year. The first state variable is the capital available in dollars to be invested. The decision variable is the amount of capital at each year that will be invested in stock A, with the remaining capital being invested in stock B. The expected dividends from the two stocks, denoted by d_A and d_B , are considered as random variables with known probability distribution functions. Hence let μ_{DA} and μ_{DB} denote the expected value of the annual dividend rates for the stocks A and B respectively. Similarly the expected stock appreciation of each stock, denoted by g_A and g_B , for stocks A and B respectively is also considered as random variables with known probability distribution functions and known mean values (denoted by μ_{GA} and μ_{GB} respectively for the stocks A and B). The distribution function and the expected values of the above two random variables are obtained from the past history of the two stocks A and B. The problem is to find the optimal policy that will maximize dividends, if the initially available capital is a .

Let

x_1^{n-1} = amount of capital available for investment at the
nth stage (year),

x_2^n = sum of dividend up to and including the nth stage
(year),

θ^n = amount of capital invested in stock A at the nth
stage (year).

Then the amount of capital invested in stock B at the nth stage will be

$x_1^{n-1} - \theta^n$. The transformation equations can be written as

$$x_1^n = (1 + g_A) \theta^n + (1 + g_B) (x_1^{n-1} - \theta^n), \quad n = 1, 2, \dots, N \quad (1)$$

$$x_1^0 = a \quad (2)$$

$$x_2^n = x_2^{n-1} + [d_A \theta^n + d_B (x_1^{n-1} - \theta^n)] \quad (3)$$

$$x_2^0 = 0 \quad (4)$$

where the dividend earned at the n th stage (year) is $d_A \theta^n + d_B (x_1^{n-1} - \theta^n)$

and x_2^N is the total amount of dividend earned in N years.

The objective function is the expected value of the total dividend; that is,

$$E(S) = E(x_2^N) \quad (5)$$

Therefore, $c_1 = 0$,

and $c_2 = 1$.

The optimization problem is to choose a set of decisions, θ^n , $n = 1, 2, \dots, N$, subject to the constraint,

$$0 \leq \theta^n \leq x_1^{n-1} \quad (6)$$

so that the objective function given by equation (5) is maximized.

According to equation (2 - 15), the Hamiltonian function is defined as

$$\begin{aligned}
 H^n &= z_1^n f_1^n + z_2^n f_2^n \\
 &= z_1^n [(1 + g_A)\theta^n + (1 + g_B)(x_1^{n-1} - \theta^n)] \\
 &\quad + z_2^n [x_2^{n-1} + d_A\theta^n + d_B(x_1^{n-1} - \theta^n)]
 \end{aligned} \tag{7}$$

Since θ^n , $n = 1, 2, \dots, N$, are constrained, we have to apply theorem 1. Hence the adjoint vectors are defined by

$$\begin{aligned}
 &(z_1^{n-1}, z_2^{n-1}) \\
 &= (z_1^n, z_2^n) \begin{pmatrix} (1 + g_B) + \{(1 + g_A) - (1 + g_B)\} \frac{\partial \theta^n}{\partial x_1^{n-1}}, & 0 \\ d_B + (d_A - d_B) \frac{\partial \theta^n}{\partial x_1^{n-1}}, & 1 \end{pmatrix}
 \end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned}
 z_1^{n-1} &= z_1^n \left\{ (1 + g_A) + [(1 + g_A) - (1 + g_B)] \frac{\partial \theta^n}{\partial x_1^{n-1}} \right\} \\
 &\quad + z_2^n \left\{ d_B + (d_A - d_B) \frac{\partial \theta^n}{\partial x_1^{n-1}} \right\} \quad n = 1, 2, \dots, N
 \end{aligned} \tag{9}$$

$$z_1^N = c_1 = 0$$

$$z_2^{n-1} = z_2^n, \quad n = 1, 2, \dots, N$$

$$z_2^N = c_2 = 1$$

Equation (9) can be further simplified to

$$\begin{aligned} z_1^{n-1} = z_1^n & \left\{ (1 + g_A) + [(1 + g_A) - (1 + g_B)] \frac{\partial \theta^n}{\partial x_1^{n-1}} \right\} \\ & + \left\{ d_B + (d_A - d_B) \frac{\partial \theta^n}{\partial x_1^{n-1}} \right\}, \quad n = 1, 2, \dots, N \end{aligned} \quad (10)$$

$$z_1^N = 0$$

$$z_2^n = 1, \quad n = 1, 2, \dots, N$$

Hence the Hamiltonian becomes

$$\begin{aligned} H^n = z_1^n & \{ (1 + g_A) \theta^n + (1 + g_B) (x_1^{n-1} - \theta^n) \} \\ & + \{ x_2^{n-1} + d_A \theta^n + d_B (x_1^{n-1} - \theta^n) \} \end{aligned} \quad (11)$$

$$\begin{aligned}
E[H^n \mid \bar{x}^{n-1}] &= E(z_1^n) \{ (1 + \mu_{GA}) \theta^n + (1 + \mu_{GB}) (x_1^{n-1} - \theta^n) \} + x_2^{n-1} \\
&\quad + \mu_{DA} \theta^n + \mu_{DB} (x_1^{n-1} - \theta^n) \\
&= x_2^{n-1} + x_1^{n-1} \{ E(z_1^n) (1 + \mu_{GB}) + \mu_{DB} \} + \theta^n \{ E(z_1^n) (\mu_{GA} - \mu_{GB}) \\
&\quad + (\mu_{DA} - \mu_{DB}) \} , \quad n = 1, 2, \dots, N
\end{aligned} \tag{12}$$

Since the values of $E(z_1^n)$, x_1^{n-1} and x_2^{n-1} at the n th stage are considered constants in extremizing the expected value of the Hamiltonian function H^n , the variable portion of H^n as given by equation (12) is

$$E[H_V^n \mid \bar{x}^{n-1}] = [E(z_1^n) (\mu_{GA} - \mu_{GB}) + (\mu_{DA} - \mu_{DB}) \theta^n] \tag{13}$$

The values of μ_{GA} , μ_{GB} , μ_{DA} , μ_{DB} and $E(z_1^n)$ are constant. Therefore, the variable portion of the Hamiltonian function, H_V^n , becomes a linear function of θ^n . The optimal value of θ^n that makes H_V^n maximum should occur at a boundary of the admissible region of θ^n , ($0 \leq \theta^n \leq x_1^{n-1}$). The sign of q^n is given by

$$q^n = E(z_1^n) (\mu_{GA} - \mu_{GB}) + (\mu_{DA} - \mu_{DB}) \tag{14}$$

decides in which one of the boundaries $\bar{\theta}^n$ lies. For a positive value of q^n , $\bar{\theta}^n$ is x_1^{n-1} , which is equivalent to investing all the money in stock A, and for a negative q^n , $\bar{\theta}^n$ is zero which is equivalent to investing all the

money in stock B. Summarizing, we have the optimal decisions at stage n as

$$\bar{\theta}^n = x_1^{n-1} \quad \text{when} \quad q^n > 0$$

$$\bar{\theta}^n = 0 \quad \text{when} \quad q^n < 0$$

$$0 \leq \bar{\theta}^n \leq x_1^{n-1} \quad \text{when} \quad q^n = 0$$

NUMERICAL EXAMPLE

Let us consider a numerical example of a five-year or stage-allocation process ($N = 5$). Let stock A be that of a growing company with a mean stock appreciation of 15 per cent ($\mu_{GA} = 0.15$) per year and a mean dividend of 8 per cent ($\mu_{DA} = 0.08$). Let stock B be that of a gold mining company whose mine is being depleted and the stock is expected to depreciate every year. Let the expected value of stock depreciation of the stock B be 20 per cent ($\mu_{GB} = -0.20$) per year, but the expected value of the dividend is 30 per cent ($\mu_{DB} = 0.30$) per year. Let the initial available capital be \$10,000 ($a = 10,000$). The problem is to find the optimal policy that will maximize the total dividend from the two stocks.

From equation (10), we obtain

$$E(z_1^5) = 0$$

$$E(z_1^4) = 0.3$$

$$E(z_1^3) = 0.54$$

$$E(z_1^2) = 0.732$$

$$E(z_1^1) = 0.8856$$

Substituting these values of $E(z_1^n)$ into equation (14), we obtain

$$q^1 = 0.08996$$

$$q^2 = 0.0362$$

$$q^3 = -0.031$$

$$q^4 = -0.115$$

$$q^5 = -0.22$$

Therefore, the optimal decisions are

$$\bar{\theta}^1 = x_1^0 = \$10,000 \quad \text{because } q^1 > 0$$

$$\bar{\theta}^2 = x_1^1 \quad \text{because } q^2 > 0$$

$$\bar{\theta}^3 = 0 \quad \text{because } q^3 < 0$$

$$\bar{\theta}^4 = 0 \quad \text{because } q^4 < 0$$

$$\bar{\theta}^5 = 0 \quad \text{because } q^5 < 0$$

Substituting these optimal decision values into equations (1) and (3) we have

$$x_1^1 = 11,500$$

$$x_1^2 = 13,225$$

$$x_1^3 = 10,580$$

$$x_1^4 = 8,464$$

$$x_1^5 = 6,771.2$$

$$x_2^1 = 800$$

$$x_2^2 = 1,720$$

$$x_2^3 = 5,687.5$$

$$x_2^4 = 8,861.5$$

$$x_2^5 = 11,400.7$$

Therefore the total dividend according to the optimal policy is \$11,400.7.

DISCUSSION OF THE RESULTS

It is found in this example that both stochastic and deterministic solutions agree with each other and thus the certainty equivalence principle is again verified. In this system the performance equations and the objective function are completely linear in random variables and in decision variables. It is seen here that the stochastic maximum principle is able to handle the constraints on the decision variable in a very similar way as in the deterministic case. Also when theorem 1 is applied for the constrained optimization problem, though the equations (14) and

(15) describing the adjoint vectors initially appear more complicated, after simplification and numerical substitution equation (16) for the adjoint vectors takes on a form that is more identical to that of the deterministic case. Finally it is felt here that though the numerical values for the optimal solution are the same for both stochastic and deterministic cases, the stochastic treatment of the problem is considered a better approach to the realistic conditions. However the effectiveness of the stochastic approach in the practical situations is more dependent on the proper evaluation of the expected values for the random variables considered in the model.

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5. A HYDROELECTRIC WATER STORAGE SYSTEM

INTRODUCTION

The big storage reservoirs of a hydroelectric system collect water during high river flows for use during low river flows. The scheduling of the use of this water in the storage makes an unconventional inventory problem. Since the future river flow is not certainly known, the problem encountered is one in which the decisions are to be made under uncertainty and in which the goal of operation must be to maximize the expected return, not the return itself. If the future river flow were known in detail a year in advance, the operations plan for water utilization could be worked by the calculus of variations method (1). Since this advanced knowledge about the river flow pattern is not available when decisions have to be made, one must use the probabilistic methods.

The scientific treatment of problems associated with water storage systems is of recent origin. The literature about the work done in water storage systems has been well reviewed by Prabhu (2). The pioneering work in this field appears to be that of Gumbel (1941), (3) which dealt with the return period of flood flows; this was later discussed by Moran (1957), (4). Also in 1954, Moran (5) gave the first probabilistic formulation of a storage model for a water reservoir system. Empirical work on the determination of optimum storage capacity of water reservoir system was done by Hurst (1951, 1956), (6), (7). In 1952, Dvoretzky, Kiefer and Wolfowitz (8) presented a general statement of hydroelectric problems as inventory problems, which was later extended to the specific case of the Grand Coulee plant on the Columbia River and a numerical solution with reference to the

Grand Coulee Dam, by Little (9) in 1955. Little constructed a mathematical model which fits into the general framework of optimization in multistage processes as discussed by Bellman (10), (11) in his papers on dynamic programming. Inventory problems arising in the combined storage systems of three plants have also been studied by Cypser (1953). More recent investigation about the hydroelectric systems is due to Koopmans (1958), (12) and Bather (1962, 1963), (13), (14). In the present investigation, a discrete stochastic model for the single storage reservoir system of a hydroelectric plant is formulated, making small changes from that proposed by Little and the maximum principle is used to obtain the solution. For the numerical example considered, the numerical values for the distribution function of the random variables involved in the model are taken from the work of Cypser (1).

STATEMENT OF THE PROBLEM

A mathematical model of a hydroelectric system is constructed and it is used in determining the optimum water use. The model chosen is as simple as possible consistent with including the main features of the problem. It comprises a single storage reservoir with a hydroelectric plant, a source of supplemental power generation, and estimated power demand and river flow characterized by probability densities. We assume that a supplementary source exists (a thermal power station) in case the entire demand can not be fully met, but these are available at a cost while the hydroelectric power is available free when there is sufficient water storage.

Time is considered broken up into N intervals. At the beginning of each a decision is made about storage use in that interval. The decision is to be made taking into account the current reservoir level and the river

flow in the preceding interval. The index of performance of the system is taken to be the cost of operation. Water is free, but if it is not available, coal must be burned in steam plants, or energy brought from outside the system. Thus the relevant cost is the cost of supplemental energy, where the latter is defined as that portion of the total demand which the system's hydroelectric plants are not supplying. A scheduling of storage water use that minimizes the expected cost of supplemental energy is sought.

The hydroelectric systems operating in conjunction with thermal sources are subdivided into two categories - the long range hydrothermal systems and the short range hydrothermal systems. In the short range problem load allocations are constrained so as to consume only a specified amount of water over a short future time interval such as one week. Variations in elevations and plant efficiencies during this short interval have small effect on the optimum operation. In systems with large storage, where appreciable cumulative changes in plant elevations and efficiencies occur, these weekly specifications must be the results of a long range optimization. Hence the systems in the long range category are characterized by appreciable influence of current operation on long range economy due to cumulative changes in storage elevation and plant efficiencies. Thus in the long range hydrothermal systems, the operation is not adequately described by power generation alone, but rather by the particular combinations of elevations and rate of change of elevation which, together with natural stream flows, determine the power generation. It follows that long range prediction of stream flows and system load enter the problem explicitly. Furthermore, unilateral constraints imposed by various hydro operating limitations must be accounted for in the mathematical model. In the following section, the mathematical model is constructed only for a long range

hydrothermal system.

PROJECT AND OPERATING LIMITATIONS

One of the most important factors in the mathematical formulation of the long range optimization problem is the operating limitations on the system performance. The principal project limitations are the maximum discharge of water that can pass through the turbines at the maximum gate opening and limitations on the minimum drawdown elevation because of the location of intakes on the limits of the storage basin. The operating limitations include minimum plant discharges for navigation or fish - life purposes and maximum storage elevation as directed by flood prospects. Many of these limitations are inequalities in form and are to be carefully analyzed in obtaining the solution for the mathematical model.

SEASONAL VARIATIONS

The long range hydrothermal problem is characterized by the seasonal shifts of load requirements and non - coinciding seasonal variations in stream flows. Usually in fall and winter the river flow decreases when precipitation in the mountains falls as snow. In spring and summer the snow melts and the river rises, filling the reservoir and satisfying all power demands. In winter, therefore, water is drawn from storage to supplement the low river flow. If drawdown is made too soon, the hydrostatic head at the plant drops and the power that can be generated by the natural river flow is severely reduced. At the same time such a drawdown is a consumption of reserves which, if the years turn out dry, can lead to power shortage. On the other hand, if the stored water is saved too long, it may never be used at all for there is always more water than can be used in

spring. Thus in spring and summer, the right decisions about water use are obvious whereas in fall and winter such decisions require a balancing of the benefits of future against immediate water use in the face of uncertain future flow. Hence because of this uncertainty in the nature of river flow and the demand for power during any period, the quantity of river flow to the dam and the demand for electricity from the consumers are considered as independently distributed stochastic random variables with known distribution in the following mathematical model presented. The probability density functions for these random variables are estimated from the past history of operation of the system using the standard statistical procedures (15).

ACCURACY OF FLOW FORECASTS

Of fundamental importance in the determination of long range optimization is the accuracy of forecasts of stream flows and power demand. Any one optimization can be only as accurate as the stream forecasts. Flow records for a long time in the past and ground - water level measurements enable flow forecasters to make predictions that flows will exceed a certain minimum, with practically a hundred percent probability. Various techniques (16), (17), (18) can be used to predict the probabilities of the flow exceeding levels above this minimum. Whether the operation of the system is to be based on the expectation of just exceeding the minimum flow, or on a lesser probability of exceeding some higher flow, is today a management decision. However, in the following consideration, it is assumed that only a "best estimate" of river flow and power demand during each period is available.

HYDROELECTRIC STORAGE SYSTEM MODEL

Let us consider an N-stage decision process as shown in Fig. 1 in which each stage represents a certain interval of time. The performance equations of the system are as follows.

$$x_1^n = x_1^{n-1} + \xi_1^n - \theta^n, \quad n = 1, 2, \dots, N \quad (1)$$

$$x_1^0 = a \quad (2)$$

$$x_2^n = \rho \theta^n [I_0 + \frac{1}{2} (x_1^{n-1} + x_1^n)] \quad (3)$$

$$x_2^0 = 0 \quad (4)$$

where

x_1^n = a state variable representing the amount of water inventory in the reservoir at the end of the nth stage

x_2^n = a state variable representing the amount of hydroelectric power generated during the nth stage (period)

θ^n = the decision variable representing the amount of water discharge during the nth stage

ξ_1^n = an independently distributed random variable with mean μ_1^n , representing the amount of river flow in the nth stage

a = a specified constant for the initial water inventory

ρ = a specified constant for the plant

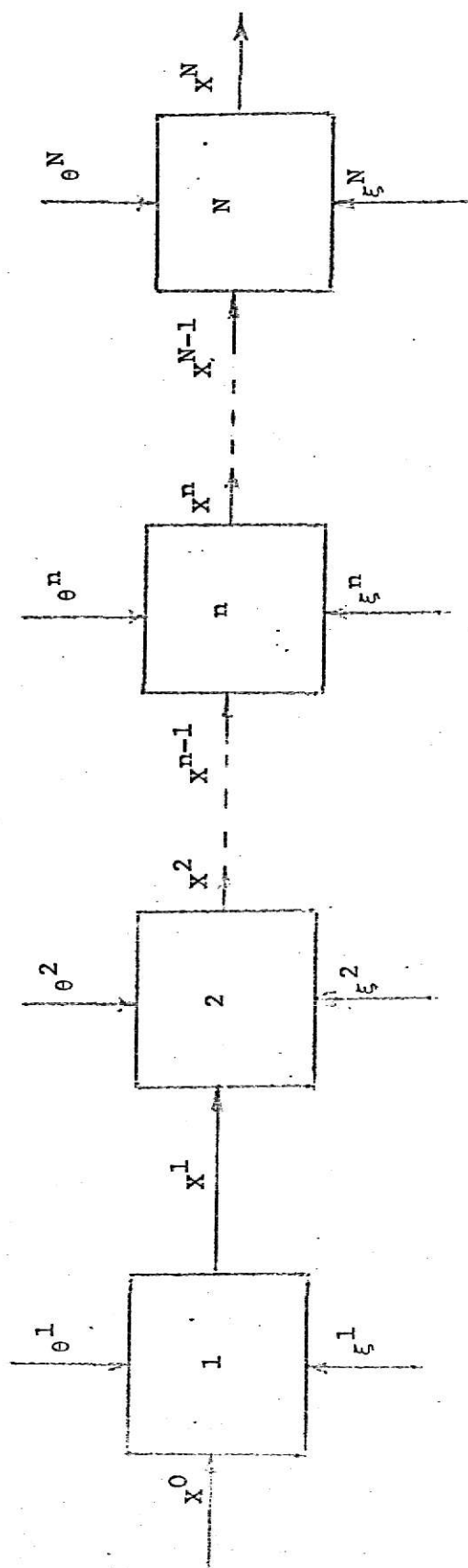


FIG. 1. A SIMPLE MULTISTAGE STOCHASTIC PROCESS

I_0 = the amount of static head difference between the level of the reservoir and the turbine position in the plant

In equation (3) the expression for hydroelectric generation during the nth stage, x_2^n , is only approximate and the bracketed term is proportional to the average net head and ρ is the constant of proportionality with an appropriate value for the plant. The average net head includes the static head difference between the reservoir and the plant location and the average water storage head. The total cost function is defined as follows.

$$S = \sum_{n=1}^N [a_0 + a_1 g^n + a_2 (g^n)^2] \quad (5)$$

where

S = Sum of cost of supplying the supplemental energy to meet the power demand upto and including the Nth stage

g^n = the supplemental energy required during the nth stage

a_0, a_1, a_2 = specified constants for the plant

The supplemental energy required is given by

$$g^n = (\xi_2^n - x_2^n)$$

where

ξ_2^n = an independently distributed random variable with mean μ_2^n , representing the amount of demand for electricity during the nth stage

Hence the total cost function is defined as

$$S = \sum_{n=1}^N [a_0 + a_1 (\xi_2^n - x_2^n) + a_2 (\xi_2^n - x_2^n)^2] \quad (6)$$

The objective function which is to be minimized is given by

$$E(S) = E\left[\sum_{n=1}^N \{a_0 + a_1 (\xi_2^n - x_2^n) + a_2 (\xi_2^n - x_2^n)^2\}\right] \quad (7)$$

The optimization problem is to choose a sequence of decisions, θ^n , $n = 1, 2, \dots, N$, which minimizes the objective function of equation (7) and which satisfies the following constraints on the state and decision variables.

$$1. \quad 0 \leq x_1^n \leq I_{\max} \quad (8)$$

$$2. \quad \theta_{\min} \leq \theta^n \leq \min(\theta_{\max}, x_1^{n-1} + \xi_1^n) \quad (9)$$

where

θ_{\min} = Minimum water storage required for navigation or fish -
life purposes

θ_{\max} = Maximum discharge of water that can pass through the turbine
at the maximum gate opening

I_{\max} = Maximum storage inventory

The solution for the above stated model by applying the discrete stochastic maximum principle is presented in the following section.

SOLUTION BY THE STOCHASTIC MAXIMUM PRINCIPLE

In order to apply the stochastic maximum principle, a new state variable is defined for the cost function as follows. Let

$$x_3^n = x_3^{n-1} + a_0 + a_1 (\xi_2^n - x_2^n) + a_2 (\xi_2^n - x_2^n)^2, \quad (10)$$

$$n = 1, 2, \dots, N$$

$$x_3^0 = 0 \quad (11)$$

where

x_3^n = Sum of the cost of supplying the supplemental energy to meet the power demand upto and including the nth stage

Then the objective function to be minimized becomes

$$E(S) = E(x_3^N) \quad (12)$$

where $c_1 = c_2 = 0$; $c_3 = 1$

To reduce to the standard form as shown by equation (2-1), the state variables x_2^n and x_3^n are rewritten as follows.

$$x_2^n = \rho \theta^n [I_0 + \frac{1}{2} (x_1^{n-1} + x_1^n)]$$

Substituting for x_1^n from equation (1), we get

$$\begin{aligned} x_2^n &= \rho \theta^n [I_0 + \frac{1}{2} (x_1^{n-1} + x_1^{n-1} + \xi_1^n - \theta^n)] \\ &= \rho \theta^n [I_0 + x_1^{n-1} + \frac{\xi_1^n}{2} - \frac{\theta^n}{2}] , \quad n = 1, 2, \dots, N \end{aligned} \quad (13)$$

$$x_2^0 = 0 \quad (14)$$

Substituting for x_2^n in equation (11), we get x_3^n in the standard form as

$$\begin{aligned}
x_3^n &= x_3^{n-1} + a_0 + a_1 \{ \xi_2^n - \rho [x_1^{n-1} \theta^n + I_0 \theta^n \\
&\quad + \frac{\xi_1^n \theta^n}{2} - \frac{(\theta^n)^2}{2}] \} + a_2 \{ \xi_2^n - \rho [x_1^{n-1} \theta^n \\
&\quad + I_0 \theta^n + \frac{\xi_1^n \theta^n}{2} - \frac{(\theta^n)^2}{2}] \}^2, \quad n = 1, 2, \dots, N
\end{aligned} \tag{15}$$

$$x_3^0 = 0$$

CASE 1. LINEAR COST FUNCTION

Let us first consider the case when the cost function is linear with respect to the random variable, ξ_2^n ; that is, let $a_2 = 0$ so that the state variable for the cost function becomes a linear function in ξ_2^n as shown below

$$\begin{aligned}
x_3^n &= x_3^{n-1} + a_0 + a_1 \{ \xi_2^n - \rho \theta^n [x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2}] \}, \\
n &= 1, 2, \dots, N
\end{aligned} \tag{16}$$

According to equation (2-15), the Hamiltonian function is defined as

$$\begin{aligned}
H^n &= z_1^n f_1^n + z_2^n f_2^n + z_3^n f_3^n \\
&= z_1^{n-1} (x_1^{n-1} + \xi_1^n - \theta^n) + z_2^n \rho \theta^n [x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2}] \\
&\quad + z_3^n \{ x_3^{n-1} + a_0 + a_1 [\xi_2^n - \rho \theta^n (x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2})] \}, \\
n &= 1, 2, \dots, N
\end{aligned} \tag{17}$$

Since θ^n , $n = 1, 2, \dots, N$ are constrained, we apply theorem 1. Therefore the adjoint vectors are defined by

$$(z_1^{n-1}, z_2^{n-1}, z_3^{n-1})$$

$$= (z_1^n, z_2^n, z_3^n) \begin{pmatrix} 1 - \frac{\partial \theta^n}{\partial x_1^{n-1}}, & -\frac{\partial \theta^n}{\partial x_2^{n-1}}, & -\frac{\partial \theta^n}{\partial x_3^{n-1}} \\ \rho \theta^n + \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}}, & \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}}, & \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} \\ -a_1 \rho \theta^n + a_1 \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}}, & \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}}, & \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} \end{pmatrix}$$

where

$$\frac{\partial f_2^n}{\partial \theta^n} = \rho (x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \theta^n)$$

and

$$\frac{\partial f_3^n}{\partial \theta^n} = -a_1 \rho (x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \theta^n)$$

Therefore

$$z_1^{n-1} = z_1^n (1 - \frac{\partial \theta^n}{\partial x_1^{n-1}}) + z_2^n \{ \rho \theta^n + \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} \}$$

$$+ z_3^n \{ -a_1 \rho \theta^n + \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} \}, \quad n = 1, 2, \dots, N \quad (18)$$

$$z_1^N = c_1 = 0 \quad (19)$$

$$z_2^{n-1} = -z_1^n \frac{\partial \theta^n}{\partial x_2^{n-1}} + z_2^n \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}} + z_3^n \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}}, \quad n = 1, 2, \dots, N \quad (20)$$

$$z_2^N = c_2 = 0 \quad (21)$$

$$z_3^{n-1} = -z_1^n \frac{\partial \theta^n}{\partial x_3^{n-1}} + z_2^n \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} + z_3^n \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}}, \quad n = 1, 2, \dots, N \quad (22)$$

$$z_3^N = c_3 = 1 \quad (23)$$

Hence

$$\begin{aligned} E[H^n | x^{n-1}] &= E(z_1^n)(x_1^{n-1} + \mu_1^n - \theta^n) + E(z_2^n)\rho\theta^n[x_1^{n-1} + I_0 + \frac{\mu_1^n}{2} - \frac{\theta^n}{2}] \\ &\quad + E(z_3^n)\{x_3^{n-1} + a_0 + a_1[\mu_2^n - \rho\theta^n(x_1^{n-1} + I_0 + \frac{\mu_1^n}{2} - \frac{\theta^n}{2})]\}, \\ n &= 1, 2, \dots, N \end{aligned} \quad (24)$$

Now a sequence of decisions θ^n , $n = 1, 2, \dots, N$, which satisfies the constraints specified by equations (8) to (10) and which minimizes the $E(H^n | x^{n-1})$ is the required optimal decision that also minimizes $E(S)$. The computation of optimal decisions for the numerical example of a four stage process is presented in the following section.

NUMERICAL EXAMPLE FOR A FOUR STAGE PROCESS

Consider a four stage process in which each stage represents a period of three months so that the performance behavior of the system through out a year is included in the four stages of the process. From the past history of the system, a set of "best estimate" curves (Fig. 2) are obtained for the river flow and power demand through out a year. From the curves the average expected values (μ 's) of the two random variables are computed and given below. The other system constants are assumed the following values as suggested by Little (8), (according to the data obtained from the Grand Coulee Dam on the Columbia River).

$$\mu_1^1 = 133.7 \text{ KCFS day}$$

$$\mu_1^2 = 36.3 \text{ KCFS day}$$

$$\mu_1^3 = 26.3 \text{ KCFS day}$$

$$\mu_1^4 = 168.3 \text{ KCFS day}$$

$$\mu_2^1 = 332.7 \text{ megawatts}$$

$$\mu_2^2 = 345.6 \text{ megawatts}$$

$$\mu_2^3 = 362.2 \text{ megawatts}$$

$$\mu_2^4 = 342.2 \text{ megawatts}$$

$$x_1^0 = 500 \text{ KCFS day}$$

$$I_0 = 260 \text{ KCFS day}$$

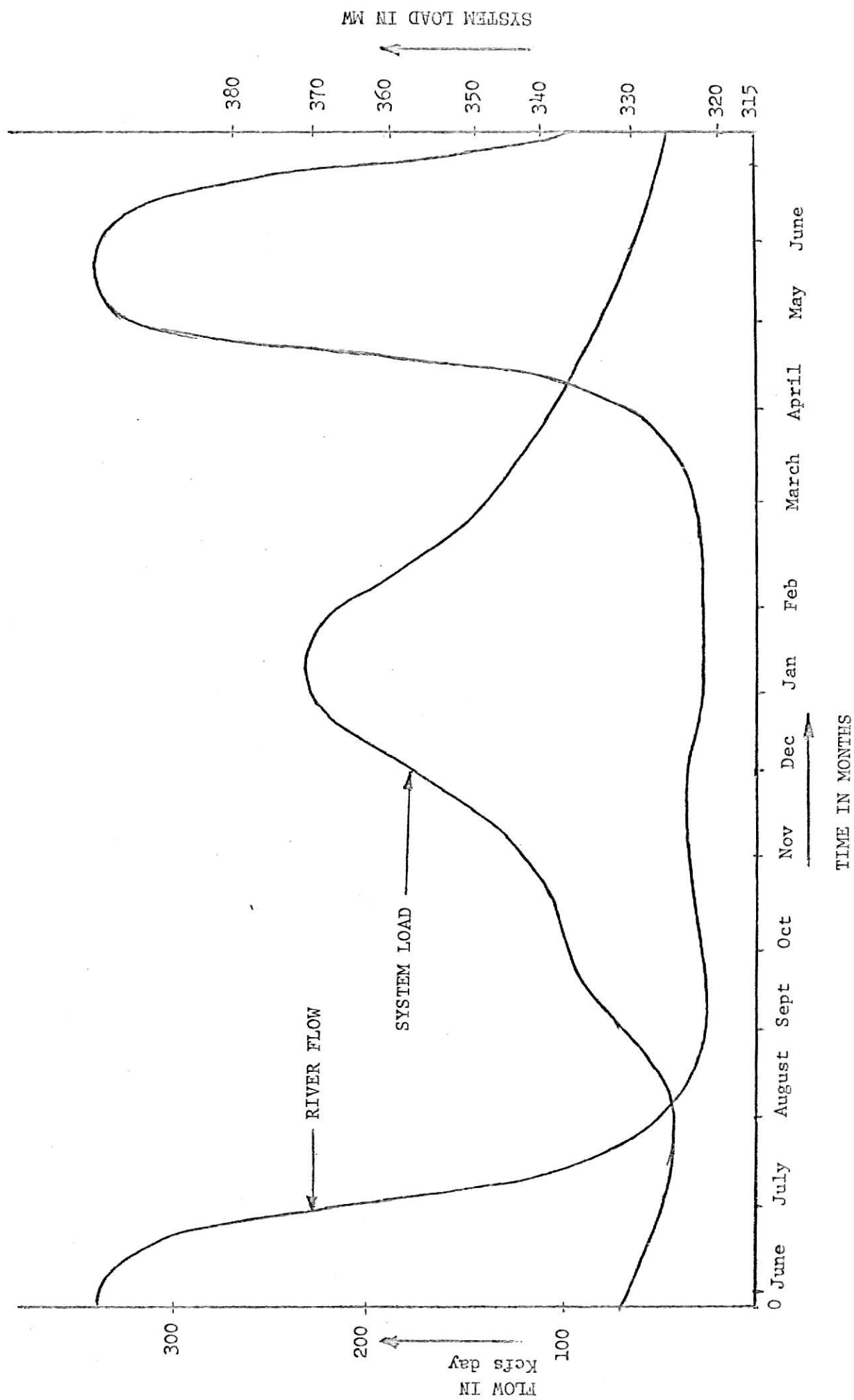


FIG. 2. FLOWS AND LOAD WITH TIME

$$\rho = 0.00228 \text{ megawatt}/(\text{KCFS day})^2$$

$$\theta_{\max} = 980 \text{ KCFS day}$$

$$\theta_{\min} = 140 \text{ KCFS day}$$

$$I_{\max} = 8750 \text{ KCFS day}$$

$$a_0 = 0$$

$$a_1 = 1 \text{ \$/megawatt}$$

(The unit Kcfs day is a volume equal to a flow of 1000 cubic ft per second for a day.)

With this set of numerical values, the optimal decision is calculated as follows.

From equations (19), (21) and (23), we get

$$E(z_1^4) = 0$$

$$E(z_2^4) = 0$$

$$E(z_3^4) = 1$$

Substituting these values in equation (24), we get

$$E(H^4 | \bar{x}^3) = x_3^3 + a_0 + a_1 [\mu_2^4 - \rho \theta^4 (x_1^3 + I_0 + \frac{\mu_1^4}{2} - \frac{(\theta^4)}{2})]$$

Now the optimal decision $\bar{\theta}^4$ is found from the condition

$$E[\frac{\partial H^4}{\partial \theta} | \bar{x}^3] = 0$$

Hence it follows that

$$E\left[\frac{\partial^4 H}{\partial \theta^4} \mid \bar{x}^3\right] = -a_1 \rho (x_1^3 + I_0 + \frac{\mu_1^4}{2} - \theta^4) = 0$$

$$\text{or } \bar{\theta}^4 = \bar{x}_1^3 + I_0 + \frac{\mu_1^4}{2}$$

$$\text{Now } E\left[\frac{\partial^2 H}{\partial (\theta^4)^2} \mid \bar{x}^3\right] = a_1 \rho > 0$$

This indicates that at $\bar{\theta}^4$, $E(H^4 \mid \bar{x}^3)$ has a minimum.

Therefore

$$\begin{aligned} \bar{\theta}^4 &= \bar{x}_1^3 + I_0 + \frac{\mu_1^4}{2} \\ &= \bar{x}_1^3 + 340.15 \end{aligned}$$

But from equation (9), θ^4 is constrained as follows.

$$140 \leq \theta^4 \leq \min(980, \bar{x}_1^3 + 168.3)$$

Since $\bar{\theta}^4$ lies outside this range, we take

$$\begin{aligned} \bar{\theta}^4 &= \text{allowable maximum of } \theta^4 \\ &= \min(980, \bar{x}_1^3 + 168.3) \\ &= (\bar{x}_1^3 + 168.3) \end{aligned}$$

Simplifying, we get

$$\bar{\theta}^1 + \bar{\theta}^2 + \bar{\theta}^3 + \bar{\theta}^4 = 1084 \quad (25)$$

From equations (18) through (23), we obtain

$$E(z_1^3) = -a_1 \rho [\bar{x}_1^3 + 344.17] \quad (26)$$

$$E(z_2^3) = 0$$

$$E(z_3^3) = 1$$

From equation (24), the Hamiltonian function for $n = 3$ is

$$\begin{aligned} E[H^3 | \bar{x}^2] &= E(z_1^3) (x_1^2 + \mu_1^3 - \theta^3) + x_3^2 + a_0 \\ &\quad + a_1 [\mu_2^3 - \rho \theta^3 (x_1^2 + I_0 + \frac{\mu_1^3}{2} - \frac{\theta^3}{2})] \end{aligned}$$

Now taking partial differentiation with respect to θ^3 and equating to zero, we get

$$\bar{\theta}^3 = \frac{E(z_1^3)}{a_1 \rho} + (\bar{x}_1^2 + 273.17)$$

Substituting for $E(z_1^3)$ from equation (26) and simplifying, we get

$$\bar{\theta}^1 + \bar{\theta}^2 + \bar{\theta}^3 = 660.21 \quad (27)$$

$$E\left[\frac{\partial^2 H^3}{\partial (\theta^3)^2} | \bar{x}^2\right] = a_1 \rho > 0 \quad \text{indicates that at } \theta^3 =$$

$$\frac{E(z_1^3)}{a_1 \rho} + (\bar{x}_1^2 + 273.17), \quad E[H^3 | \bar{x}^2] \text{ has a minimum value.}$$

From equations (18) through (23), we obtain

$$E(z_1^2) = -a_1 \rho (\bar{x}_1^2 + 273.17) \quad (28)$$

$$E(z_2^2) = 0$$

$$E(z_3^2) = 1$$

From equation (24), the Hamiltonian function for $n = 2$ is

$$\begin{aligned} E[H^2 | \bar{x}^1] &= E(z_1^2)(x_1^1 + \mu_1^2 - \theta^2) + x_3^1 + a_0 \\ &\quad + a_1[\mu_2^2 - \rho\theta^2(x_1^1 + I_0 + \frac{\mu_1^2}{2} - \frac{\theta^2}{2})] \end{aligned}$$

Taking partial differentiation with respect to θ^2 and equating to zero, we get

$$\bar{\theta}^2 = \frac{E(z_1^2)}{a_1\rho} + (\bar{x}_1^1 + 278.17)$$

Substituting for $E(z_1^2)$ from equation (28) and simplifying, we get

$$\bar{\theta}^1 + \bar{\theta}^2 = 372.4 \quad (29)$$

$$E\left[\frac{\partial^2 H^2}{\partial(\theta^2)^2} \mid \bar{x}^1\right] = a_1\rho > 0 \text{ indicates that at } \bar{\theta}^2 = \frac{E(z_1^2)}{a_1\rho} + (\bar{x}_1^1 + 278.17),$$

$E[H^2 | \bar{x}^1]$ has a minimum. From equations (18) through (23), we obtain

$$E(z_1^1) = -a_1\rho[\bar{x}_1^1 + 278.17] \quad (30)$$

$$E(z_2^1) = 0$$

$$E(z_3^1) = 1$$

From equation (24) the Hamiltonian function for $n = 1$ is

$$E[H^1|\bar{x}^0] = E(z_1^1)(x_1^0 + \mu_1^1 - \theta^1) + x_3^0 + a_0 \\ + a_1[\mu_2^1 - \rho\theta^1(x_1^0 + I_0 + \frac{\mu_1^1}{2} - \frac{\theta^1}{2})]$$

Taking partial differentiation with respect to θ^1 and equating to zero, we get

$$\bar{\theta}^1 = \frac{E(z_1^1)}{a_1\rho} + 826.9$$

Substituting for $E(z_1^1)$ from equation (30) and simplifying, we get

$$\bar{\theta}^1 = 165.7$$

Substituting for $\bar{\theta}^1$ in equations (25), (27) and (29) and solving the equations, the optimal decisions are

$$\bar{\theta}^1 = 165.7 \text{ Kcfs day}$$

$$\bar{\theta}^2 = 206.7 \text{ Kcfs day}$$

$$\bar{\theta}^3 = 247.3 \text{ Kcfs day}$$

$$\bar{\theta}^4 = 424.3 \text{ Kcfs day}$$

Substituting these optimal decisions into equations (1), (3) and (16) yields

$$\bar{x}_1^1 = 409.4 \text{ Kcfs day}$$

$$\bar{x}_1^2 = 367.6 \text{ Kcfs day}$$

$$\bar{x}_1^{-3} = 296.3 \text{ Kcfs day}$$

$$\bar{x}_1^{-4} = 0$$

$$\bar{x}_2^{-1} = 268.5 \text{ megawatts}$$

$$\bar{x}_2^{-2} = 291.0 \text{ megawatts}$$

$$\bar{x}_2^{-3} = 351.7 \text{ megawatts}$$

$$\bar{x}_2^{-4} = 412.0 \text{ megawatts}$$

$$\bar{x}_3^{-1} = \$127.34$$

$$\bar{x}_3^{-2} = \$294.60$$

$$\bar{x}_3^{-3} = \$412.73$$

$$\bar{x}_3^{-4} = \$492.62$$

When the optimal policy is used, the supplemental energy required during the different periods is as follows.

$$g^1 = 64.20 \text{ megawatts}$$

$$g^2 = 54.60 \text{ megawatts}$$

$$g^3 = 10.50 \text{ megawatts}$$

$$g^4 = 0$$

CASE 2. QUADRATIC COST FUNCTION

Now the case when the cost function is quadratic with respect to the demand random variable, ξ_2^n , is considered; that is, the state variable for the cost function, x_3^n , is expressed as a quadratic function in ξ_2^n as shown in equation (15). Therefore from equation (15), we obtain

$$x_3^n = x_3^{n-1} + a_0 + a_1[\xi_2^n - \rho\theta^n(x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2})] + a_2[\xi_2^n - \rho\theta^n(x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2})]^2,$$

$$n = 1, 2, \dots, N$$

$$x_3^0 = 0$$

The Hamiltonian function becomes

$$\begin{aligned} E[H^n | \bar{x}^{n-1}] &= E(z_1^n)(x_1^{n-1} + \mu_1^n - \theta^n) + E(z_2^n)\rho\theta^n[x_1^{n-1} \\ &\quad + I_0 + \frac{\mu_1^n}{2} - \frac{\theta^n}{2}] + E(z_3^n)\{x_3^{n-1} + a_0 + a_1[\mu_2^n \\ &\quad - \rho\theta^n(x_1^{n-1} + I_0 + \frac{\mu_1^n}{2} - \frac{\theta^n}{2})] + a_2E[\xi_2^n - \rho\theta^n(x_1^{n-1} \\ &\quad + I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2})]^2\}, \quad n = 1, 2, \dots, N \end{aligned} \quad (31)$$

By theorem 1, the adjoint vectors are defined by

$$(z_1^{n-1}, z_2^{n-1}, z_3^{n-1})$$

$$= (z_1^n, z_2^n, z_3^n) \begin{pmatrix} 1 - \frac{\partial \theta^n}{\partial x_1^{n-1}} & , & -\frac{\partial \theta^n}{\partial x_2^{n-1}} & , & -\frac{\partial \theta^n}{\partial x_3^{n-1}} \\ \rho \theta^n + \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} & , & \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}} & , & \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} \\ \frac{\partial f_3^n}{\partial x_1^{n-1}} + \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} & , & \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}} & , & \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} \end{pmatrix} \quad (32)$$

where

$$\frac{\partial f_2^n}{\partial \theta^n} = \rho(x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \theta^n)$$

$$\frac{\partial f_3^n}{\partial x_1^{n-1}} = -a_1 \rho \theta^n + 2a_2 \rho \theta^n \{ \rho \theta^n x_1^{n-1} - [\xi_2^n - \rho \theta^n (I_0 + \frac{\xi_1^n}{2} - \frac{\theta^n}{2})] \} \quad (33)$$

and

$$\begin{aligned} \frac{\partial f_3^n}{\partial \theta^n} = & -a_1 \rho (x_1^{n-1} + I_0 + \frac{\xi_1^n}{2} - \theta^n) + a_2 \rho \{ 2\rho \theta^n (x_1^{n-1} + I_0 + \frac{\xi_1^n}{2}) \\ & - 2\xi_2^n (x_1^{n-1} + I_0 + \frac{\xi_1^n}{2}) + \rho (\theta^n)^3 + [2\theta^n \xi_2^n - 3\rho x_1^{n-1} (\theta^n)^2 - 3\rho I_0 (\theta^n)^2 \\ & - 3\rho I_0 (\theta^n)^2 - \frac{3\rho \xi_1^n}{2} (\theta^n)^2] \} \end{aligned} \quad (34)$$

Therefore we have

$$\begin{aligned}
 z_1^{n-1} = & z_1^n \left(1 - \frac{\partial \theta^n}{\partial x_1^{n-1}} \right) + z_2^n \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} \\
 & + z_3^n \left\{ \frac{\partial f_3^n}{\partial x_1^{n-1}} + \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} \right\}
 \end{aligned} \tag{35}$$

$$z_1^N = c_1 = 0 \tag{36}$$

$$z_2^{n-1} = -z_1^n \frac{\partial \theta^n}{\partial x_2^{n-1}} + z_2^n \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}} + z_3^n \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_2^{n-1}} \tag{37}$$

$$z_2^N = c_2 = 0 \tag{38}$$

$$z_3^{n-1} = -z_1^n \frac{\partial \theta^n}{\partial x_3^{n-1}} + z_2^n \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} + z_3^n \left(1 + \frac{\partial f_3^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_3^{n-1}} \right) \tag{39}$$

$$z_3^N = c_3 = 1 \tag{40}$$

The sequence of optimal decisions, θ^n , $n = 1, 2, \dots, N$, which minimizes $E(H^n | x^{n-1})$ and which also satisfies the constraints specified by equations (8) to (10) is found by the following computational scheme. The computational procedure is explained for the same numerical example of four stage process as shown in case 1.

NUMERICAL EXAMPLE

The four stage process with the same numerical values for the various system constants as assumed previously in case 1 are considered here. The additional constants now introduced with the quadratic terms are assumed the following values. The values for variance, $(\sigma_1^n)^2$ of the random variable, ξ_1^n , $n = 1, 2, 3, 4$ are computed from the best estimate curve (Fig. 2) obtained for the river flow and power demand through out an year.

$$a_2 = 1$$

$$(\sigma_1^1)^2 = 200 \text{ Kcfs day}$$

$$(\sigma_1^2)^2 = 20 \text{ Kcfs day}$$

$$(\sigma_1^3)^2 = 20 \text{ Kcfs day}$$

$$(\sigma_1^4)^2 = 180 \text{ Kcfs day}$$

With this set of numerical values, the optimal decisions, $\bar{\theta}^n$, $n = 1, 2, 3, 4$ are calculated as follows.

From equations (35) through (40), we get for $N = 4$

$$E(z_1^4) = 0$$

$$E(z_2^4) = 0$$

$$E(z_3^4) = 1$$

Substituting these values in equation (31), we get

$$\begin{aligned}
E[H^4 | \bar{x}^3] &= x_3^3 + a_0 + a_1 [\mu_2^4 - \rho \theta^4 (x_1^3 + I_0 + \frac{\mu_1^4}{2} - \frac{\theta^4}{2})] + a_2 \{E(\xi_2^4)^2 \\
&\quad + \rho^2 (\theta^4)^2 [(x_1^3 + I_0)^2 + \frac{E(\xi_1^4)^2}{4} + \mu_1^4 (x_1^3 + I_0)] \\
&\quad - 2\rho \mu_2^4 \theta^4 (x_1^3 + I_0 + \frac{\mu_1^4}{2}) + \frac{\rho^2 (\theta^4)^4}{4} + \rho (\theta^4)^2 [\mu_2^4 - \rho \theta^4 (x_1^3 + I_0 \\
&\quad + \frac{\mu_1^4}{2})]\} \quad (41)
\end{aligned}$$

where

$$E(\xi_1^4)^2 = (\sigma_1^4)^2 + (\mu_1^4)^2$$

and

$$E(\xi_2^4)^2 = (\sigma_2^4)^2 + (\mu_2^4)^2$$

The optimal decision $\bar{\theta}^4$ is found from the condition

$$E\left[\frac{\partial H^4}{\partial \theta} | \bar{x}^3\right] = 0$$

Therefore differentiating equation (41) with respect to θ^4 and equating to zero, we get the following equation

$$\begin{aligned}
&a_2 \rho (\theta^4)^3 - 3a_2 \rho (\theta^4)^2 (x_1^3 + I_0 + \frac{\mu_1^4}{2}) + \theta^4 \{a_1 + 2a_2 \rho [(x_1^3 + I_0)^2 \\
&\quad + \frac{1}{4} ((\sigma_1^4)^2 + (\mu_1^4)^2) + \mu_1^4 (x_1^3 + I_0)] + 2a_2 \mu_2^4\} - (x_1^3 + I_0 \\
&\quad + \frac{\mu_1^4}{2}) (a_1 + 2a_2 \mu_2^4) = 0 \quad (42)
\end{aligned}$$

Substituting the numerical values and simplifying, we get

$$\begin{aligned}
 & .00228 (\theta^4)^3 - .00684 (\theta^4)^2 (1040 - \theta^1 - \theta^2 - \theta^3) \\
 & + \theta^4 \{ .00456 [(956 - \theta^1 - \theta^2 - \theta^3)^2 - 168 (\theta^1 + \theta^2 + \theta^3) \\
 & + 168026] + 685 \} - 712000 + 685 (\theta^1 + \theta^2 + \theta^3) = 0
 \end{aligned} \tag{43}$$

Now θ^4 as determined from equation (43) is the optimal decision, $\bar{\theta}^4$, if it satisfies the following conditions:

$$1) \quad E\left[\frac{\partial^2 H^4}{\partial (\theta^4)^2} \mid \bar{x}^3\right] \bigg|_{\theta^4 = \bar{\theta}^4} > 0$$

$$2) \quad 140 \leq \theta^4 \leq \min(980, x_1^3 + 168.3)$$

From equation (35) through (40), we obtain for $n = 4$

$$E(z_1^3) = E\left(\frac{\partial f_3^4}{\partial x_1^3} + \frac{\partial f_3^4}{\partial \theta^4} \frac{\partial \theta^4}{\partial x_1^3}\right) \tag{44}$$

$$E(z_2^3) = 0$$

$$E(z_3^3) = 1$$

where

$$1). \quad \frac{\partial f_3^4}{\partial x_1^3} \text{ is obtained from equation (33), which after simplification gives}$$

$$\frac{\partial^4 f_3}{\partial x_1^3} = - .00000502 (\theta^4)^3 + .00000502 (\theta^4)^2 (2015.7 - 2\theta^1 - 2\theta^2 - 2\theta^3) - 1.562712\theta^4 \quad (45)$$

2). $\frac{\partial^4 f_3}{\partial \theta^4}$ is obtained from equation (34), which after simplification gives

$$\begin{aligned} \frac{\partial^4 f_3}{\partial \theta^4} = & .00000502 (\theta^4)^3 - .00001506 (\theta^4)^2 (1037.9 - \theta^1 - \theta^2 - \theta^3) \\ & + .00228 \theta^4 \{685.4 + .00456(1037.9 - \theta^1 - \theta^2 - \theta^3)^2\} \\ & - 1.562712 (1037.9 - \theta^1 - \theta^2 - \theta^3) \end{aligned} \quad (46)$$

3). $\frac{\partial \theta^4}{\partial x_1^3}$ is obtained by differentiating equation (42) with respect to x_1^3 , which after simplification gives

$$\begin{aligned} \frac{\partial \theta^4}{\partial x_1^3} = & \frac{3(\theta^4)^2 - 2\theta^4 (2015.7 - 2\theta^1 - 2\theta^2 - 2\theta^3) + 300000}{3(\theta^4)^2 - 6\theta^4 (1037.9 - \theta^1 - \theta^2 - \theta^3) + 2(693.7 - \theta^1 - \theta^2 - \theta^3)} \\ & + 1376.6(693.7 - \theta^1 - \theta^2 - \theta^3) + 551423.3 \end{aligned} \quad (47)$$

From equation (31) the Hamiltonian function for $n = 3$ is

$$\begin{aligned} E[H^3 | x^2] = & E(z_1^3) (x_1^2 + \mu_1^3 - \theta^3) + x_3^2 + a_0 + a_1 [\mu_2^3 - \rho \theta^3 (x_1^2 + I_0 + \frac{\mu_1^3}{2} \\ & - \frac{\theta^3}{2})] + a_2 \{E(\xi_2^3)^2 + \rho^2 (\theta^3)^2 [(x_1^2 + I_0)^2 + \frac{E(\xi_1^3)^2}{4} + \mu_1^3 (x_1^2 \\ & + I_0)] - 2\rho \mu_2^3 \theta^3 (x_1^2 + I_0 + \frac{\mu_1^3}{2}) + \frac{\rho^2 (\theta^3)^4}{4} + \rho (\theta^3)^2 [\mu_2^3 \\ & - \rho \theta^3 (x_1^2 + I_0 + \frac{\mu_1^3}{2})]\} \end{aligned} \quad (48)$$

The optimal decision $\bar{\theta}^3$ is obtained from the condition

$$E\left[\frac{\partial H}{\partial \theta^3} \mid \bar{x}^2\right] = 0$$

Therefore differentiating equation (48) with respect to θ^3 and equating to zero, we get

$$\begin{aligned} & a_2 \rho^2 (\theta^3)^3 - 3a_2 \rho^2 (\theta^3)^2 (x_1^2 + I_0 + \frac{\mu_1^3}{2}) + \rho \theta^3 \{a_1 + 2a_2 \rho [(x_1^2 + I_0)^2 \\ & + \frac{1}{4} ((\sigma_1^3)^2 + (\mu_1^3)^2) + \mu_1^3 (x_1^2 + I_0)] + 2a_2 \mu_2^3\} - [E(z_1^3) + \rho(a_1 \\ & + 2a_2 \mu_2^3) (x_1^2 + I_0 + \frac{\mu_1^3}{2})] = 0 \end{aligned} \quad (49)$$

Substituting the numerical values and simplifying, we get

$$\begin{aligned} & .00000502 (\theta^3)^3 - .00001506 (\theta^3)^2 (943.15 - \theta^1 - \theta^2) + .00228 \theta^3 \{53.6 \\ & + .00456 [(930 - \theta^1 - \theta^2)^2 + 24617 - 26.3 (\theta^1 + \theta^2)]\} - [E(z_1^3) \\ & + .1222 (943.15 - \theta^1 - \theta^2)] = 0 \end{aligned} \quad (50)$$

where $E(z_1^3)$ is obtained from equations (44) through (47). Now θ^3 as determined by equation (50) is the optimal decision, $\bar{\theta}^3$, if it satisfies the following conditions:

$$1). \quad E\left[\frac{\partial^2 H}{\partial (\theta^3)^2} \mid \bar{x}^2\right] > 0$$

$$2). \quad 140 \leq \theta^3 \leq \min(980, x_1^2 + 26.3)$$

For $n = 3$, from equations (35) through (40), we obtain

$$E(z_1^2) = E \left\{ \frac{\partial f_3^3}{\partial x_1^2} + \frac{\partial f_3^3}{\partial \theta^3} \frac{\partial \theta^3}{\partial x_1^2} \right\} \quad (51)$$

$$E(z_2^2) = 0$$

$$E(z_3^2) = 1$$

where

- 1). $\frac{\partial f_3^3}{\partial x_1^2}$ is obtained from equation (33), which after simplification gives

$$\begin{aligned} \frac{\partial f_3^3}{\partial x_1^2} = & - .00000502 (\theta^3)^3 + .00000502 (\theta^3)^2 (1886.3 - 2\theta^1 - 2\theta^2) \\ & - .83265 \theta^3 \end{aligned} \quad (52)$$

- 2). $\frac{\partial f_3^3}{\partial \theta^3}$ is obtained from equation (34), which after simplification gives

$$\begin{aligned} \frac{\partial f_3^3}{\partial \theta^3} = & .00000502 (\theta^3)^3 - .00001506 (\theta^3)^2 (943.15 - \theta^1 - \theta^2) \\ & + .00228 \theta^3 [725.4 + .00456(943 - \theta^1 - \theta^2)^2] - .832656(943.15 \\ & - \theta^1 - \theta^2) \end{aligned} \quad (53)$$

- 3). $\frac{\partial \theta^3}{\partial x_1^2}$ is obtained by differentiating equation (50) with respect to x_1^2 , which after simplification gives

$$\frac{\partial \theta^3}{\partial x_1^2} = \frac{3(\theta^3)^2 - 2\theta^3(1886.3 - \theta^1 - \theta^2) + 318000}{3(\theta^3)^2 - 6\theta^3(670 - \theta^1 - \theta^2) + 2(670 - \theta^1 - \theta^2)^2 + 466453} \quad (54)$$

The Hamiltonian function for $n = 2$ is

$$\begin{aligned} E[H^2 | \bar{x}^1] = & E(z_1^2)(x_1^1 + \mu_1^2 - \theta^2) + x_3^1 + a_0 + a_1[\mu_2^2 - \rho\theta(x_1^1 + I_0 \\ & + \frac{\mu_1^2}{2} - \frac{\theta^2}{2})] + a_2\{E(\xi_2^2)^2 + \rho^2(\theta^2)^2[(x_1^1 + I_0)^2 + \frac{E(\xi_1^2)^2}{4} \\ & + \mu_1^2(x_1^1 + I_0)] - 2\rho\mu_2^2\theta^2(x_1^1 + I_0 + \frac{\mu_1^2}{2}) + \frac{\rho^2(\theta^2)^4}{4} \\ & + \rho(\theta^2)^2[\mu_2^2 - \rho\theta^2(x_1^1 + I_0 + \frac{\mu_1^2}{2})]\} \end{aligned} \quad (55)$$

Differentiating equation (55) with respect to θ^2 and equating to zero, we get

$$\begin{aligned} a_2 \rho^2 (\theta^2)^3 - 3a_2 \rho (\theta^2)^2 (x_1^1 + I_0 + \frac{\mu_1^2}{2}) + \rho\theta^2\{a_1 + 2a_2 \rho [(x_1^1 + I_0)^2 \\ + \frac{1}{4} ((\sigma_1^2)^2 + (\mu_1^2)^2) + \mu_1^2(x_1^1 + I_0)] + 2a_2\mu_2^2\} - [E(z_1^2) + \rho(a_1 \\ + 2a_2\mu_2^2)(x_1^1 + I_0 + \frac{\mu_1^2}{2})] = 0 \end{aligned} \quad (56)$$

Substituting the numerical values and simplifying, we get

$$\begin{aligned} .00000502 (\theta^2)^3 - .00001506 (\theta^2)^2 (912 - \theta^1) + .00228\theta^1\{692.2 + .00456 \\ [(893.7 - \theta^1)^2 + 32734 - 36.3\theta^1]\} - \{E(z_1^2) + 1.579(912 - \theta^1)\} = 0 \end{aligned} \quad (57)$$

where $E(z_1^2)$ is obtained from equations (51) through (54). Now θ^2 as determined by equation (57) is the optimal decision, if it satisfies the following conditions.

$$1). \quad E\left[\frac{\partial^2 H^2}{\partial(\theta^2)^2} \mid \bar{x}^1\right] > 0$$

$$2) \quad 140 \leq \theta^2 \leq \min(980, x_1^1 + 36.3)$$

For $n = 2$, from equations (35) through (40), we obtain

$$E(z_1^1) = E\left(\frac{\partial f_3^2}{\partial x_1^1} + \frac{\partial f_3^2}{\partial \theta^2} \frac{\partial \theta^2}{\partial x_1^1}\right) \quad (58)$$

$$E(z_2^1) = 0$$

$$E(z_3^1) = 1$$

where

$$1). \quad \frac{\partial f_3^2}{\partial x_1^1} \text{ is obtained from equation (33), which after simplification gives}$$

$$\frac{\partial f_3^2}{\partial x_1^1} = - .00000502 (\theta^2)^3 + .00000502 (\theta^2)^2 (1823.7 - 2\theta^1) - 1.578216\theta^2 \quad (59)$$

$$2). \quad \frac{\partial f_3^2}{\partial \theta^2} \text{ is obtained from equation (34), which after simplification gives}$$

$$\begin{aligned} \frac{\partial^2 f_3}{\partial \theta^2} = & .00000502(\theta^2)^3 - .00001506(\theta^2)^2(911.85 - \theta^1) \\ & + .00228\{692.2 + .00456(911.85 - \theta^1)^2\} - 1.578216(911.85 - \theta^1) \quad (60) \end{aligned}$$

3). $\frac{\partial^2}{\partial x_1^2}$ is obtained by differentiating equation (56) with respect to x_1^1 , which after simplification gives

$$\frac{\partial^2}{\partial x_1^2} = \frac{3(\theta^2)^2 - 2\theta^2(1190 - \theta^1) + 303700}{3(\theta^2)^2 - 6^2(912 - \theta^1) + 2(633.7 - \theta^1)^2 - 1112.6\theta^1 + 1163257.4} \quad (61)$$

The Hamiltonian function for $n = 1$ is

$$\begin{aligned} E[H^1 | \bar{x}] = & E(z_1^1)(x_1^0 + \mu_1^1 - \theta^1) + x_3^0 + a_0 + a_1[\mu_2^1 - \rho\theta^1(x_1^0 + I_0 \\ & + \frac{\mu_1^1}{2} - \frac{\theta^1}{2})] + a_2\{E(\xi_2^1)^2 + \rho^2(\theta^1)^2[(x_1^0 + I_0)^2 + \frac{E(\xi_1^1)^2}{4} \\ & + \mu_1^1(x_1^0 + I_0)] - 2\rho\mu_2^1\theta^1(x_1^0 + I_0 + \frac{\mu_1^1}{2}) + \frac{\rho^2(\theta^1)^4}{4} \\ & + \rho(\theta^1)^2[\mu_2^1 - \rho\theta^1(x_1^0 + I_0 + \frac{\mu_1^1}{2})]\} \quad (62) \end{aligned}$$

Differentiating equation (62) with respect to θ^1 and equating to zero, we get

$$\begin{aligned} & a_2\rho^2(\theta^1)^3 - 3a_2\rho^2(\theta^1)^2(x_1^0 + I_0 + \frac{\mu_1^1}{2}) + \rho\theta^1\{a_1 + 2a_2\rho[(x_1^0 + I_0)^2 \\ & + \frac{1}{4}((\sigma_1^1)^2 + (\mu_1^1)^2) + \mu_1^1(x_1^0 + I_0) + 2a_2\mu_2^1\} - [E(z_1^1) + \rho(a_1 + 2a_2\mu_2^1)(x_1^0 \\ & + I_0 + \frac{\mu_1^1}{2})] = 0 \quad (63) \end{aligned}$$

Substituting the numerical values and simplifying, we get

$$.00000502 (\theta^1)^3 - .013459122 (\theta^1)^2 + 1.527\theta^1 - [E(z_1^1) + 1351] = 0 \quad (64)$$

where $E(z_1^1)$ is obtained from equations (58) through (61). Now θ^1 as determined by equation (64) is the optimal decision, if it satisfies the following conditions:

- 1) $E\left[\frac{\partial^2 H^1}{\partial (\theta^1)^2} \mid \bar{x}^0\right] > 0$
- 2) $140 \leq \theta^1 \leq \min(980, x_1^0 + 133.7)$

Summarizing the sequence of optimal decisions, $\bar{\theta}^n$, $n = 1, 2, 3, 4$, is to be obtained from the solution of four simultaneous equations (43), (50), (57), and (64). To obtain the optimal decision values the following two techniques are used:

- 1). a numerical computational method
- 2). the simplex search technique

The computational procedure by both methods is presented in the following section.

COMPUTATIONAL PROCEDURE

The numerical computational method which is very useful in obtaining the optimal values by the maximum principle is first explained. By using equations (43) through (47), (50) through (54) and (57) through (60), the optimal sequence of decision variables, $\bar{\theta}^n$, can be found. The particular

algorithm used to accomplish this is as follows:

Step 1. Assume a set of values θ^1 , θ^2 , θ^3 and θ^4 as trial.

Step 2. Use equations (1), (3) and (15) to obtain the value of the state variables x_1^n , x_2^n and x_3^n at each stage of the process. Start at $n = 1$ and proceed to $n = 4$.

Step 3. Calculate the values of the adjoint variables, z_1^n , z_2^n and z_3^n . Work backward at $n = 4$ and proceeding to $n = 1$.

Step 4. Calculate $\partial H^n / \partial \theta^n$ and $\partial^2 H^n / \partial (\theta^n)^2$ by equations (31), (43), (50), (57) and (64), using the values of x_1^n , x_3^n and z_1^n , z_2^n , and z_3^n obtained above.

Step 5. Compute a new sequence of decision variables θ^n from the following equation.

$$(\theta^n)_{\text{revised}} = (\theta^n)_{\text{old}} + \Delta \theta^n \quad (65)$$

where $\Delta \theta^n$ is given by

$$\Delta \theta^n = - \frac{\partial H^n}{\partial \theta^n} / \frac{\partial^2 H^n}{\partial (\theta^n)^2} \quad (66)$$

Step 6. Return to step 2 and repeat the procedure until the new set of decision variables is sufficiently close to the previous set to indicate adequate convergence.

It is worth noting that when the optimal point is not reached, the revised set of decision given by equation (65) are assumed and the computations are repeated. For minimization of the Hamiltonian, H^n , the second derivative

of the Hamiltonian with respect to the chosen decision variable,

$$\frac{\partial^2 H^n}{\partial (\theta^n)^2}$$

is positive. When the first derivative of the Hamiltonian with respect to the decision variable,

$$\frac{\partial H^n}{\partial \theta^n}$$

is negative, then the increment for the decision variable, $\Delta \theta^n$, should be positive, and if

$$\frac{\partial H^n}{\partial \theta^n}$$

is positive, $\Delta \theta^n$ should be negative in order that the decision variable approaches to the optimal point. The magnitude and sign of the increment $\Delta \theta^n$ is given by equation (66).

The computer program written for applying this algorithm, the table of notations, the numerical results obtained for different iterations and the optimal solution are presented in Appendix. A number of iterations with different assumed values for θ^n are needed before locating the optimal point. The assumed values of $\Delta \theta^n$ are varied at different stages, using larger values for $\Delta \theta^n$ at the beginning and smaller values as the iteration converges. A minimum increment of .001 is used to arrive at the optimal point. The optimal decision values obtained from the solution of the above equations when all

values of $\partial H^n / \partial \theta^n$ are less than the allowable error preassigned to them are

$$\bar{\theta}^1 = 139.85$$

$$140 \text{ Kcfs day}$$

$$\bar{\theta}^2 = 167.25 \text{ Kcfs day}$$

$$\bar{\theta}^3 = 240.32 \text{ Kcfs day}$$

$$\bar{\theta}^4 = 494.46 \text{ Kcfs day}$$

For the optimal policy, the minimum total cost of supplying the supplemental energy is

$$\bar{x}_3^4 = \$26,107.56$$

The second method used to find the optimal decision values of the problem is the simplex search technique. The details of the simplex method for function minimization can be found from reference (19). The computer program for the simplex search technique and the results obtained are presented in the Appendix. The optimal solution obtained by simplex method is given below.

$$\bar{\theta}^1 = 138.639 \text{ Kcfs day}$$

$$\bar{\theta}^2 = 167.057 \text{ Kcfs day}$$

$$\bar{\theta}^3 = 240.183 \text{ Kcfs day}$$

$$\bar{\theta}^4 = 494.741 \text{ Kcfs day}$$

$$\bar{x}_3^4 = \$26105.8$$

DISCUSSION OF THE RESULTS

From the solution obtained it is found that for the linear case, the same solution can be obtained by using the deterministic maximum principle also when the random variables are replaced by their expected values. In other words, the certainty equivalence principle is found to hold good for the linear cost function while it is found to be not applicable for the quadratic cost function case. The reason could be due to the use of the parameter, variance of the random variable, in the quadratic cost function solution. Another noticeable difference between the solutions of the two cost functions is that the expected value of the demand random variable does not directly affect the optimal decision value in the linear cost function while for the quadratic cost function, the optimal decisions in each stage are directly influenced by the expected values of the demand random variable in each stage.

From the results obtained it is seen that there is a wide difference in the values for the minimum cost by the linear and quadratic cost functions. A high value of cost is expected in the quadratic cost function because of the squared terms involved in the cost function expression, which have higher magnitudes at the optimal condition. Hence it is seen that suitability of a linear or a quadratic cost function for any specific system depends upon its own requirements and nature of the system.

Also a proper determination of the system constants (like a_1 , a_2 , ρ , etc.) is needed since they considerably affect the cost values obtained.

It is felt that in the above presented solution the effect of the system constants a_1 and a_2 are not fully brought out on the cost values (since they are considered equal to one), which needs more attention in any further work. It is found from the numerical solution obtained that the specified system constraints have more influence in the determination of the decision values in the linear cost function case than in the case of the quadratic cost function. On the other hand, if the optimal decisions obtained using the stationality condition fall outside the range of the constraints in the quadratic cost function case, the computation of optimal decisions and the corresponding determining equations would have been entirely different.

For the quadratic cost function case, the two methods used for the numerical computation of optimal decisions give an opportunity to compare the adaptability of these two methods for use in optimization problems. With an initial assumed value of 100 for all θ 's and using a step size of unity, the number of iterations required to reach the optimal point by the simplex search technique method was 26 while the number of iterations required by the numerical computational algorithm was 39. Though the latter method took a larger number of iterations to reach the optimum, the total computer execution time required by that method was only 1.25 minutes on the IBM-1620 computer while the computer execution time required for the simplex search technique was 1.09 minutes on the IBM-360 computer (which is considered as at least ten times faster than the IBM-1620 computer). However the numerical computational algorithm requires an initial judgement of the range of optimal decisions and based on the judgement a few initial runs are to be carried out with different base point values to locate the

optimal point. Another point worth noting is that both methods give the same optimal solution with reasonable accuracy. The optimal policy obtained is also found to be the global optimal policy with different sets of base point values by the search technique method.

SCOPE FOR FUTURE WORK

The hydroelectric water storage model which is considered an important optimization model can be extensively studied further with modifications on the following factors:

- (1). A bigger problem formulated with more stages (probably a twelve stage process with each stage representing one month interval) can be considered and the solution by the maximum principle be compared with that obtained by the dynamic programming method (8).
- (2). Instead of a single reservoir problem, a combined storage system of three or more plants (as used in reference (1) for a three plant system by gradient technique) can be investigated.
- (3). Better statistical procedures can be used to determine the mean and variance of the random variables from collecting more information on the random variables from the past history of the system.
- (4). Cost function can be modified so that it is made to represent realistic conditions in a better way. Presently the validity of the assumed cost function is not known which can be analyzed in future.

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6. A PRODUCTION AND INVENTORY CONTROL PROBLEM WITH DISCRETE PROBABILITY DISTRIBUTION

INTRODUCTION

The maximum principle and dynamic programming are considered to be two most important optimization techniques which can be effectively used for the optimization of multistage processes. Each of these two methods, however, has comparative advantages over the other, depending on the problem to be solved (1). Here a production and inventory control problem with given discrete values for the cost function and a discrete probability distribution for the random variable (demand) for which the stochastic dynamic programming can be directly used is modified by fitting a regression equation for the discrete values and then the stochastic maximum principle is applied to obtain the solution. The solution obtained by the two methods are then compared. Hence this problem indicates how a multistage discrete valued problem for which the dynamic programming is normally used can be modified to obtain the solution by the maximum principle. The problem considered here is taken from reference (2) and the solution by the two methods are presented in the following sections.

STATEMENT OF THE PROBLEM

The problem considered is a production decision problem associated with aircraft industry. A company that produces autopilots for aircraft can produce 0, 1, or 2 autopilots per month. The company is planning its production schedule for the next three months, (say): February, March and April. At the beginning of February, it will have one autopilot in the inventory. During February, March and April it anticipates demand of

1, 2 or 3 autopilots respectively during each month with corresponding estimated probabilities of demand as shown in Table 1. The production and inventory costs are given in Table 2 where inventory cost refers to the beginning of the month inventory. In addition, if demand is not met in any month, there is a shortage cost of \$10,000 per unit (shortages in any month are not carried over to the next month). Also a certain inventory cost is associated with the inventory at the end of April which is shown in Table 3.

The optimization problem is to find the production schedule for the next three months that will minimize the sum of production, inventory and shortage costs. The formulation and solution of the above stated problem by the stochastic dynamic programming is presented first in the next section, followed by the maximum principle solution.

STOCHASTIC DYNAMIC PROGRAMMING SOLUTION

In what follows, we shall present briefly the approach of the stochastic dynamic programming method (3) for the optimization of multistage decision processes as shown in Fig. 1. Suppose that we want to minimize

$$E(S) = E[(c)^T x^0] \quad (1)$$

with respect to

$$\theta^n = \theta^n(x^n), \quad n = 1, 2, \dots, N \quad (2)$$

The system equations are

$$x^{n-1} = f^n(x^n, \theta^n, \xi^n), \quad n = 1, 2, \dots, N \quad (3)$$

TABLE 1

Estimated Probability of Demand For Each Month

Demand	Probability of Demand		
	February	March	April
0	$\frac{1}{4}$	0	$\frac{2}{3}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$
2	$\frac{1}{4}$	$\frac{1}{2}$	0
3	0	$\frac{1}{4}$	0

TABLE 2

The Production and Inventory Costs For the Autopilots

Production (autopilots/month)	0	1	2	
Total Cost (in \$1000)	15	20	35	
Inventory (# of autopilots in stock)	0	1	2	3
Total Inventory Cost (in \$1000/month)	2	5	9	15

TABLE 3

End of April Inventory Cost For the Autopilots

Inventory	0	1	2	3
Cost (in \$1000)	10	0	5	10

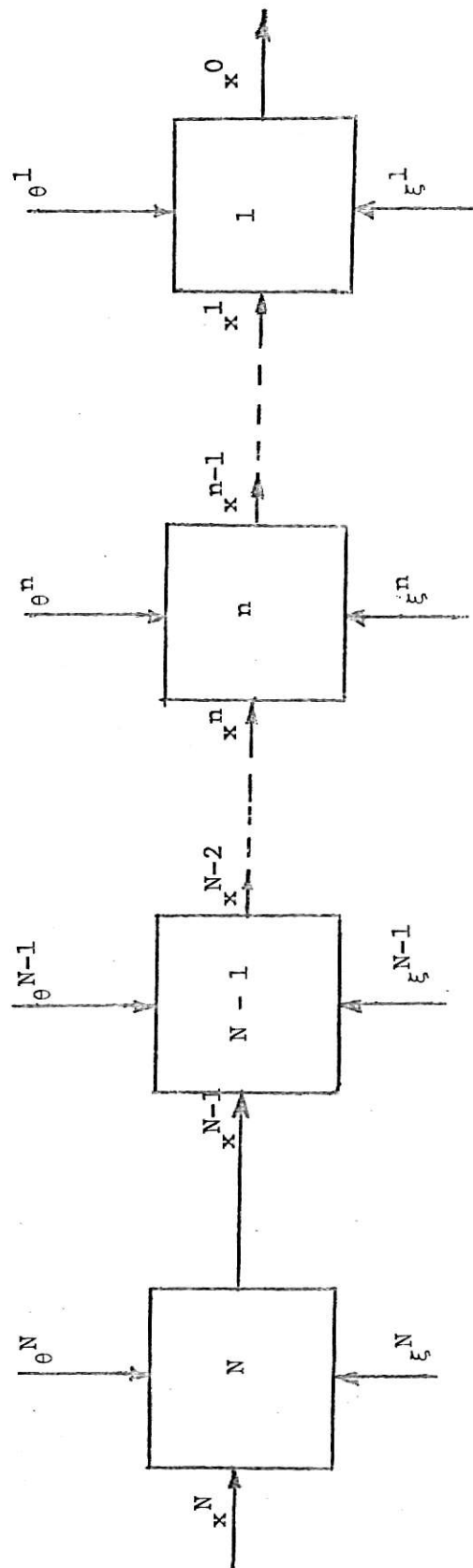


FIG. 1. A SIMPLE MULTISTAGE STOCHASTIC PROCESS WITH BACKWARD NUMBERING

where ξ^n , $n = 1, 2, \dots, N$ are assumed to be mutually independent with known distribution density functions. By defining

$$R^n(x^n) = \min_{\{\theta^n, \theta^{n-1}, \dots, \theta^1\}} E[(c)^T x^0 | x^n], \quad n = 1, 2, \dots, N \quad (4)$$

we have

$$\begin{aligned} R^n(x^n) &= \min_{\{\theta^n, \theta^{n-1}, \dots, \theta^1\}} E[(c)^T x^0 | x^n] \\ &= \min_{\{\theta^n, \theta^{n-1}, \dots, \theta^1\}} E\{E[(c)^T x^0 | x^{n-1}] | x^n\} \\ &= \min_{\theta^n} E\left\{\min_{(\theta^{n-1}, \dots, \theta^1)} E[(c)^T x^0 | x^{n-1}] | x^n\right\} \\ &= \min_{\theta^n} E\{R^{n-1}(x^{n-1}) | x^n\} \end{aligned} \quad (5)$$

Equation (5) is essentially the mathematical statement of the principle of optimality.

By using the definition of equation (4) for $n = N-1$ and the recursive relation of equation (5), we can express the $\min E(S)$ as follows:

$$\min_{\{\theta^1, \theta^2, \dots, \theta^N\}} E[S] = \min_{\{\theta^N, \theta^{N-1}, \dots, \theta^2, \theta^1\}} E[(c)^T x^0]$$

$$= \min_{\{\theta^N, \theta^{N-1}, \dots, \theta^2\}} E[R^0(x^0) | x^1]$$

$$= \min_{\{\theta^N, \theta^{N-1}, \dots, \theta^3\}} E[R^1(x^1) | x^2]$$

.....

$$= \min_{\{\theta^N\}} E[R^{N-1}(x^{N-1}) | x^N] \quad (6)$$

Working equation (5) backward starting with $R^1(x^1)$, we obtain a set of optimal decisions, $\bar{\theta}^n = \bar{\theta}^n(x^n)$, $n = 1, 2, \dots, N$.

Using the above stated algorithm, the solution for the problem is obtained. Each month is defined as a stage (Fig. 1) with backward numbering. The computation of minimum cost path in each stage for the different initial inventory level at the beginning of the stage is started with the first stage (ie, with the month of April). Let

x_1^n = state vector representing the number of units in the inventory at the beginning of the nth stage (month)

D^n = decision vector representing the number of units produced
in the nth stage

$\xi^n(p^n)$ = an independently distributed random variable representing
the demand in number of units in the nth stage with a prob-
ability of p^n .

Then the performance equation for the system can be written as

$$x_1^{n-1} = x_1^n + D^n - \xi^n, \quad n = 1, 2, 3 \quad (7)$$

The expected cost function is defined as follows.

Let $f_1^n(x_1^n)$ = the minimum expected cost for the n stage process for a given
value of x_1^n

$$= \min_{D^n} E[r^n(x_1^n, D^n, \xi^n) + f_1^{n-1}(x_1^{n-1}) | x_1^n],$$

for $n = 2, 3$

$$= \min E[r^1(x_1^1, D^1, \xi^1) + r^0(x_1^0) | x_1^1]$$

for $n = 1$ (8)

where

$r^n(x_1^n, D^n, \xi^n)$ = sum of production, inventory and shortage costs for
the nth stage

$r_1^0(x_1^0)$ = End of April inventory cost for the first stage (April)
only for a given value of x_1^0

$f_1^{n-1}(x_1^{n-1})$ = The minimum expected cost for the (n-1) stage process
for a given value of x_1^{n-1}

For each stage, the possible production rates for each of the possible initial inventory levels are calculated and the corresponding total cost values are also computed. Then the production rate that gives minimum expected cost is the optimal decision for the particular inventory level. For example with an initial inventory of zero (ie, $x_1^1 = 0$) at the beginning of April, the different possible production rates (D^1) and the corresponding total costs are shown below.

x_1^1	$\xi^1(p^1)$	D^1	x_1^0	$[r_1(x_1^1, D^1, \xi^1) + r_0(x_1^0) x_1^1 = 0]$
0	$(\frac{2}{3})$	0	0	$15 + 10 + 2 = 27$
		1	1	$0 + 20 + 2 = 22$
		2	2	$5 + 35 + 2 = 42$
1	$(\frac{1}{3})$	0	0	$10 + 15 + 2 + 10 = 37$
		1	0	$10 + 20 + 2 + 10 = 42$
		2	1	$0 + 35 + 2 + 0 = 37$

Hence for $x_1^1 = 0$,

the expected total cost with $D^1 = 0$

$$= 27 \times \frac{2}{3} + 37 \times \frac{1}{3} = 30 \frac{1}{3}$$

the expected total cost with $D^1 = 1$

$$= 22 \times \frac{2}{3} + 32 \times \frac{1}{3} = 25 \frac{1}{3}$$

the expected total cost with $D^1 = 2$

$$= 42 \times \frac{2}{3} + 37 \times \frac{1}{3} = 40 \frac{1}{3}$$

Therefore

$$\begin{aligned} f_1(x_1^1 = 0) &= \min_{D^1} E[r_1(x_1^1, D^1, \xi^1) + r_0(x_1^0) | x_1^1 = 0] \\ &= 25 \frac{1}{3} \text{ with } D^1 = 1 \end{aligned}$$

Similarly the optimal decision, D^1 , for $x_1^1 = 1, 2$, and 3 is calculated as shown below.

x_1^1	$\xi^1(p^1)$	D^1	x_1^0	$[r_1(x_1^1, D^1, \xi^1) + r_0(x_1^0) x_1^1]$
	$0 (\frac{2}{3})$	0	1	$0 + 15 + 5 + 0 = 20$
		1	2	$5 + 20 + 5 + 0 = 30$
1		2	3	$10 + 35 + 5 + 0 = 50$
	$1 (\frac{1}{3})$	0	0	$10 + 15 + 5 = 30$

$$1 \quad 1 \quad 0 + 20 + 5 = 25$$

$$2 \quad 2 \quad 5 + 35 + 5 = 45$$

$$\text{Therefore } f_1(x_1^1 = 1) = \min_{D^1} E[r_1(x_1^1, D^1, \xi^1) + r_0(x_1^0) | x_1^1 = 1]$$

$$= 20 \times \frac{2}{3} + 30 \times \frac{1}{3}$$

$$= 23 \frac{1}{3} \text{ with } D^1 = 0$$

$$x_1^1 \quad \xi^1(p^1) \quad D^1 \quad x_1^0 \quad [r_1(x_1^1, D^1, \xi^1) + r_0(x_1^0) | x_1^1]$$

$$0 \left(\frac{2}{3}\right) \quad 0 \quad 2 \quad 5 + 15 + 9 + 0 = 29$$

$$1 \quad 3 \quad 10 + 20 + 9 + 0 = 39$$

$$2 \quad 0 \quad 1 \quad 0 + 15 + 9 = 24$$

$$1 \left(\frac{1}{3}\right) \quad 1 \quad 2 \quad 5 + 20 + 9 = 34$$

$$2 \quad 3 \quad 10 + 35 + 9 = 54$$

$$\text{Therefore } f_1(x_1^1 = 2) = 29 \times \frac{2}{3} + 24 \times \frac{1}{3}$$

$$= 27 \frac{1}{3} \text{ with } D^1 = 0$$

$$x_1^1 \quad \xi^1(p^1) \quad D^1 \quad x_1^0 \quad [r_1(x_1^1, D^1, \xi^1) + r_0(x_1^0) | x_1^1]$$

$$3 \quad 0 \left(\frac{2}{3}\right) \quad 0 \quad 3 \quad 10 + 15 + 15 = 40$$

$$1 \left(\frac{2}{3}\right) \quad 0 \quad 2 \quad 5 + 15 + 15 = 35$$

$$1 \quad 3 \quad 10 + 20 + 15 = 45$$

Therefore $f_1(x_1^1 = 3) = 40 \times \frac{2}{3} + 35 \times \frac{1}{3}$

$$= 38 \frac{1}{3} \text{ with } D^1 = 0$$

Thus with these values the optimal network for the one stage process is constructed in Fig. 2.

Now for the two stage process consisting of stages 1 and 2, the optimal decision, D^2 , associated with each possible initial inventory level (x_1^2) at the beginning of the second stage (March) is obtained as follows.

x_1^2	$\xi^2(p^2)$	D^2	x_1^1	$[r_2(x_1^2, D^2, \xi^2) + f_1(x_1^1) x_1^2]$
1 $(\frac{1}{4})$	0	0	0	$15 + 2 + 10 + 25 \frac{1}{3} = 52 \frac{1}{3}$
		1	0	$20 + 2 + 0 + 25 \frac{1}{3} = 47 \frac{1}{3}$
		2	1	$35 + 2 + 23 \frac{1}{3} = 60 \frac{1}{3}$
2 $(\frac{1}{2})$	0	0*		$15 + 2 + 20 + 25 \frac{1}{3} = 62 \frac{1}{3}$
		1	0*	$20 + 2 + 10 + 25 \frac{1}{3} = 57 \frac{1}{3}$
		2	0	$35 + 2 + 0 + 25 \frac{1}{3} = 62 \frac{1}{3}$

(The * symbol indicates shortage over the demand occurs.)

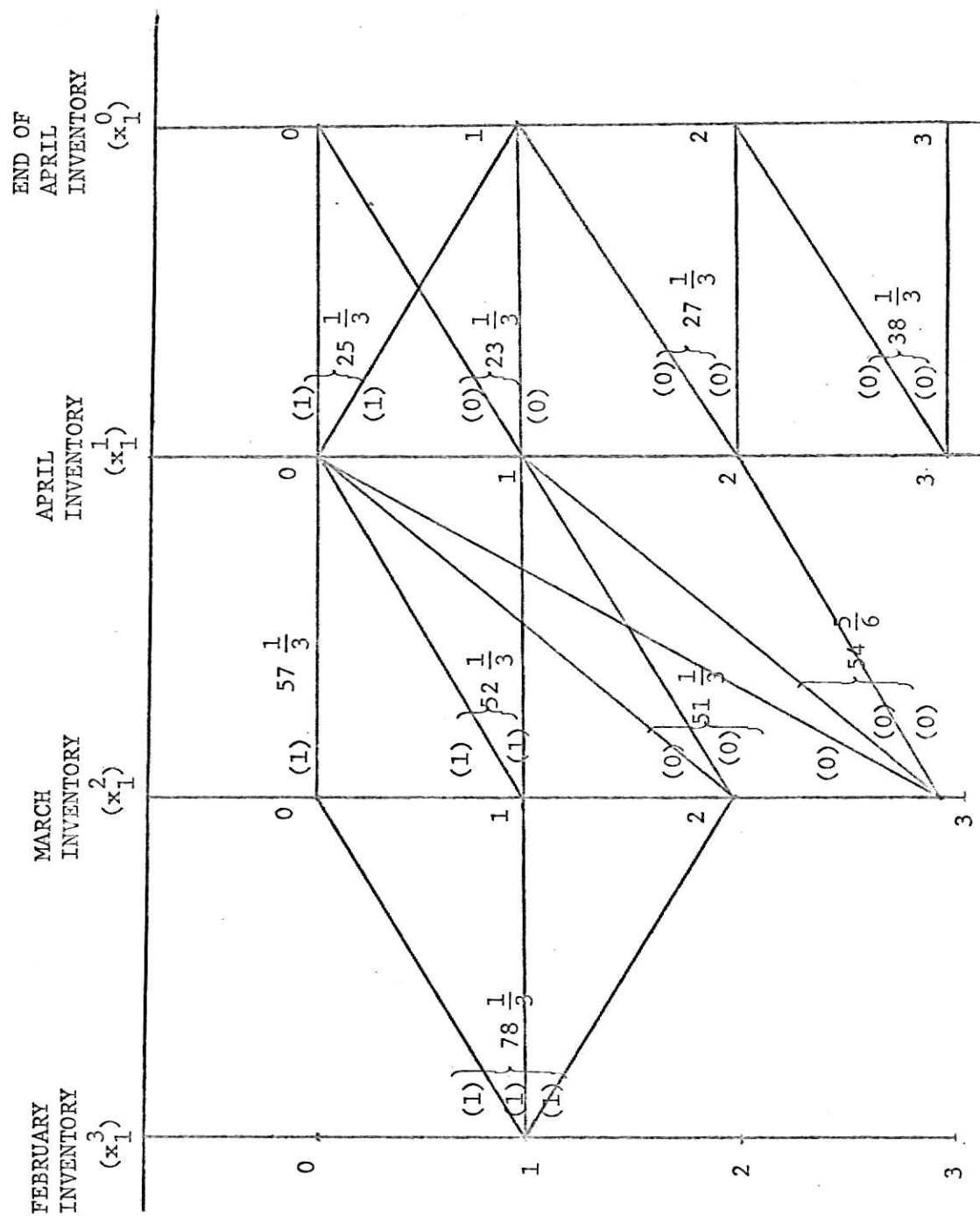


FIG. 2. OPTIMAL NET WORK BY DYNAMIC PROGRAMMING

$$3 \left(\frac{1}{4}\right) \quad 0 \quad 0^* \quad 15 + 2 + 30 + 25 \frac{1}{3} = 72 \frac{1}{3}$$

$$1 \quad 0^* \quad 20 + 2 + 20 + 25 \frac{1}{3} = 67 \frac{1}{3}$$

$$2 \quad 0^* \quad 35 + 2 + 10 + 25 \frac{1}{3} = 72 \frac{1}{3}$$

For $x_1^2 = 0$,

$$\text{Expected cost with } D^2 = 0 = 52 \frac{1}{3} \times \frac{1}{4} + 62 \frac{1}{3} \times \frac{1}{2} + 72 \frac{1}{3} \times \frac{1}{4}$$

$$= 62 \frac{1}{3}$$

$$\text{Expected cost with } D^2 = 1 = 47 \frac{1}{3} \times \frac{1}{4} + 57 \frac{1}{3} \times \frac{1}{2} + 67 \frac{1}{3} \times \frac{1}{4}$$

$$= 57 \frac{1}{3}$$

$$\text{Expected cost with } D^2 = 2 = 60 \frac{1}{3} \times \frac{1}{4} + 62 \frac{1}{3} \times \frac{1}{2} + 72 \frac{1}{3} \times \frac{1}{4}$$

$$= 64 \frac{1}{3}$$

$$\text{Therefore, } f_2(x_1^2 = 0) = \min_{D^2} E[r_2(x_1^2, D^2, \xi^2) + f_1(x_1^1) | x_1^2]$$

$$= 57 \frac{1}{3} \text{ with } D^2 = 1$$

Similarly the optimal decision D^2 and minimum expected cost, $f_2(x_1^2)$ for $x_1^2 = 1, 2$, and 3 are obtained as shown below.

$$x_1^2 \quad \xi^2 (p^2) \quad D^2 \quad x_1^2 \quad [r^2(x_1^2, D^2, \xi^2) + f_1(x_1^1) | x_1^2]$$

$$1 \left(\frac{1}{4}\right) \quad 0 \quad 0 \quad 5 + 15 + 0 + 25 \frac{1}{3} = 45 \frac{1}{3}$$

$$1 \quad 1 \quad 5 + 20 + 0 + 23 \frac{1}{3} = 48 \frac{1}{3}$$

$$2 \quad 2 \quad 5 + 35 + 0 + 27 \frac{1}{3} = 67 \frac{1}{3}$$

$$1 \quad 2 \left(\frac{1}{2}\right) \quad 0 \quad 0^* \quad 5 + 15 + 10 + 25 \frac{1}{3} = 55 \frac{1}{3}$$

$$1 \quad 0 \quad 5 + 20 + 0 + 25 \frac{1}{3} = 50 \frac{1}{3}$$

$$2 \quad 1 \quad 5 + 35 + 0 + 23 \frac{1}{3} = 63 \frac{1}{3}$$

$$3 \left(\frac{1}{4}\right) \quad 0 \quad 0^* \quad 5 + 15 + 20 + 25 \frac{1}{3} = 65 \frac{1}{3}$$

$$1 \quad 0^* \quad 5 + 20 + 10 + 25 \frac{1}{3} = 60 \frac{1}{3}$$

$$2 \quad 0 \quad 5 + 35 + 0 + 25 \frac{1}{3} = 65 \frac{1}{3}$$

$$\text{Therefore, } f_2(x_1^2 = 1) = 48 \times \frac{1}{3} + 50 \frac{1}{3} \times \frac{1}{2} + 60 \frac{1}{3} \times \frac{1}{4}$$

$$= 52 \frac{1}{3} \text{ with } D^2 = 1$$

$$x_1^2 \quad \xi^2 (p^2) \quad D^2 \quad x_1^1 \quad [r_2(x_1^2, D^2, \xi^2) + f_1(x_1^1) | x_1^2]$$

$$1 \left(\frac{1}{4}\right) \quad 0 \quad 1 \quad 9 + 15 + 0 + 23 \frac{1}{3} = 47 \frac{1}{3}$$

		1	2	$9 + 20 + 0 + 27 \frac{1}{3} = 56 \frac{1}{3}$
		2	3	$9 + 35 + 0 + 38 \frac{1}{3} = 82 \frac{1}{3}$
2	$2 (\frac{1}{2})$	0	0	$9 + 15 + 0 + 25 \frac{1}{3} = 49 \frac{1}{3}$
		1	1	$9 + 20 + 0 + 23 \frac{1}{3} = 52 \frac{1}{3}$
		2	2	$9 + 35 + 0 + 27 \frac{1}{3} = 71 \frac{1}{3}$
3	$(\frac{1}{4})$	0	0*	$9 + 15 + 10 + 25 \frac{1}{3} = 59 \frac{1}{3}$
		1	0	$9 + 20 + 0 + 25 \frac{1}{3} = 54 \frac{1}{3}$
		2	1	$9 + 35 + 0 + 25 \frac{1}{3} = 69 \frac{1}{3}$

Therefore, $f_2(x_1^2 = 2) = 47 \frac{1}{3} \times \frac{1}{4} + 49 \frac{1}{3} \times \frac{1}{2} + 57 \frac{1}{3} \times \frac{1}{4}$

$$= 51 \frac{1}{3} \text{ with } D^2 = 0$$

x_1^2	$\xi(p^2)$	D^2	x_1^1	$[r_2(x_1^2, D^2, \xi^2) + f_1(x_1^1) x_1^2]$
	$1 (\frac{1}{4})$	0	2	$15 + 15 + 0 + 27 \frac{1}{3} = 57 \frac{1}{3}$
		1	3	$15 + 20 + 0 + 38 \frac{1}{3} = 73 \frac{1}{3}$
3	$2 (\frac{1}{2})$	0	1	$15 + 15 + 0 + 23 \frac{1}{3} = 53 \frac{1}{3}$
		1	2	$15 + 20 + 0 + 27 \frac{1}{3} = 62 \frac{1}{3}$

	2	3	$15 + 35 + 0 + 38 \frac{1}{3} = 88 \frac{1}{3}$
$3 (\frac{1}{4})$	0	0	$15 + 15 + 0 + 25 \frac{1}{3} = 55 \frac{1}{3}$
	1	1	$15 + 20 + 0 + 23 \frac{1}{3} = 58 \frac{1}{3}$
	2	2	$15 + 35 + 0 + 27 \frac{1}{3} = 77 \frac{1}{3}$

Thus with these values, the optimal net work for the two stage process is constructed in Fig. 2.

Now for the three stage process consisting of stages 1, 2, and 3, the optimal decision associated with the given initial inventory level ($x_1^3 = 1$) at the beginning of the third stage (February) is obtained as follows.

x_1^3	$\xi^3(p^3)$	D^3	x_1^2	$[r_3(x_1^3, D^3, \xi^3) + f_2(x_1^2) x_1^3]$
	$0(\frac{1}{4})$	0	1	$5 + 15 + 0 + 52 \frac{1}{3} = 72 \frac{1}{3}$
1		1	2	$5 + 20 + 0 + 51 \frac{1}{3} = 76 \frac{1}{3}$
		2	3	$5 + 35 + 0 + 54 \frac{5}{6} = 94 \frac{5}{6}$
	$1(\frac{1}{2})$	0	0	$5 + 15 + 0 + 57 \frac{1}{3} = 77 \frac{1}{3}$
		1	1	$5 + 20 + 0 + 52 \frac{1}{3} = 77 \frac{1}{3}$
		2	2	$5 + 35 + 0 + 51 \frac{1}{3} = 91 \frac{1}{3}$

$$2 \left(\frac{1}{4}\right) \quad 0 \quad 0^* \quad 5 + 15 + 10 + 57 \frac{1}{3} = 87 \frac{1}{3}$$

$$1 \quad 0 \quad 5 + 20 + 0 + 57 \frac{1}{3} = 82 \frac{1}{3}$$

$$2 \quad 1 \quad 5 + 35 + 0 + 52 \frac{1}{3} = 92 \frac{1}{3}$$

$$\text{Therefore, } f_3(x_1^3 = 1) = 76 \frac{1}{3} \times \frac{1}{4} + 77 \frac{1}{3} \times \frac{1}{2} + 82 \frac{1}{3} \times \frac{1}{4}$$

$$= 78 \frac{1}{3} \text{ with } D^3 = 1$$

With these calculated values, the optimal net work for the three stage process is completed in Fig. 2. The figure gives the optimal path to be followed for a known inventory level at the beginning of each stage with an assigned probability. Thus this optimal net work indicates the optimal policy with an estimated probability of occurrence.

STOCHASTIC MAXIMUM PRINCIPLE SOLUTION

To apply the stochastic maximum principle to obtain the optimal solution the following procedure is followed:

- (1) Fit appropriate regression equations for the given discrete values of the different costs so that the total cost function equation is obtained in terms of the decision and state variables in each stage.
- (2) Calculate the expected value of the random variable (demand) in each stage.
- (3) Apply the stochastic maximum principle to the system with the total cost function equation obtained by regression and using the calculated expected values for the demand find the optimal solution.

FITTING REGRESSION EQUATION FOR THE COST FUNCTION

The method of fitting linear and multiple regression equations for discrete valued functions can be well understood from reference (4).

Hence for the following quantities appropriate regression equations are to be fitted to define the total cost function.

- (1) Production cost: From the data given in Table 6, the production cost is found to have a parabolic relationship with the number of units produced. Hence a quadratic regression equation can be fitted for the given values.
- (2) Inventory cost: This can also be expressed as a quadratic function in terms of the inventory available at the beginning of each stage.
- (3) Inventory cost for the end of April inventory: This cost, which is applicable only for the month of April, can also be expressed as a quadratic function of the end of April inventory.

Before presenting the regression analysis, the performance equation of the system (as shown in Fig. 3) for the application of the maximum principle is explained below. Let

x_1^n = state variable representing the number of units in inventory
at the end of the nth stage (month)

θ^n = decision variable representing the number of units produced
in nth stage

ξ^n = an independently distributed random variable representing the
demand in nth stage

Then,

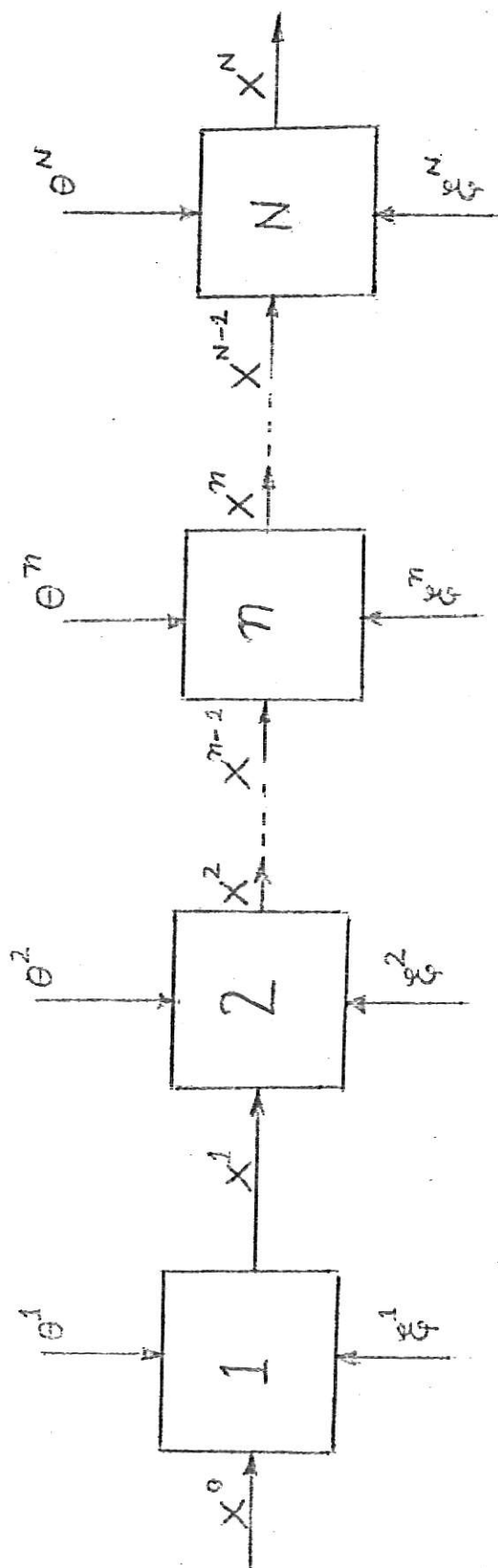


FIG. 3. A SIMPLE MULTI-STAGE STOCHASTIC PROCESS

$$x_1^n = x_1^{n-1} + \theta^n - \xi^n, \quad n = 1, 2, \dots, N \quad (9)$$

$$x_1^0 = 1 \quad (10)$$

Let $g_1^n(\theta^n)$ = the production cost in the nth stage and let

$$g_1^n(\theta^n) = a + b(\theta^n)^2$$

The values of the regression coefficients, a and b , are calculated as follows.

Let $Y = g_1^n(\theta^n)$ and $X = \theta^n$

Then $Y = a + b X^2$

Let $X_1 = X^2$ so that the simple regression equation becomes

$$Y = a + b X_1$$

The deviations from regression are given by $(X_1 - \bar{X}_1)$ and $(Y - \bar{Y})$ where \bar{X}_1 and \bar{Y} are the average values of X_1 and Y .

From Table 4, substituting for the values, we obtain

$$b = \frac{\sum (X_1 - \bar{X}_1)(Y - \bar{Y})}{\sum (X_1 - \bar{X}_1)^2} = \frac{43.33}{8.67} = 5.00$$

$$a = \bar{Y} - b\bar{X}_1 = 23.33 - 5 \times 1.66 = 15.00$$

$$\text{Therefore, } g_1^n(\theta^n) = 15 + 5(\theta^n)^2 \quad (11)$$

TABLE 4

SIMPLE REGRESSION EQUATION FOR PRODUCTION COST

Y	$(Y - \bar{Y})$	X	$X_1 = X^2$	$(X_1 - \bar{X}_1)$	$(X_1 - \bar{X})^2$	$(X_1 - \bar{X}_1)$ $(Y - \bar{Y})$
15	-8.33	0	0	-1.66	2.78	13.89
20	-3.33	1	1	-.66	.45	2.22
35	11.66	2	4	2.33	5.45	27.22

$$\begin{aligned} \bar{Y} &= 23.33 & \bar{X}_1 &= 1.66 & \sum (X_1 - \bar{X}_1)^2 &= 8.67 & \sum (X_1 - \bar{X}_1)(Y - \bar{Y}) &= 43.33 \end{aligned}$$

This regression equation for the production cost is found to hold good exactly in predicting the production cost values for given values of θ^n .

Now let $g_2^n(x_1^{n-1})$ = Inventory cost in the nth stage for the
inventory at the beginning of the nth stage
(month)

and let $g_2^n(x_1^{n-1}) = a + bx_1^{n-1} + c(x_1^{n-1})^2$

The regression coefficients, a, b and c are calculated as follows.

Let $Y = g_2^n(x_1^{n-1})$ and $X_1 = x_1^{n-1}$

Let $(X_1)^2 = X_2$ so that the multiple regression equation becomes

$$Y = a + bX_1 + cX_2$$

The deviations from regression are given by $(X_1 - \bar{X}_1)$, $(X_2 - \bar{X}_2)$ and $(Y - \bar{Y})$ where \bar{X}_1 , \bar{X}_2 and \bar{Y} represent the average values of X_1 , X_2 and

Y. The normal equations for the regression are

$$b \sum (X_1 - \bar{X}_1)^2 + c \sum (X_1 - \bar{X}_1)(X_2 - \bar{X}_2) = \sum (X_1 - \bar{X}_1)(Y - \bar{Y})$$

$$b \sum (X_1 - \bar{X}_1)(X_2 - \bar{X}_2) + c \sum (X_2 - \bar{X}_2)^2 = \sum (X_2 - \bar{X}_2)(Y - \bar{Y})$$

Substituting for the values from Table 5 in the normal equations and solving for the constants b and c, we obtain

TABLE 5

MULTIPLE REGRESSION EQUATION FOR INVENTORY COST

Y	(Y- \bar{Y})	X ₁	(X ₁ - \bar{X}_1)	X ₂	(X ₂ - \bar{X}_2)	(X ₁ - \bar{X}_1) ²	(X ₂ - \bar{X}_2) ²	(X- \bar{X}) (Y- \bar{Y})	(X ₂ - \bar{X}_2) (Y- \bar{Y})	(X ₁ - \bar{X}_1) (X ₂ - \bar{X}_2)
2	-5.75	0	-1.5	0	-3.5	2.25	12.24	8.64	20.17	5.25
5	-2.75	1	-.5	1	-2.5	.25	6.25	1.375	6.88	1.25
9	1.25	2	.5	4	.5	.25	.25	.625	.625	.25
15	7.25	3	1.5	9	5.5	2.25	30.3	10.89	39.85	8.25
<hr/>										
$\bar{Y}=7.75$		$\bar{X}_1=1.5$		$\bar{X}_2=3.5$		$\sum (X_1-\bar{X}_1)^2$	$\sum (X_2-\bar{X}_2)^2$	$\sum (X_1-\bar{X}_1) \sum (X_2-\bar{X}_2)$	$\sum (X_2-\bar{X}_2)$	$\sum (X_1-\bar{X}_1)$
						= 5.00	= 49.04	(Y- \bar{Y})	(Y- \bar{Y})	(X ₂ - \bar{X}_2)
								=21.53	=67.525	=15.00

$$b = 2.082; \quad c = 0.7414$$

$$a = \bar{Y} - b\bar{X}_1 - c\bar{X}_2$$

$$= 7.75 - 2.082 \times 1.5 - .7414 \times 3.5$$

$$= 2.0321$$

Therefore the regression equation for inventory cost is

$$g_2^n(x_1^{n-1}) = 2.0321 + 2.082x_1^{n-1} + .7414(x_1^{n-1})^2 \quad (12)$$

The above regression equation is able to predict the specified values of the inventory cost with an average percentage deviation between the estimated and specified values of 3.3% in the specified range for x_1^{n-1} .

Now let $g_3^N(x_1^N)$ = Inventory cost for the inventory at the end of the
Nth stage

$$\text{Let } g_3^N(x_1^N) = a + bx_1^N + c(x_1^N)^2$$

The regression coefficients a , b and c are calculated as follows.

$$\text{Let } Y = g_3^N(x_1^N) \text{ and } X_1 = x_1^N$$

Let $(x_1^N)^2 = X_2$ so that the multiple regression equation becomes

$$Y = a + bX_1 + cX_2$$

The deviations from the regression are given by $(X_1 - \bar{X}_1)$, $(X_2 - \bar{X}_2)$ and $(Y - \bar{Y})$ where \bar{X}_1 , \bar{X}_2 and \bar{Y} represent the average values of X_1 , X_2 and Y .

The normal equations for regression are

$$b \sum (x_1 - \bar{x}_1)^2 + c \sum (x_1 - \bar{x}_1)(\bar{x}_2 - \bar{x}_2) = \sum (x_1 - \bar{x}_1)(y - \bar{y})$$

$$b \sum (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + c \sum (x_2 - \bar{x}_2)^2 = \sum (x_2 - \bar{x}_2)(y - \bar{y})$$

Substituting the values from Table 6 in the normal equations and solving for the constants b and c, we obtain

$$b = -10.612; \quad c = 3.704$$

$$a = \bar{y} - b\bar{x}_1 - c\bar{x}_2$$

$$= 6.25 + 10.612 \times 1.5 - 3.704 \times 3.5 = 9.198$$

Therefore the regression equation is

$$g_3^N(x_1^N) = 9.198 - 10.612x_1^N + 3.704(x_1^N)^2 \quad (13)$$

The above regression equation for the end of April inventory cost is able to predict the specified values of the inventory cost at the last stage with an average percentage deviation between the estimated and specified values of 8.3% in the specified range for x_1^N .

CALCULATION OF EXPECTED VALUES

The expected value of the random variable, demand, during each stage is calculated from the discrete probability values given in Table 1 following the method indicated in reference (5) as shown below.

For a discrete type random variable X, the expected value of X is given by

TABLE 6

MULTIPLE REGRESSION EQUATION FOR END OF APRIL INVENTORY COST

Y	(Y- \bar{Y})	X ₁	(X ₁ - \bar{X}_1)	X ₂	(X ₂ - \bar{X}_2)	(X ₁ - \bar{X}_1) ²	(X ₂ - \bar{X}_2) ²	(X ₁ - \bar{X}_1)	(X ₂ - \bar{X}_2)	(X ₁ - \bar{X}_1)	(X ₂ - \bar{X}_2)	(X ₁ - \bar{X}_1)	(X ₂ - \bar{X}_2)
								(Y- \bar{Y})		(Y- \bar{Y})		(X ₁ - \bar{X}_1)	(X ₂ - \bar{X}_2)
10	3.75	0	-1.5	0	-3.5	2.25	12.24	-5.63	-13.13	5.25			
0	-6.25	1	-.5	1	-2.5	.25	6.25	3.125	15.61	1.25			
5	-1.25	2	.5	4	.5	.25	.25	-.625	-.625	.25			
10	3.75	3	1.5	9	5.5	2.25	30.3	5.63	20.6	8.25			
<hr/>													
$\bar{Y}=6.25$		$\bar{X}_1=1.5$		$\bar{X}_2=3.5$		$\sum (X_1-\bar{X}_1)^2$	$\sum (X_2-\bar{X}_2)^2$	$\sum (X_1-\bar{X}_1)$	$\sum (X_2-\bar{X}_2)$	$\sum (X_1-\bar{X}_1)$	$\sum (X_2-\bar{X}_2)$		
						= 5.00	= 49.04	(Y- \bar{Y})	(Y- \bar{Y})	(X ₂ - \bar{X}_2)			
								= 2.5	=22.455	= 15.0			

$$E[X] = \sum_X XP(X)$$

where $P(X)$ = probability density for the particular value of X . Hence the expected value of demand during the first stage is

$$\begin{aligned} E(\xi^1) &= \mu^1 = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times 0 \\ &= 1 \end{aligned} \tag{14}$$

Similarly

$$\begin{aligned} E(\xi^2) &= \mu^2 = 0 + 1 \times \frac{1}{4} + 2 \times \frac{1}{2} + 3 \times \frac{1}{4} \\ &= 2 \end{aligned} \tag{15}$$

$$\begin{aligned} E(\xi^3) &= \mu^3 = 0 \times \frac{2}{3} + 1 \times \frac{1}{3} + 2 \times 0 + 3 \times 0 \\ &= \frac{1}{3} \approx 0 \end{aligned} \tag{16}$$

Using these values of expected demand in each stage, the optimal decisions are computed by the stochastic maximum principle.

NUMERICAL SOLUTION

The shortage cost for the n th stage, which occurs when the demand exceeds the sum of the units available in the inventory and the units produced in the n th stage, can be defined as

Shortage cost for the n th stage

$$= \begin{cases} 10(\xi^n - x_1^{n-1} - \theta^n), & \text{if } \xi^n > x_1^{n-1} + \theta^n \\ 0, & \text{if } \xi^n \leq x_1^{n-1} + \theta^n \end{cases} \quad (17)$$

$$n = 1, 2, \dots, N$$

Summing up the equations (11),(12),(13) and (17), the state variable for the total cost function is now defined. Let

$$x_2^n = \text{Sum of total cost upto and including the } n\text{th stage}$$

Therefore we obtain

$$x_2^n = \begin{cases} x_2^{n-1} + g_1^n(\theta^n) + g_2^n(x_1^{n-1}) + 10(\xi^n - x_1^{n-1} - \theta^n), & \text{if } \xi^n > x_1^{n-1} + \theta^n \\ x_2^{n-1} + g_1^n(\theta^n) + g_2^n(x_1^{n-1}), & \text{if } \xi^n \leq x_1^{n-1} + \theta^n \end{cases} \quad (18)$$

$$n = 1, 2, \dots, N-1$$

$$x_2^0 = 0$$

$$x_2^N = \begin{cases} x_2^{N-1} + g_1^N(\theta^N) + g_2^N(x_1^{N-1}) + 10(\xi^N - x_1^{N-1} - \theta^N) \\ + g_3^N(x_1^N), & \text{if } \xi^N > x_1^{N-1} + \theta^N \\ x_2^{N-1} + g_1^N(\theta^N) + g_2^N(x_1^{N-1}) + g_3^N(x_1^N), & \text{if } \xi^N \leq x_1^{N-1} + \theta^N \end{cases} \quad (19)$$

The objective function to be minimized can be written as

$$E[S] = E[x_2^N] \quad (20)$$

$$\text{with } c_1 = 0; \quad c_2 = 1$$

The Hamiltonian function and the adjoint vectors are defined as follows.

$$H^n = z_1^n f_1^n + z_2^n f_2^n, \quad n = 1, 2, \dots, N \quad (21)$$

$$(z_1^{n-1}, z_2^{n-1}) = (z_1^n, z_2^n) \begin{pmatrix} 1 + \frac{\partial \theta^n}{\partial x_1^{n-1}} & 0 \\ \frac{\partial f_2^n}{\partial x_1^{n-1}} + \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}}, & 1 \end{pmatrix} \quad (22)$$

$$z_1^{n-1} = z_1^n \left(1 + \frac{\partial \theta^n}{\partial x_1^{n-1}} \right) + z_2^n \left(\frac{\partial f_2^n}{\partial x_1^{n-1}} + \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} \right), \quad (23)$$

$$n = 1, 2, \dots, N$$

$$z_1^N = c_1 = 0$$

$$z_2^{n-1} = z_2^n, \quad n = 1, 2, \dots, N$$

$$z_2^N = c_2 = 1$$

Equation (23) can be further simplified to

$$z_1^{n-1} = z_1^n \left(1 + \frac{\partial \theta^n}{\partial x_1^{n-1}} \right) + \left(\frac{\partial f_2^n}{\partial x_1^{n-1}} + \frac{\partial f_2^n}{\partial \theta^n} \frac{\partial \theta^n}{\partial x_1^{n-1}} \right), \quad n = 1, 2, \dots, N \quad (24)$$

$$z_1^N = c_1 = 0$$

$$z_2^n = 1, \quad n = 1, 2, \dots, N$$

Hence the Hamiltonian reduces to

$$H^n = z_1^n (x_1^{n-1} + \theta^n - \xi^n) + f_2^n, \quad n = 1, 2, \dots, N \quad (25)$$

where f_2^n is given by equations (18) and (19)

Now for $N = 3$,

$$z_1^3 = c_1 = 0$$

Therefore, $H^3 = f_2^3$

Since $\mu^3 = 0$, μ^3 is $\leq (x_1^2 + \theta^3)$

Hence from equation (19), we obtain

$$f_2^3 = x_2^2 + g_1^3(\theta^3) + g_2^3(x_1^2) + E[g_3^3(x_1^3)] \quad (26)$$

Therefore,

$$E[H^3 | \bar{x}^2] = x_2^2 + g_1^3(\theta^3) + g_2^3(x_1^2) + E[g_3^3(x_1^3)] \quad (27)$$

The optimal decision $\bar{\theta}^3$ is obtained from the stationality condition as follows.

$$E \left[\frac{\partial H^3}{\partial \theta^3} | \bar{x}^2 \right] = 0 \quad \text{yields}$$

$$\bar{\theta}^3 = .61 - .4252 \bar{x}_1^2 \quad (28)$$

Now $E \left[\frac{\partial^2 H^3}{(\partial \theta^3)^2} | \bar{x}^2 \right] = 10 > 0$ indicates that at $\bar{\theta}^3 = (.61 - .4252 \bar{x}_1^2)$,

$E[H^3|x^2]$ has a minimum.

For $n = 2$, let us assume that $\mu^2 > (x_1^1 + \theta^2)$.

Then from equation (18), we obtain

$$f_2^2 = x_2^1 + g_1^2(\theta^2) + g_2^2(x_1^1) + 10(\xi^2 - x_1^2 - \theta^2) \quad (29)$$

Therefore,

$$\begin{aligned} E[H^2|x^1] &= E(z_1^2)(x_1^1 + \theta^2 - \mu^2) + x_2^1 + g_1^2(\theta^2) \\ &\quad + g_2^2(x_1^1) + 10(\mu^2 - x_1^1 - \theta^2) \end{aligned} \quad (30)$$

where $E(z_1^2)$ is obtained as shown below.

From equation (24), we have

$$E(z_1^2) = \frac{\partial f_2^3}{\partial x_1^2} + \frac{\partial f_2^3}{\partial \theta^3} \frac{\partial \theta^3}{\partial x_1^2} \quad (31)$$

From equation (26),

$$\frac{\partial f_2^3}{\partial x_1^2} = 8.53 + 8.890.8 x_1^2 + 7.408 \theta^3$$

$$\frac{\partial f_2^3}{\partial \theta^3} = 17.408 \theta^3 + 7.408 x_1^2 - 10.612$$

From equation (28),

$$\frac{\partial \theta^3}{\partial x_1^2} = - .4252$$

Substituting these values in equation (31) and simplifying, we get

$$E(z_1^2) = 15.39 + 5.7348 x_1^2 + .006 \theta^3 \quad (32)$$

Now $E\left[\frac{\partial H^2}{\partial \theta^2} \mid \bar{x}^1\right] = 0$ gives

$$\bar{\theta}^2 = 1 - \frac{E(z_1^2)}{10} \quad (33)$$

where $E(z_1^2)$ is obtained from equation (32).

$$E\left[\frac{\partial^2 H^2}{\partial (\theta^2)^2} \mid \bar{x}^1\right] = 10 > 0 \text{ indicates that at } \bar{\theta}^2 = 1 - \frac{E(z_1^2)}{10},$$

$E[H^2 \mid \bar{x}^1]$ has a minimum.

$$\text{For } n = 1, \quad H^1 = z_1^1 f_1^1 + z_2^1 f_2^1 \quad (34)$$

Since $\mu^1 = 1$ and $x_1^0 = 1$, μ^1 is $\leq (x_1^0 + \theta^1)$

Then from equation (18), we obtain

$$f_2^1 = x_2^0 + g_1^1(\theta^1) + g_2^1(x_1^0) \quad (35)$$

Therefore,

$$E[H^1 \mid \bar{x}^0] = E(z_1^1)(x_1^0 + \theta^1 - \mu^1) + g_1^1(\theta^1) + g_2^1(x_1^0) \quad (36)$$

where $E(z_1^1)$ is obtained as follows.

From equation (24),

$$E(z_1^1) = E(z_1^2) + \frac{\partial f_2^2}{\partial x_1^1} \quad (37)$$

From equation (29),

$$\frac{\partial f_2^2}{\partial x_1^1} = 2.082 + 1.428 x_1^1 - 10$$

Substituting in equation (37) for $\frac{\partial f_2^2}{\partial x_1^1}$, $E(z_1^1)$ is calculated.

$$\text{Now } E\left(\frac{\partial H^1}{\partial \theta^1} \mid \bar{x}^0\right) = 0 \text{ gives}$$

$$\bar{\theta}^1 = \frac{-E(z_1^1)}{10} \quad (38)$$

where $E(z_1^1)$ is obtained from equation (37).

$$E\left(\frac{\partial^2 H^1}{(\partial \theta^1)^2} \mid \bar{x}^0\right) = 10 > 0 \text{ indicates that at } \bar{\theta}^1 = \frac{-E(z_1^1)}{10},$$

$E[H^1 \mid x^0]$ has a minimum.

Simplifying equations (28), (33) and (38), we get the following three simultaneous equations for the optimal decisions.

$$.425 \bar{\theta}^1 + .425 \bar{\theta}^2 + \bar{\theta}^3 = 1.46 \quad (39)$$

$$.574 \bar{\theta}^1 + 1.574 \bar{\theta}^2 + .001 \bar{\theta}^3 = .608 \quad (40)$$

$$1.722 \bar{\theta}^1 + .574 \bar{\theta}^2 + .001 \bar{\theta}^3 = .4 \quad (41)$$

Solving these equations, we get

$$\bar{\theta}^1 = .1178 \approx 0$$

$$\bar{\theta}^2 = .343 \approx 0$$

$$\bar{\theta}^3 = 1.2643 \approx 1$$

For the optimal policy, the corresponding inventory level and total cost at the end of each stage are

$$x_1^1 = 0$$

$$x_1^2 = 0$$

$$x_1^3 = 1$$

$$x_2^1 = 19.8554$$

$$x_2^2 = 56.8874$$

$$x_2^3 = 81.2094$$

DISCUSSION OF THE RESULTS

The minimum cost obtained by stochastic dynamic programming method is 78,333 while that by the stochastic maximum principle approach is 80,2094, the costs being shown in \$1000. This difference in the optimal values by the two methods is expected due to the approximation in fitting the quadratic regression equations. The optimal net work obtained by stochastic dynamic programming indicates the different optimal policies for the different values of starting inventory levels while the stochastic maximum principle gives the optimal policy for one particular value of initial inventory level. For different initial inventory levels, the same procedure is to be repeated to obtain the optimal solution.

An advantage in the stochastic dynamic programming method is found to be that it gives integer optimal decision values needed in the problem while in the stochastic maximum principle solution the optimal decision is to be rounded to the nearest integer value for use. The consequence of this procedure and the effect of this approximation on the optimal solution is yet to be investigated in the future. From the procedure followed it is expected that for a problem with many stages and with more number of discrete cost values specified the solution by the stochastic dynamic programming may involve considerable amount of numerical computation. On the other hand, the solution by the stochastic maximum principle may have comparatively less computation except for little additional calculation in the regression analysis. Also where there are more state variables (more than three or four) the stochastic dynamic programming approach becomes extremely difficult and complicated even for a computer with a large memory capacity. On the other hand, the stochastic maximum

principle application for many state variables is essentially the same as that followed in the above problem with one state variable.

Finally it is concluded that for discrete valued stochastic function optimization, the stochastic dynamic programming can be directly used while the optimization can also be effectively done by the stochastic maximum principle with some additional computation and approximation involved in fitting regression equations for the discrete values of the objective function.

FURTHER DISCUSSION

The dynamic programming solution (Figure 2) indicates the different optimal policies associated with each possible inventory level at the beginning of each stage whereas the maximum principle solution obtained gives only one optimal policy for the specified inventory at the beginning of the process (first stage). The same optimal solution of dynamic programming can be shown obtainable by the maximum principle also for the last stage (April) as given below.

The expected value of the Hamiltonian for the last stage (Fig. 3) can be expressed as

$$\begin{aligned}
 E[H^3|\bar{x}^2] = & \frac{2}{3} [x_2^2 + g_1^3(\theta^3) + g_2^3(x_1^2) + g_3^3(x_1^3)|_{\xi^3=0}] + \frac{1}{3} [x_2^2 + g_1^3(\theta^3) \\
 & + g_2^3(x_1^2) + g_3^3(x_1^3)|_{\xi^3=1} + 10(1 - x_1^2 - \theta^3) U(1 - x_1^2 - \theta^3)]
 \end{aligned}
 \tag{42}$$

where $U(1 - x_1^2 - \theta^3)$ represents a unit step function for the shortage cost and is defined as

$$U(1 - x_1^2 - \theta^3) = \begin{cases} 1, & \text{if } x_1^2 + \theta^3 < 1 \\ 0, & \text{if } x_1^2 + \theta^3 \geq 1 \end{cases}$$

The optimal decision, $\bar{\theta}^3$, is to be determined such that $E(H^3|\bar{x}^2)$ is minimized. Since x_1^2 and x_2^2 are given values, minimization of $E(H^3|\bar{x}^2)$ depends on the value of θ^3 only. Hence the variable portion of the Hamiltonian can be written as follows.

$$E[H^3 | \bar{x}^2]^* = g_1^3(\theta^3) + \frac{2}{3} g_3^3(x_1^3) |_{\xi^3=0} + \frac{1}{3} g_3^3(x_1^3) |_{\xi^3=1} + \frac{1}{3} [10(1 - x_1^2 - \theta^3) U(1 - x_1^2 - \theta^3)] \quad (43)$$

Now the optimal decision $\bar{\theta}^3$ can be calculated for minimizing equation (43) for the various possible values of x_1^2 as shown below.

x_1^2	θ^3	$E[H^3 \bar{x}^2]^*$	$\min E[H^3 \bar{x}^2]^*$	$\bar{\theta}^3$
0	0	26.8559		
	1	22.133	22.133	1
	2	38.11		
1	0	20.788		
	1	23.11	20.788	0
	2	43.028		
2	0	22.4162		
	1	23.1362	22.4162	0
	2	60.0231		
3	0	13.163		
	1	40.94	13.163	0
	2	76.15		

The optimal decision values shown above for the different starting inventory levels are the same as that obtained previously by dynamic programming. Thus the net work of optimal policies can also be constructed for the last stage by the stochastic maximum principle method. But unfortunately the above procedure is not possible to apply for the second and first stages. The difficulty is caused in the calculation of the adjoint variable. For example, for the second stage the expected value of the Hamiltonian function can be written as

$$\begin{aligned} E[H^2 | x^1] = & E[z_1^2 (x_1^1 + \theta^2 - \xi^2)] + \frac{1}{4} [x_2^1 + g_1^2 (\theta^2) + g_2^2 (x_1^1) + 10 (1 - x_1^1 \\ & - \theta^2) U(1 - x_1^1 - \theta^2)] + \frac{1}{2} [x_2^1 + g_1^2 (\theta^2) + g_2^2 (x_1^1) + 10 (2 - x_1^1 \\ & - \theta^2) U(2 - x_1^1 - \theta^2)] + \frac{1}{4} [x_2^1 + g_1^2 (\theta^2) + g_2^2 (x_1^1) + 10 (3 - x_1^1 \\ & - \theta^2) U(3 - x_1^1 - \theta^2)] \end{aligned}$$

where z_1^2 is to be obtained from equation (23) as

$$z_1^2 = \frac{\partial f_2^3}{\partial x_1^2} \bigg|_{\bar{x}^2, \bar{\theta}^3} + \left(\frac{\partial f_2^3}{\partial \theta^3} \frac{\partial \theta^3}{\partial x_1^2} \right) \bigg|_{\bar{x}^2, \bar{\theta}^3}$$

In the above expression for z_1^2 , a possible procedure is not yet found to evaluate the partial derivatives of f_2^3 with respect to x_1^2 and θ^3 at their optimal values. At the optimal point the function f_2^3 is expressed as a set of discrete values (it is integer here) for different x_1^2 values and so f_2^3 is not easily differentiable with respect to x_1^2 and θ^3 . Thus this requirement of differentiable functions limits the use of the stochastic

maximum principle. However, a successful procedure is yet to be found out in future work for differentiation of discrete functions.

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APPENDIX

Computer flow diagrams, programs, symbol table and results obtained for the numerical computational method and the simplex search technique for the hydroelectric water storage system.

Fig. 1. Computer flow diagram for the numerical computational method

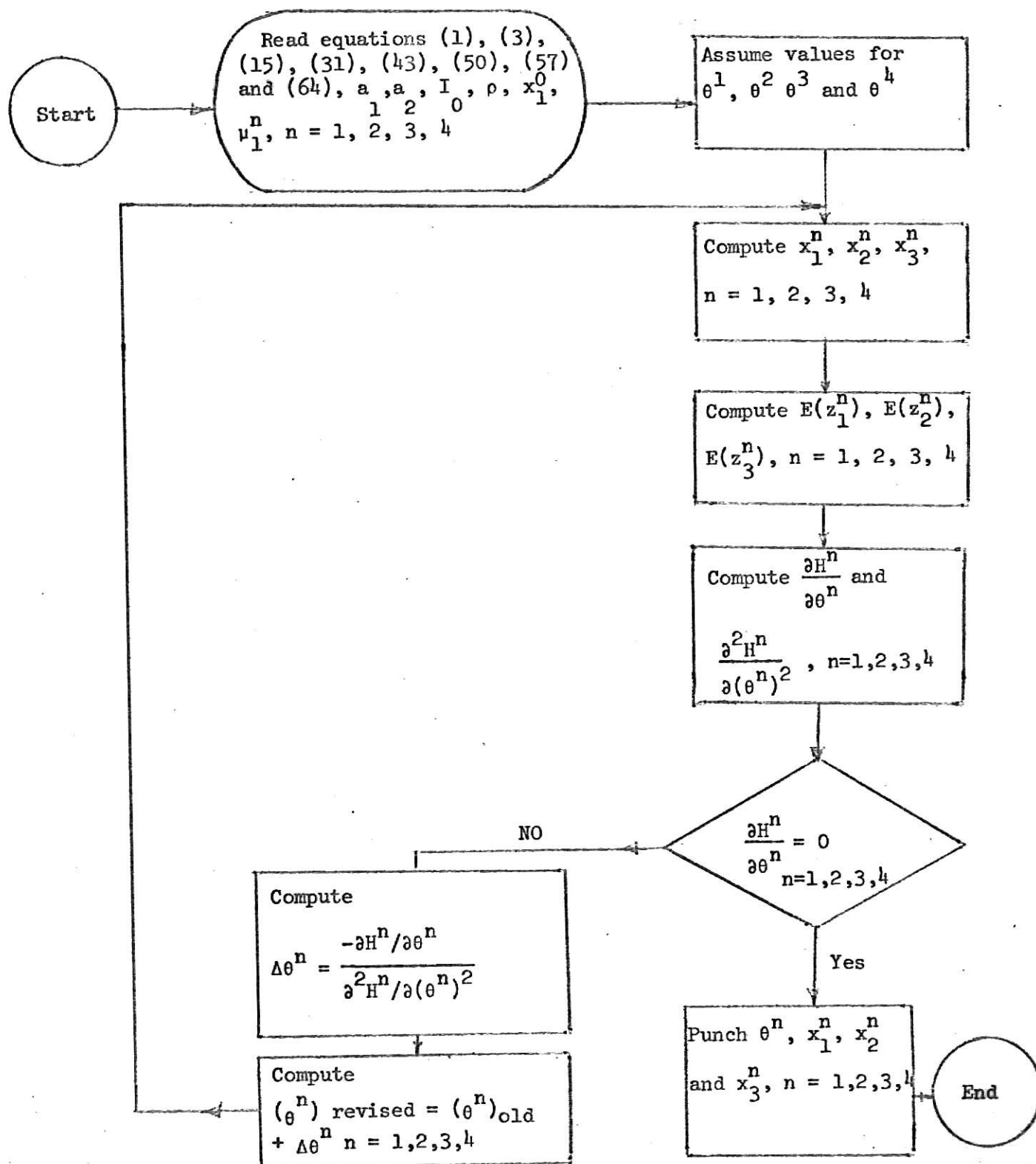


TABLE 1. COMPUTER PROGRAM FOR THE NUMERICAL
COMPUTATIONAL METHOD

C C OPTIMIZATION BY MAXIMUM PRINCIPLE

PUNCH 100

PUNCH 101

THE1=139.86

THE2=167.26

THE3=240.33

THE4=494.47

DELTA=.001

11 DFT1=.00000502*(THE4**3)

DFT2=-.00001506*(THE4**2)*(1037.9-THE1-THE2-THE3)

DFT3=.00228*THE4*(685.4+.00456*((1037.9-THE1-THE2-THE3)**2))

DFT4=-1.562712*(1037.9-THE1-THE2-THE3)

DF4T4=DFT1+DFT2+DFT3+DFT4

DFT5=-.00000502*(THE4**3)-1.56712*THE4

DFT6=.00000502*(THE4**2)*(2015.7-2.*THE1-2.*THE2-2.*THE3)

DF4X3=DFT5+DFT6

DFT7=3.*(THE4**2)+300000.

DFT8=-2.*THE4*(2015.7-2.*THE1-2.*THE2-2.*THE3)

DFT9=3.*(THE4**2)-6.*THE4*(1037.85-THE1-THE2-THE3)

DFT10=2.*(693.7-THE1-THE2-THE3)

DFT11=1376.6*(693.7-THE1-THE2-THE3)+551423.3

DT4X3=(DFT7+DFT8)/(DFT9+DFT10+DFT11)

Z13=DF4X3+(DF4T4*DT4X3)

DFT12=-.00000502*(THE3**3)-.832656*THE3

DFT13=.00000502*(THE3**2)*(1886.3-2.*THE1-2.*THE2)

DF3X2=DFT12+DFT13

DFT14=.00000502*(THE3**3)-.832656*(943.15-THE1-THE2)

DFT15=-.00001506*(THE3**2)*(943.15-THE1-THE2)

DFT16=.00228*THE3*(725.4+.00456*((943.15-THE1-THE2)**2))

DF3T3=DFT14+DFT15+DFT16

DFT17=3.*(THE3**2)-2.*THE3*(1886.3-2.*THE1-2.*THE2)+318000.

DFT18=3.*(THE3**2)-6.*THE3*(670.-THE1-THE2)

DFT19=2.*((670.-THE1-THE2)**2)+466453.

DT3X2=DFT17/(DFT18+DFT19)

Z12=DF3X2+(DF3T3*DT3X2)

DFT20=-.00000502*(THE2**3)

DFT21=.00000502*(THE2**2)*(1823.7-2.*THE1)-1.578216*THE2

DF2X1=DFT20+DFT21

DFT22=.00000502*(THE2**3)-.00001506*(THE2**2)*(911.85-THE1)

DFT23=.00228*THE2*(692.2+.00456*((911.85-THE1)**2))

DFT24=-1.578216*(911.85-THE1)

DF2T2=DFT22+DFT23+DFT24

DFT25=3.*(THE2**2)-2.*THE2*(1190.-THE1)+303700.

DFT26=3.*(THE2**2)-6.*THE2*(912.-THE1)

DFT27=2.*((633.7-THE1)**2)-1112.6*THE1+1163257.1

DT2X1=DFT25/(DFT26+DFT27)

Z11=DF2X1+(DF2T2*DT2X1)

TABLE 1 (CONTINUED)

```

DFT28=.00228*(THE4**3)-.00684*(THE4**2)*(1040.-THE1-THE2-THE3)
DFT29=((956.-THE1-THE2-THE3)**2)-168.*(THE1+THE2+THE3)+168026.)
DFT30=THE4*(.00456*DFT29+685.)
DFT31=-712000.+685.*(THE1+THE2+THE3)
F1=DFT28+DFT30+DFT31-1464.
DFT32=.00000502*(THE3**3)
DFT33=-.00001506*(THE3**2)*(943.15-THE1-THE2)
DFT34=((930.-THE1-THE2)**2)+24617.-26.3*(THE1+THE2)
DFT35=.00228*THE3*(53.6+(.00456*DFT34))
DFT36=-(Z13+.1222*(943.15-THE1-THE2))
F2=DFT32+DFT33+DFT35+DFT36-559.3
DFT37=.00000502*(THE2**3)-.00001506*(THE2**2)*(912.-THE1)
DFT38=((893.7-THE1)**2)+32734.-36.3*THE1)
DFT39=.00228*THE2*(692.2+.00456**DFT38)
DFT40=-(Z12+1.579*(912.-THE1))
F3=DFT37+DFT39+DFT40+1231.4
DFT41=.00000502*(THE1**3)-.013459122*(THE1**2)
DFT42=1.527*THE1-(Z11+1351.)
F4=DFT41+DFT42+1307.31
A=THE1
B=THE2
C=THE3
D=THE4
E=Z11
F=Z12
G=Z13
PUNCH 1,A,B,C,D,E,F,G,F1,F2,F3,F4
IF(F1)3,3,6
3 IF(F2)4,4,6
4 IF(F3)5,5,6
5 IF(F4)6,8,8
2 THE1=THE1+DELTA
THE2=THE2+DELTA
THE3=THE3+DELTA
THE4=THE4+DELTA
GO TO 11
6 THE1=THE1-DELTA
THE2=THE2-DELTA
THE3=THE3-DELTA
THE4=THE4-DELTA
GO TO 11
1 FORMAT(4F9.3,3F10.2,4(1X,F12.2)
100 FORMAT(4X,4HTHE1,5X,4HTHE2,5X,4HTHE3,5X,4HTHE4,7X,3HZ13,7X,3HZ12)
101 FORMAT(1H11X,3HZ11,11X,2HF1,8X,2HF2,6X,2HF3,6X,2HF4)
8 STOP
END

```

TABLE 2 Notations Used in the Computer Program By the Numerical Computational Method

$$\text{THE1} = \theta^1$$

$$\text{THE2} = \theta^2$$

$$\text{THE3} = \theta^3$$

$$\text{THE4} = \theta^4$$

$$\text{DELTA} = \Delta\theta$$

$$\text{DFT4} = \frac{\partial^4 f_3}{\partial \theta^4}$$

$$\text{DF4X3} = \frac{\partial^4 f_3}{\partial x_1^3}$$

$$\text{DT4X3} = \frac{\partial^4 \theta}{\partial x_1^5}$$

$$\text{Z13} = E(z_1^3)$$

$$\text{DF3X2} = \frac{\partial^3 f_3}{\partial x_1^2}$$

$$\text{DF3T3} = \frac{\partial^3 f_3}{\partial \theta^3}$$

$$\text{DT3X2} = \frac{\partial^3 \theta}{\partial x_1^2}$$

$$\text{Z12} = E(z_1^2)$$

$$\text{DF2X1} = \frac{\partial^2 f_3}{\partial x_1}$$

$$\text{DF2T2} = \frac{\partial^2 f_3}{\partial \theta^2}$$

$$\text{DT2X1} = \frac{\partial^2 \theta}{\partial x_1}$$

$$\text{Z11} = E(z_1^1)$$

$$\text{F1} = E\left[\frac{\partial H^1}{\partial \theta^1}\right]$$

$$\text{F2} = E\left[\frac{\partial H^2}{\partial \theta^2}\right]$$

$$\text{F3} = E\left[\frac{\partial H^3}{\partial \theta^3}\right]$$

$$\text{F4} = E\left[\frac{\partial H^4}{\partial \theta^4}\right]$$

TABLE 3. SOLUTION OBTAINED BY NUMERICAL COMPUTATIONAL

THE1	THE2	THE3	THE4	Z13	Z12	Z11
145.850	173.250	246.320	500.460	-76.93	-19.02	-158.92
144.850	172.250	245.320	499.460	-77.53	-20.25	-149.41
143.850	171.250	244.320	498.460	-78.13	-21.45	-138.99
142.850	170.250	243.320	497.460	-78.73	-22.62	-127.40
141.850	169.250	242.320	496.460	-79.34	-23.76	-114.28
140.850	168.250	241.320	495.460	-79.95	-24.88	-99.07
139.850	167.250	240.320	494.460	-80.56	-25.97	-80.92
138.850	166.250	239.320	493.460	-81.18	-27.03	-58.35
137.850	165.250	238.320	492.460	-81.79	-28.05	-28.65
136.850	164.250	237.320	491.460	-82.41	-29.05	13.72
135.850	163.250	236.320	490.460	-83.04	-30.02	82.24
134.850	162.250	235.320	489.460	-83.67	-30.96	220.47
133.850	161.250	234.320	488.460	-84.30	-31.86	691.05
132.850	160.250	233.320	487.460	-84.93	-32.74	-3365.61

THE1	THE2	THE3	THE4	Z13	Z12	Z11
140.850	168.250	241.320	495.460	-79.95	-24.88	-99.07
140.750	168.150	241.220	495.360	-80.01	-24.99	-97.41
140.650	168.050	241.120	495.260	-80.07	-25.10	-95.72
140.550	167.950	241.020	495.160	-80.13	-25.21	-93.99
140.450	167.850	240.920	495.060	-80.19	-25.32	-92.23
140.350	167.750	240.820	494.960	-80.25	-25.43	-90.44
140.250	167.650	240.720	494.860	-80.32	-25.53	-88.61
140.150	167.550	240.620	494.760	-80.38	-25.64	-86.75
140.050	167.450	240.520	494.660	-80.44	-25.75	-84.85
139.950	167.350	240.420	494.560	-80.50	-25.86	-82.91
139.850	167.250	240.320	494.460	-80.56	-25.97	-80.92
139.750	167.150	240.220	494.360	-80.62	-26.07	-78.90
139.650	167.050	240.120	494.260	-80.68	-26.18	-76.83
139.550	166.950	240.020	494.160	-80.74	-26.29	-74.71
139.450	166.850	239.920	494.060	-80.81	-26.39	-72.54
139.350	166.750	239.820	493.960	-80.87	-26.50	-70.32
139.250	166.650	239.720	493.860	-80.93	-26.61	-68.04
139.150	166.550	239.620	493.760	-80.99	-26.71	-65.71
139.050	166.450	239.520	493.660	-81.05	-26.82	-63.32
138.950	166.350	239.420	493.560	-81.11	-26.92	-60.87
138.850	166.250	239.320	493.460	-81.18	-27.03	-58.35
138.750	166.150	239.220	493.360	-81.24	-27.13	-55.76

D

F1	F2	F3	F4
4239.69	55.56	-6.12	-14.78
3393.91	49.91	-4.97	-12.11
2604.37	43.33	-3.84	-9.47
1870.59	35.57	-2.75	-6.84
1192.18	26.27	-1.70	-4.23
568.74	14.87	-.68	-1.65
.03	.50	.30	.92
-514.49	-18.30	1.25	3.47
-975.21	-44.24	2.15	5.99
-1382.36	-82.87	3.02	8.50
-1736.46	-147.68	3.85	10.99
-2037.89	-282.20	4.64	13.45
-2286.92	-749.10	5.39	15.90
-2484.04	3311.23	6.10	18.33

F1	F2	F3	F4
568.74	14.87	-.68	-1.65
509.40	13.58	-.58	-1.39
450.63	12.27	-.48	-1.13
392.40	10.92	-.38	-.88
334.74	9.54	-.28	-.62
277.61	8.13	-.18	-.36
220.95	6.68	-.09	-.11
164.93	5.19	.01	.15
109.36	3.67	.11	.41
54.44	2.11	.21	.66
.03	.50	.30	.92
-53.88	-1.15	.40	1.17
-107.27	-2.84	.49	1.43
-160.02	-4.58	.59	1.68
-212.32	-6.37	.69	1.94
-264.02	-8.22	.78	2.19
-315.24	-10.11	.87	2.45
-365.83	-12.07	.97	2.70
-415.98	-14.08	1.06	2.96
-465.54	-16.16	1.15	3.21
-514.49	-18.30	1.25	3.47

TABLE 3 (CONTINUED)

THE1	THE2	THE3	THE4	Z13	Z12	Z11
139.950	167.350	240.420	494.560	-80.50	-25.86	-82.91
139.940	167.340	240.410	494.550	-80.51	-25.87	-82.71
139.930	167.330	240.400	494.540	-80.51	-25.88	-82.51
139.920	167.320	240.390	494.530	-80.52	-25.89	-82.32
139.910	167.310	240.380	494.520	-80.52	-25.90	-82.12
139.900	167.300	240.370	494.510	-80.53	-25.91	-81.92
139.890	167.290	240.360	494.500	-80.54	-25.92	-81.72
139.880	167.280	240.350	494.490	-80.54	-25.93	-81.52
139.870	167.270	240.340	494.480	-80.55	-25.94	-81.32
139.860	167.260	240.330	494.470	-80.55	-25.96	-81.12
139.850	167.250	240.320	494.460	-80.56	-25.97	-80.92
139.840	167.240	240.310	494.450	-80.57	-25.98	-80.72
139.830	167.230	240.300	494.440	-80.57	-25.99	-80.52
139.820	167.220	240.290	494.430	-80.58	-26.00	-80.32
139.810	167.210	240.280	494.420	-80.59	-26.01	-80.12
139.800	167.200	240.270	494.410	-80.59	-26.02	-79.92
139.790	167.190	240.260	494.400	-80.60	-26.03	-79.71
139.780	167.180	240.250	494.390	-80.60	-26.04	-79.51
139.770	167.170	240.240	494.380	-80.61	-26.05	-79.31
139.760	167.160	240.230	494.370	-80.62	-26.06	-79.10

THE1	THE2	THE3	THE4	Z13	Z12	Z11
139.860	167.260	240.330	494.470	-80.55	-25.96	-81.12
139.859	167.259	240.329	494.469	-80.56	-25.96	-81.10
139.858	167.258	240.328	494.468	-80.56	-25.96	-81.08
139.857	167.257	240.327	494.467	-80.56	-25.96	-81.06
139.856	167.256	240.326	494.466	-80.56	-25.96	-81.04
139.855	167.255	240.325	494.465	-80.56	-25.96	-81.02
139.854	167.254	240.324	494.464	-80.56	-25.96	-81.00
139.853	167.253	240.323	494.463	-80.56	-25.96	-80.98
139.852	167.252	240.322	494.462	-80.56	-25.96	-80.96
139.851	167.251	240.321	494.461	-80.56	-25.97	-80.94
139.850	167.250	240.320	494.460	-80.56	-25.97	-80.92
139.849	167.249	240.319	494.459	-80.56	-25.97	-80.90
139.848	167.248	240.318	494.458	-80.56	-25.97	-80.88
139.847	167.247	240.317	494.457	-80.56	-25.97	-80.86
139.846	167.246	240.316	494.456	-80.56	-25.97	-80.84
139.845	167.245	240.315	494.455	-80.56	-25.97	-80.82
139.844	167.244	240.314	494.454	-80.56	-25.97	-80.80
139.843	167.243	240.313	494.453	-80.56	-25.97	-80.78
139.842	167.242	240.312	494.452	-80.57	-25.98	-80.76
139.841	167.241	240.311	494.451	-80.57	-25.98	-80.74

F1	F2	F3	F4
54.44	2.11	.21	.66
49.00	1.95	.22	.69
43.50	1.79	.23	.71
38.04	1.63	.24	.74
32.60	1.47	.24	.76
27.10	1.31	.25	.79
21.74	1.15	.26	.82
16.24	.99	.27	.84
10.88	.83	.28	.87
5.40	.66	.29	.89
.03	.50	.30	.92
-5.45	.34	.31	.94
-10.81	.18	.32	.97
-16.24	.01	.33	1.00
-21.65	-.15	.34	1.02
-27.05	-.32	.35	1.05
-32.38	-.48	.36	1.07
-37.83	-.65	.37	1.10
-43.12	-.81	.38	1.12
-48.54	-.98	.39	1.15

F1	F2	F3	F4
5.40	.66	.29	.89
4.89	.65	.29	.90
4.36	.63	.30	.90
3.81	.62	.30	.90
3.26	.60	.30	.90
2.70	.58	.30	.91
2.17	.57	.30	.91
1.62	.55	.30	.91
1.04	.53	.30	.91
.54	.52	.30	.92
.03	.50	.30	.92
-.49	.49	.30	.92
-1.10	.47	.30	.92
-1.62	.45	.31	.93
-2.16	.44	.31	.93
-2.73	.42	.31	.93
-3.24	.41	.31	.93
-3.81	.39	.31	.94
-4.32	.37	.31	.94
-4.82	.36	.31	.94

TABLE 4. COMPUTER PROGRAM FOR THE SIMPLEX SEARCH TECHNIQUE

```

//SARCH      JOB 10B740409G001,2,2*,SUBRAMA,MSGLEVEL=1
//GRFXWYZQ   EXEC PROC=GOWATFOR
//GO.SYSIN DD *
$JOB          GILBERT,RUN=NOCHECK,TIME=15,PAGES=400,LINES=60,KP=26
C
C
C  PURPOSE
C    TO FIND THE BEST FUNCTION VALUE OF A FUNCTION WITH N INDEPENDENT
C    VARIABLES AND THE SET OF INDEPENDENT VARIABLES WHICH PRODUCES
C    THIS OUTCOME.
C
C  USAGE
C
C    A PART OF THE SUBROUTINE CALLED SUBNAM SHOULD BE WRITTEN AND
C    PLUGGED IN THE PROVIDED SUBROUTINE DECK TOGETHER WITH SOME
C    ARRANGEMENTS BY THE USER, IF NECESSARY.
C
C  DESCRIPTION OF PARAMETERS
C
C    ALPHA.. REFLECTION FACTOR WITH A VALUE BETWEEN 1.0 AND 1.5 .
C
C    BETA.. CONTRACTION FACTOR BETA= 0.5 HAS BEEN SET IN THE
C           SEARCH DECK. ITS RANGE LIES BETWEEN 0 AND 1 .
C
C    C(J).. THE WEIGHT OF THE JTH VERTEX OF THE PATTERN.  ..DIMEN-
C           SION..K-1 .
C
C    CNTRCX(I)..THE ITH DECISION VARIABLE AT THE CENTROID OF THE
C               PATTERN.  ..DIMENSION..K.
C
C    DCVX(I,J)..THE ITH INDEPENDENT VARIABLE AT THE JTH VERTEX OF
C               THE N DIMENSIONAL SPACE.  ...DIMENSION...NDIM,K(=
C               NDIM+JMCHEN).. WHERE JMCHEN=1 IN THE NEW METHOD AND
C               THE SIMPLEX, =NDIM IN BOX METHOD.
C
C    DLTvx(1,J)..THE INCREMENT OF THE ITH INDEPENDENT VARIABLE FROM
C               THE INITIAL VERTEX TO THE LTH REMAINING VERTEX OF
C               THE INITIAL PATTERN.  ..DIMENSION..NDIM,K..
C
C    FRROR.. THE PRESCRIBED ACCURACY OF THE FUNCTION VALUE FOR
C            =2 ..THE SIMPLEX METHOD.
C            =3 ..THE BOX METHOD
C
C    NDIM.. NO. OF DECISION VARIABLES, N.

```

TABLE 4 (CONTINUED)

STOPPING THE COMPUTATION

GAMMA.. EXPANSION FACTOR GAMMA=2 HAS BEEN SET .

K.. MAXIMUM VERTICES USED FOR SETTING UP THE INITIAL PATTERN

KK.. NO OF ACTUAL FUNCTION EVALUATION.

MAXNC.. MAX. NO. OF FUNCTION EVALUATION SET BY THE USER FOR
TERMINATING THE COMPUTATION WHEN THE NO. OF FUNCTION
EVALUATION EXCEEDS THIS GIVEN VALUE. MAX,VALUE=99998.

METHOD.. =1 ..THE NEW DEVELOPED SEARCH TECHNIQUE.

NDIMP1.. THE NO. OF VERTICES OTHER THAN THE STARTING PT. IN
FORMING THE INITIAL PATTERN. NDIMP1=K-1 .

NOPT.. NO. OF VERTICES OF THE PATTERN TO WHICH THE DESIRED
INFORMATION WILL BE WRITTEN OUT. MAX. NO. =K.

<(J).. THE FUNCTION VALUE OF THE JTH VERTEX. ..DIMENSION..K(=
NDIM+JMCHEN).

<UPLIM.. THIS IS A SUPERLIMIT SET BY THE USER IN CONSTRAINED
OPTIMIZATION PROBLEMS, WHICH IS POSITIVELY INFINITE
IN MINIMIZATION (NEGATIVELY INFINITELY INF. IN
MAXIMIZATION) WHEN THE CONSTRAINTS ARE VIOLATED.

REMARKS

THE DIMENSION STATEMENT IN THE DECK HAS BEEN BUILT FOR A
FUNCTION WITH 27 DECISION VARIABLES WHEN METHOD 1 AND 2 ARE
USED. IF METHOD 3 IS USED, IT CAN ONLY USED FOR A FCN. WITH
14 DECISION VARIABLES.

THE DATA OF THE PARAMETERS SHOULD BE PROVIDED BY THE USER ARE

- 1) NDIM
- 2) DCVX(1,1)
- 3) DLTIX(1,J), I=1,NDIM AND J=1,NDIMP1
- 4) NOPT
- 5) MAXNC
- 6) METHOD
- 7) ERROR (E.G. 1.0E-08)

8) SUPLIM =0 IN UNCONSTRAINED PROBLEM, IN CONSTRAINED PROBLEM

TABLE 4 (CONTINUED)

SEE DESCRIPTION OF SUPLIM.

** THE USERS ARE ENCOURAGED TO READ THROUGH CAREFULLY THE COMMENT STATEMENTS IN THE PROVIDED DECK.

** THE SEARCH DECK HAS BEEN BUILT FOR MINIMIZATION PROBLEM. HOWEVER THE SAME DECK CAN ALSO BE USED FOR MAX. PROBLEM IF $-S(J)$ INSTEAD OF $S(J)$ ($S(J)$ IS THE REAL FCN. VALUE OF THE MAX. PROBLEM) IS USED, I.E. $S(J)=-T$.

ILLUSTRATION...UNCONSTRAINED PROBLEM

TO MINIMIZE $S(X,Y)=X*X+Y*Y+1$. WITH AN ARBITRARY STARTING PT., $S(10,5)$.

THE INPUT DATA ARE

- 1) NDIM=2
- 2) NDIMP1=2 FOR METHOD 1 AND 2, FOR METHOD 3 NDIMP1=3
- 3) DCVX(1,1)=10., DCVX(2,1)=0.
- 4) DLT VX(1,1)=0.5 ,DLTVX(2,1)=0.
DLTVX(1,2)=0. ,DLTVX(2,1)=0.25 .
- 5) NOPT=2
- 6) MAXNO=1000
- 7) ERROR=1.0E-08
- 8) METHOD=1
- 9) SUPLIM=0.

THE PART OF SUBROUTINE TO BE WRITTEN AND PLUGGED IN IS

$T=X(1)*X(1)+X(2)*X(2)+1$

IN OTHER WORDS, THE CONTINUOUS THREE CARDS ARE

7 CONTINUE

$T=X(1)*X(1)+X(2)*X(2)+1$

$S(J)=T$

THIS IS THE MAIN PROGRAM FOR PROVIDING THE NECESSARY DATA OF THE PARAMETERS.

DIMENSION DLT VX(27,28),S(30),DCVX(27,30)

COMMON X(27),X1(20),X2(20),X3(20),Q(20),C,D,E,M,G,V,XX(27),X4(9),P

101 FORMAT(10I5)

3 FORMAT(15,6F10.3)

102 FORMAT(7E10.4)

103 FORMAT(/16H EVALUATION NO =I5/)

104 FORMAT(6E13.6)

READ(1,101)NDIM,NOPT,NDIMP1,MAXNO,METHOD

TABLE 4 (CONTINUED)

```

READ(1,102)ERROR,SUPLIM
READ(1,102)((DLTVX(I,J),I=1,NDIM),J=1,NDIMP1)
READ(1,3) M, C, D, E,G,V,P
READ(1,102)(Q(N),N=1,M)
WRITE(3,101)NDIM,NCPT,NDIMP1,MAXNC,METHOD
WRITE(3,104)ERROR,SUPLIM
WRITE(3,104)((DLTVX(I,J),I=1,NDIM),J=1,NDIMP1)
WRITE(3,3) M, C, D, E,G,V,P
WRITE(3,102) (Q(N), N=1,M)
READ(1,102)(DCVX(I,1),I=1,NDIM)
WRITE(3,104)(DCVX(I,1),I=1,NDIM)
CALL GKCHEN(NDIM,METHOD,MAXNC,ERROR,SUPLIM,DLTVX,DCVX,S,KK)
WRITE(3,104)S(NDIM+2),(DCVX(I,NDIM+2),I=1,NDIM)
WRITE(3,104)((DCVX(I,J),I=1,NDIM),J=1,NCPT)
WRITE(3,104)(S(I),I=1,NCPT)
WRITE(3,103)KK
STOP
END

```

THE FOLLOWING PROGRAM HAS BEEN WRITTEN IN FORTRAN II AND PUNCHED IN KEYPUNCHER 26 FOR IMMENSE USAGE.

```

SUBROUTINE GKCHEN(NDIM,METHOD,MAXNC,ERROR,SUPLIM,DLTVX,DCVX,S,KK)
DIMENSION DLT VX(27,28),C(28),DCVX(27,30),S(30),CNTROX(27)
110 FORMAT(/19H THIS IS NEW METHOD/)
111 FORMAT(/16H THIS IS SIMPLEX/)
112 FORMAT(/12H THIS IS BOX/)
113 FORMAT(/16H ****WARNING****/)
114 FORMAT( 50H INADEQUATE GIVEN MAX. NO FOR FUNCTION EVALUATION,)
115 FORMAT(47H INCREASING THE MAXNC OR CHANGING THE STEP SIZE/)
GO TO (116,117,118),METHOD

```

THE SEARCH BEGINS WITH THE CHOSEN METHOD.

THIS IS THE NEW METHOD.

```

116 JMCHEN=1
KCHEN=1
ALPHO=1.0
BETA=.5

```

TABLE 4 (CONTINUED)

```

COEFF=1.2
GAMMA=2.0
WRITE(3,110)
GO TO 1

```

THIS IS THE SIMPLEX.

```

117 JMCHEN=1
    KCHEN=2
    ALPHO=1.0
    BETA=0.5
    GAMMA=2.0
    WRITE(3,111)
    GO TO 1

```

THIS IS BOX.

```

118 JMCHEN=NDIM
    ALPHO=1.3
    BETA=0.5
    WRITE(3,112)

```

NO STATEMENTS FROM NOW ON CAN BE REMOVED EXCEPT YOU ARE CERTAINLY SURE WHAT TO DO.

SET UP THE INITIAL PATTERN

1) EVALUATION OF THE GIVEN INITIAL POINT

```

1 J=1
  KK=1
  CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)
  K=NDIM+JMCHEN
  KLT1=K-1

```

2) EVALUATION OF THE REMAINING POINTS OF THE INITIAL PATTERN

```

DO 3 J=2,K
DO 2 I=1,NDIM
2 DCVX(I,J)=DCVX(I,1)+DLTVX(I,J-1)
  CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)
3 CONTINUE
4 M=K
  ALPHA=ALPHO

```

TABLE 4 (CONTINUED)

ORDERING THE FUNCTION VALUES OF THE PATTERN

CALL ORDER(M,NDIM,S,DCVX)

DEFINING THE CENTROID TO OBTAIN THE FURTHER SEARCH

DO 5 I=1,KLT1

5 C(I)=1.

CALL CNTRCD(NDIM,KLT1,C,CNTRCX,DCVX)

REFLECTING OPERATION

6 DO 7 I=1,NDIM

7 DCVX(I,K+1)=CNTRCX(I)+ALPHA*(CNTRCX(I)-DCVX(I,K))

J=K+1

CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)

IF(KK-MAXNC)8,8,36

8 GO TO (9,9,23),METHOD

NO EXPANSION IN BOX METHOD, THAT IS THE SIGNIFICANT DIFFERENCE

9 IF(S(K+1)-S(1))10,10,23

EXPANDING OPERATION

10 DO 11 I=1,NDIM

11 DCVX(I,K+2)=CNTRCX(I)+GAMMA*(DCVX(I,K+1)-CNTRCX(I))

J=K+2

CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)

IF(KK-MAXNC)12,12,36

THE DIFFERENCE OF THE NEW METHOD FROM THE SIMPLEX

12 GO TO (16,13),KCHEN

13 IF(S(K+2)-S(1))14,14,21

14 S(K)=S(K+2)

DO 15 L=1,NDIM

15 DCVX(L,K)=DCVX(L,K+2)

GO TO 35

16 IF(S(K+2)-S(K+1))17,17,21

17 S(K)=S(K+2)

DO 18 L=1,NDIM

18 DCVX(L,K)=DCVX(L,K+2)

M=K

CALL ORDER(M,NDIM,S,DCVX)

CALL SCHECK(K,SUM,NDIM,S)

TABLE 4 (CONTINUED)

```

C      IF(SUM-ERROR)37,37,19
C
C      DEFINING THE NEW CNTRCD ACCORDING TO THE IDEA OF THE NEW METHOD
C
19  CVALUE=2*NDIM-1
    DO 20 I=1,KLT1
      C(I)=CVALUE
20  CVALUE=2*NDIM-2
    CALL CNTRCD(NDIM,KLT1,C,CNTRCX,DCVX)
    ALPHA=ALPHO*COEFF
    GO TO 6
21  S(K)=S(K+1)
    DO 22 L=1,NDIM
22  DCVX(L,K)=DCVX(L,K+1)
    GO TO 35
23  IF(S(K+1)-S(K-1))21,21,24
24  IF(S(K+1)-S(K))25,25,27
25  S(K)=S(K+1)
    DO 26 I=1,NDIM
26  DCVX(I,K)=DCVX(I,K+1)
C
C      CONTRACTING OPERATION
C
27  DO 28 I=1,NDIM
28  DCVX(I,K+1)=CNTRCX(I)+BETA*(DCVX(I,K)-CNTRCX(I))
    J=K+1
    CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)
    IF(KK-MAXNO)29,29,36
29  IF(S(K+1)-S(K))30,30,32
30  S(K)=S(K+1)
    DO 31 I=1,NDIM
31  DCVX(I,K)=DCVX(I,K+1)
    GO TO 35
C
C      SHRINKING THE PATTERN DUE TO A BAD CONTRACTION
C
32  DO 34 J=2,K
    DO 33 I=1,NDIM
33  DCVX(I,J)=(DCVX(I,1)+DCVX(I,J))/2.
    CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)
34  CONTINUE
    IF(KK-MAXNO)35,35,36
35  CALL SCHECK(K,SUM,NDIM,S)
    IF(SUM-ERROR)37,37,4
C
C      THE SEARCH IS INCOMPLETE ACCORDING TO THE GIVEN INADEQUATE MAXNO.

```


TABLE 4 (CONTINUED)

```

36 WRITE(3,113)
   WRITE(3,114)
   WRITE(3,115)
   GO TO 40

```

THE SEARCH IS COMPLETED, RETURN TO THE MAIN PROGRAM AFTER EVALUATING THE CNTROD OF THE PATTERN.

```

37 DO 38 I=1,KLT1
38 C(I)=1.
   CALL CNTROD(NDIM,KLT1,C,CNTROX,DCVX)
   DO 39 I=1,NDIM
39 DCVX(I,K+1)=CNTROX(I)
   J=K+1
   CALL SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)
40 RETURN
   END

```

THIS SUBROUTINE SUBNAM SHOULD PROVIDED BY USER FOR OBTAINING THE REQUIRED OBJECTIVE FUNCTION VALUE.

KCONT...A CONTROL NUMBER SET FOR OUTPUT. FOR EVERY KCONT NO. OF FUNCTION EVALUATIONS THE COMPUTER WILL WRITE OUT THE DATA ONCE.

ERR.....A FUNCTION VALUE SET FOR THE DATA TO BE WEITTEN OUT AS THE COMPUTED FUNCTION VALUE DROPPED. A TENTH ORDER EACH TIME.

SOPT...THE BEST FUNCTION VALUE HAS BEEN FOUND AT EACH STAGE OF COMPUTATION.

XCPT(I)...THE CORRESPONDING ITH DECISION VARIABLE OF SOPT.

```

SUBROUTINE SUBNAM(NDIM,J,SUPLIM,S,DCVX,KK)
DIMENSION S(30),DCVX(27,30),XCPT(27)
COMMON X(27),X1(20),X2(20),X3(20),Q(20),C,D,E,M,G,V,XX(27),X4(9),P
203 FORMAT(E12.5)
204 FORMAT(I5,4F15.3)
1 FORMAT(31H THE OPTIMUM FUNCTION VALUE IS E13.6)
2 FORMAT(6E13.6)
3 FORMAT(10I4)
   IF(J-1)4,4,5
4 KCONT=10
   ERR=10.
   GO TO 6

```

TABLE 4 (CONTINUED)

5 KK=KK+1

TRANSLLOCATION OF THE VALUES OF THE INDEPENDENT VARIABLES FROM THE SEARCH DECK TO THOSE USED IN THIS SPECIAL PROBLEM, WHERE X(I) IS THE ITH INDEPENDENT VARIABLE IN THE USER PROBLEM.

6 DO 7 I=1,NDIM

X(I)=DCVX(I,J)

7 CONTINUE

THE USER SHOULD PROVIDE A PART OF THIS SUBROUTINE FOR OBTAINING THE REQUIRED FUNCTION VALUE AT EACH VERTEX BETWEEN THIS COMMENT STATEMENT AND THE FOLLOWING STATEMENT IN WHICH T MEANS THE REQUIRED FUNCTION VALUE.

IF(X(1).LE.0.)GO TO 16

IF(X(2).LE.0.)GO TO 16

IF(X(3).LE.0.)GO TO 16

IF(X(4).LE.0.)GO TO 16

PX11=633.7-X(1)

PX12=670.0-X(1)-X(2)

PX13=696.3-X(1)-X(2)-X(3)

PX21=.00114*X(1)*(1653.8-X(1))

PX22=.00228*X(2)*(278.2+PX11-(X(2)/2.0))

PX23=.00228*X(3)*(273.2+PX12-(X(3)/2.0))

PX24=.00228*X(4)*(344.2+PX13-(X(4)/2.0))

T11=1382.7-(PX21+PX22+PX23+PX24)+(342.2-PX24)**2

T22=(362.2-PX23)**2+(345.6-PX22)**2+(332.7-PX21)**2

T=T11+T22

S(J)=T

STORAGE OF BETTER FUNCTION VALUE WITH THE CORRESPONDING INDEPENDENT AND DEPENDENT VARIABLES, IF NECESSARY.

IF(J-1)9,9,11

9 DO 10 I=1,NDIM

XCPT(I)=X(I)

10 CONTINUE

SCPT=T

IF(J-1)17,17,12

11 IF(S(1)-S(J))12,9,9

12 IF(KK-KCONT)14,13,13

13 WRITE(3,1)SCPT

WRITE(3,2)(XCPT(I),I=1,NDIM)

WRITE(3,3)KK

DO 21 N=1,M

21 WRITE(3,204) N,X1(N),X3(N),X(N),X2(N)

TABLE 4 (CONTINUED)

```

      KCONT=KCONT+10
14  IF(S(J)-ERR)15,15,17
15  WRITE(3,1)SCPT
      WRITE(3,2)(XCPT(I),I=1,NDIM)
      DO 20 N=1,M
20  WRITE(3,204) N,X1(N),X3(N),X(N),X2(N)
      WRITE(3,3)KK
      ERR=ERR*0.1
      GO TO 17
16  S(J)=SUPLIM
17  RETURN
      END

```

THE SUBROUTINE FOR ORDERING THE FUNCTION VALUES OF THE PATTERN

```

SUBROUTINE ORDER(M,NDIM,S,DCVX)
  DIMENSION S(30),DCVX(27,30)
  K=M
  KLT1=K-1
  DO 5 I=1,KLT1
    M=M-1
    DO 4 J=1,M
      IF(S(M+1)-S(J))2,2,4
2    A=S(M+1)
      S(M+1)=S(J)
      S(J)=A
      DO 3 L=1,NDIM
        B=DCVX(L,M+1)
        DCVX(L,M+1)=DCVX(L,J)
        DCVX(L,J)=B
3    CONTINUE
4    CONTINUE
5    CONTINUE
  RETURN
  END

```

A NECESSARY PART OF THE WHOLE SEARCH DECK BUILT FOR OBTAINING THE CENTROID OF THE PATTERN EXCLUSIVE OF THE WORST POINT.

```

SUBROUTINE CNTRCD(NDIM,KLT1,C,CNTRCX,DCVX)
  DIMENSION C(28),CNTRCX(27),DCVX(27,30)
  SEARCHING FOR THE BETTER POINT OF THE SPACE
  CSUM=0.
  DO 1 I=1,KLT1

```

TABLE 4 (CONTINUED)

```

1 CSUM=CSUM+C(I)
  DO 3 I=1,NDIM
    AXIS=0.
    DO 2 J=1,KLT1
      CNTROX(I)=AXIS+C(J)*DCVX(I,J)
      AXIS=CNTROX(I)
2 CONTINUE
  CNTROX(I)=CNTROX(I)/CSUM
3 CONTINUE
  RETURN
  END

```

```

C
C
C   THE BUILT-IN SUBROUTINE FOR CHECKING WHETHER THE OPTIMUM POINT HAS
C   BEEN ACHIEVED
C   THE CRITERION USED IS SQRT(((AVG.(S)-S(J))**2/NDIM),J=1,K) .L.E.
C   ERROR.
C

```

```

SUBROUTINE SCHECK(K,SUM,NDIM,S)
  DIMENSION S(30)
  SAVG=0.
  DO 1 L=1,K
1 SAVG=S(L)+SAVG
  AK=K
  SAVG=SAVG/AK
  SUM=0.
  DO 2 L=1,K
2 SUM=SUM+(S(L)-SAVG)**2
  ANDIM=NDIM
  SUM=SUM**0.5/ANDIM
  RETURN
  END

```

```

$ENTRY

```

4	4	4	2000	1
.1000E-07	.1000E+50			
0.1000E01	.0000E00	.0000E00	.0000E00	.0000E00
.0000E00	.0000E00	.0000E00	.0000E00	.1000E01
.0000E00	.0000E00	.0000E00	.1000E01	
.1000E03	.1000E03	.1000E03	.1000E03	

```

$STOP

```

```

/*

```

THIS IS NEW METHOD

THE OPTIMUM FUNCTION VALUE IS 0.128393E 06
0.100009E 03 0.993050E 02 0.102055E 03 0.102055E 03

10

THE OPTIMUM FUNCTION VALUE IS 0.115798E 06
0.100847E 03 0.102212E 03 0.110795E 03 0.116405E 03

20

THE OPTIMUM FUNCTION VALUE IS 0.509256E 05
0.966567E 02 0.127711E 03 0.200990E 03 0.268911E 03

30

THE OPTIMUM FUNCTION VALUE IS 0.386492E 05
0.977529E 02 0.143424E 03 0.264178E 03 0.375107E 03

40

THE OPTIMUM FUNCTION VALUE IS 0.367666E 05
0.100487E 03 0.153900E 03 0.289948E 03 0.414960E 03

50

THE OPTIMUM FUNCTION VALUE IS 0.303327E 05
0.121351E 03 0.171083E 03 0.277177E 03 0.402729E 03

60

THE OPTIMUM FUNCTION VALUE IS 0.294004E 05
0.129185E 03 0.175652E 03 0.269839E 03 0.395649E 03

70

THE OPTIMUM FUNCTION VALUE IS 0.290422E 05
0.130768E 03 0.170748E 03 0.253204E 03 0.371544E 03

80

THE OPTIMUM FUNCTION VALUE IS 0.289349E 05
0.133480E 03 0.172776E 03 0.254395E 03 0.374009E 03

90

THE OPTIMUM FUNCTION VALUE IS 0.289037E 05
0.132868E 03 0.170123E 03 0.259212E 03 0.378389E 03

100

THE OPTIMUM FUNCTION VALUE IS 0.288309E 05
0.136734E 03 0.172151E 03 0.260587E 03 0.382311E 03

110

THE OPTIMUM FUNCTION VALUE IS 0.287383E 05
0.135129E 03 0.167985E 03 0.261190E 03 0.383082E 03

120

THE OPTIMUM FUNCTION VALUE IS 0.286110E 05
0.138784E 03 0.166263E 03 0.257942E 03 0.381019E 03

130

THE OPTIMUM FUNCTION VALUE IS 0.282502E 05
0.139688E 03 0.158368E 03 0.254926E 03 0.391655E 03

140

THE OPTIMUM FUNCTION VALUE IS 0.274792E 05
0.134266E 03 0.155910E 03 0.241568E 03 0.405004E 03

150

THE OPTIMUM FUNCTION VALUE IS 0.263516E 05
0.138681E 03 0.163247E 03 0.247641E 03 0.462930E 03

160

THE OPTIMUM FUNCTION VALUE IS 0.261759E 05
0.135477E 03 0.166627E 03 0.238008E 03 0.503741E 03

370

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

380

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

390

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

400

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

410

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

420

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

430

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

440

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

450

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

460

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

470

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

480

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

490

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

500

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138637E 03 0.167053E 03 0.240182E 03 0.494751E 03

170

THE OPTIMUM FUNCTION VALUE IS 0.261060E 05
0.138513E 03 0.167199E 03 0.240046E 03 0.495177E 03

180

THE OPTIMUM FUNCTION VALUE IS 0.261060E 05
0.138630E 03 0.167091E 03 0.240032E 03 0.494712E 03

190

THE OPTIMUM FUNCTION VALUE IS 0.261059E 05
0.138649E 03 0.167024E 03 0.240083E 03 0.495177E 03

200

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138652E 03 0.167059E 03 0.240075E 03 0.495036E 03

210

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138652E 03 0.167059E 03 0.240075E 03 0.495036E 03

220

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138652E 03 0.167059E 03 0.240075E 03 0.495036E 03

230

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

240

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

250

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

260

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

270

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

280

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

290

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

300

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

310

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

320

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

330

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

340

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

350

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

360

THE OPTIMUM FUNCTION VALUE IS 0.261058E 05
0.138654E 03 0.167056E 03 0.240075E 03 0.495035E 03

OPTIMIZATION OF SOME MULTISTAGE STOCHASTIC MANAGEMENT SYSTEMS

by

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AN ABSTRACT OF A MASTER'S REPORT

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Manhattan, Kansas

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The objective of this report is to demonstrate the applicability of the discrete stochastic maximum principle to the case of some multi-stage stochastic systems frequently encountered in management and in industry. The basic algorithm of the discrete stochastic maximum principle is stated. The special theorems which are very useful in determining the necessary condition for optimality and in calculating the adjoint vectors are also explained.

First a production scheduling model is solved by applying the stochastic maximum principle for two types of cost functions - a linear cost function and a nonlinear cost function. For the linear cost function the certainty equivalence principle is verified. A linear resource allocation model is next solved for which also the certainty equivalence principle is found to hold good.

The third multistage stochastic model formulated for and solved by the stochastic maximum principle is the hydroelectric water storage system. Solutions for linear and nonlinear cost functions are obtained separately. For the nonlinear case, the numerical computation of optimal decisions is done by using two techniques - the simplex search technique and a gradient technique. The optimal policy by the two methods are found to be the same.

The last problem considered is a production and inventory control problem with given discrete probability values for the demand random variable and given discrete cost function values. The solution by stochastic dynamic programming is first presented. Then a stochastic model is formulated for the system after fitting regression equation for the cost function and the solution by the stochastic maximum principle is compared with that obtained by the stochastic dynamic programming.