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Research Article

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A symmetry result for strictly convex domains

Abstract: Let $D \subset \mathbb{R}^2$ be a strictly convex domain with C^2-smooth boundary. Assume that $\int_D e^{ix} y^n \,dxdy = 0$ for all sufficiently large n. In this paper, we will prove that D is a disc.

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1 Introduction

We assume throughout that $D \subset \mathbb{R}^2$ is a *strictly convex* domain and its boundary S is C^2-smooth. Suppose that

$$\int_D e^{ix} y^n \,dxdy = 0, \quad n = 0, 1, 2, \dots . \tag{1}$$

Our result is stated as Theorem 1.

Theorem 1. *If D is a strictly convex bounded domain in $\mathbb{R}^2$ and* (1) *holds, then D is a disc.*

This result the author obtained while studying the Pompeiu problem, see, for example, [3, Chapter 11]. The result of Theorem 1 can also be established if the following is assumed in place of equation (1):

$$\int_D e^{iy} x^n \,dxdy = 0, \quad n = 0, 1, 2, \dots .$$

This follows from the proof of Theorem 1.

The proof shows that equation (1) is used only for sufficiently large n because asymptotic formula (3) is used in the proof.

2 Proof of Theorem 1

Let ℓ be an arbitrary unit vector, L_1 be the support line to D (at the point $s_1 \in S$) parallel to ℓ, and L_2 be the support line to D (at the point $q_1 \in S$) parallel to L_1, where $q_1 = q_1(s_1)$. Since D is strictly convex, one can introduce the equations $y = f(x)$ and $y = g(x)$ of the boundary S between the support points s_1 and q_1. For definiteness and without loss of generality let us assume that the orthogonal projection of the point s_1 onto the line L_1 lies not lower than the projection of the point q_1 onto L_1, and let the x-axis pass through s_1 and be orthogonal to L_1. The graph of f is located above the graph of g. Since S is strictly convex the function f has a unique point of maximum x_1, where $x_1 \in (a, b)$, and $f(x_1) > f(x)$ for $x \in [a, b]$, $f(x_1) > 0$ and $f''(x_1) < 0$. Here a and b are the x-coordinates of the points q_1 and s_1, $a < b$. Let us denote by s the value of the natural parameter (arc length on S) corresponding to the maximum point of f, that is, to the point x_1. The function g has a unique point of minimum x_2, $x_2 \in (a, b)$, $g(x) > g(x_2)$, $g(x_2) < 0$ and $g''(x_2) > 0$. From the strict convexity of S it follows that these maximum and minimum are non-degenerate, that is, $f''(x_1) \neq 0$, and $g''(x_2) \neq 0$. Denote by q the value of the natural parameter corresponding to the minimum point of g. Let us write equation (1) as

$$\int_D e^{ix} y^n \,dxdy = \int_a^b e^{ix} \frac{f^{n+1}(x) - g^{n+1}(x)}{n+1}\,dx = 0, \quad n = 0, 1, 2, \ldots. \tag{2}$$

The factor $n+1$ in the denominator can be canceled because the integral in (2) equals zero. We want to take $n \to \infty$ and use the Laplace method for evaluating the main term of the asymptotic of the integral. Let us recall this known result, the formula for the asymptotic of the integral

$$F(\lambda) := \int_a^b \phi(x) e^{\lambda S(x)}\,dx = \Big(\frac{2\pi}{\lambda |S''(\xi)|}\Big)^{\frac{1}{2}} \phi(\xi) e^{\lambda S(\xi)} (1 + o(1)), \quad \lambda \to \infty,$$

see, for example, [1]. In this formula $\xi \in (a, b)$ is a unique point of a non-degenerate maximum of a real-valued twice continuously differentiable function $S(x)$ on $[a, b]$, $S''(\xi) < 0$, and ϕ is a continuous function on $[a, b]$, possibly complex-valued. We apply this formula with

$$S(x) = \ln |f|, \quad \lambda := 2m := n + 1 \to \infty, \quad \phi = e^{ix},$$

and take $n = 2m - 1$ to ensure that $n + 1 = 2m$ is an even number, so that f^{2m} and g^{2m} are positive, and $\ln f^{2m}$ and $\ln g^{2m}$ are well defined. The point x_2 of minimum of g becomes a point of local maximum of the function g^{2m}. Note that

$$\big|(\ln |f|)''\big| = \frac{|f''(x_1)|}{|f(x_1)|}$$

at the point x_1 where $f'(x_1) = 0$, $f(x_1) > 0$ and $f''(x_1) < 0$.

Taking the above into consideration, one obtains from (2) the following asymptotic formula:

$$\int_D e^{ix} y^n \,dxdy = \Big[e^{ix_1 + 2m \ln |f(x_1)|} \Big(\frac{\pi |f(x_1)|}{m |f''(x_1)|}\Big)^{\frac{1}{2}} - e^{ix_2 + 2m \ln |g(x_2)|} \Big(\frac{\pi |g(x_2)|}{m |g''(x_2)|}\Big)^{\frac{1}{2}}\Big] (1 + o(1)) = 0, \quad n \to \infty, \tag{3}$$

where $2m = n + 1$, $x_1 \in (a, b)$ and $x_2 \in (a, b)$. It follows from the above formula that the expression in the brackets, that is, the main term of the asymptotic, must vanish for all sufficiently large m. This implies that $f(x_1) = |f(x_1)| = |g(x_2)|$ and $|f''(x_1)| = g''(x_2) = |g''(x_2)|$, because $f(x_1) > 0$, $g(x_2) < 0$, $f''(x_1) < 0$ and $g''(x_2) > 0$. It also follows from formula (3) that $e^{ix_1} = e^{ix_2}$. This implies $x_1 = x_2 + 2\pi p$, where p is an integer. The integer p does not depend on s because p is locally continuous and cannot have jumps. Thus,

$$x_1 - x_2 := 2\pi p; \quad |f(x_1)| = |g(x_2)|; \quad |f''(x_1)| = |g''(x_2)|. \tag{4}$$

We prove in Lemma 2 (see below) that $p = 0$. Another proof of this is given in Remark 3 below the proof of Lemma 2.

Consider the support lines L_3 at the point s and L_4 at the point q, where L_3 and L_4 are orthogonal to ℓ. Denote by $L = L(s)$ the distance between L_3 and L_4, that is, the width of D in the direction parallel to ℓ. Note that $L = f(x_1) - g(x_2) > 0$, and

$$L = \big(r(s) - r(q), \ell\big),$$

where $r = r(s)$ is the radius vector (position vector) corresponding to the point on S which is defined by the parameter s. This point will be called point s. The same letter s is used for the point $s \in S$ and for the corresponding natural parameter. Let $R = R(s)$ denote the radius of curvature of the curve S at the point s and let $\kappa = \kappa(s)$ denote the curvature of S at this point. Then one has

$$R^{-1} = \kappa = |f''(x_1)|,$$

because $\kappa = |f''(x_1)| [1 + |f'(x_1)|^2]^{-\frac{3}{2}}$ and $f'(x_1) = 0$ since x_1 is a point of maximum of f.

From (4) we will derive that

$$L(s) = 2R(s) \quad \text{for all } s \in S. \tag{5}$$

It will be proved in Lemma 2 that equation (5) implies that D is a disc. Thus, the conclusion of Theorem 1 will be established.

We denoted by $r = r(s)$ the equation of S, where s is the natural parameter on S and r is the radius vector of the point on S, corresponding to s. One has $r'(s) = t$, where $t = t(s)$ is a unit vector tangential to S at the point s. We have chosen s so that $t(s)$ is orthogonal to ℓ. Since ℓ is arbitrary, the point $s \in S$ is arbitrary. The point $q \in S$, $q = q(s)$, is uniquely determined by the requirement that $t(q) = -t(s)$, because S is strictly convex. One has $(r(s) - r(q), \ell) = L$, where $L = L(s)$ is the width of D in the direction parallel to ℓ. Since $r'(s) = t(s)$, the first formula in (4) implies

$$(r(q) - r(s), r'(s)) = 2\pi p, \quad (r(q) - r(s), r'(q)) = -2\pi p \quad \text{for all } s \in S. \tag{6}$$

Differentiate the first equation in (6) with respect to s and get

$$\Big(r'(q)\frac{dq}{ds} - r'(s), r'(s)\Big) + (r(q) - r(s), r''(s)) = 0 \quad \text{for all } s \in S. \tag{7}$$

Note that $r'(s) = t(s) = -t(q) = -r'(q)$ and $r''(s) = \kappa(s)\nu(s)$, where $\nu(s)$ is the unit normal to S (at the point corresponding to s) directed into D, and $(r(s) - r(q), \ell) = L(s) = (r(q) - r(s), \nu(s))$, because $\nu(s)$ is directed along $-\ell$. Consequently, it follows from (7) that

$$-\frac{dq}{ds} - 1 + \kappa(s)L(s) = 0 \quad \text{for all } s \in S. \tag{8}$$

One has $L(s) = L(q)$, and it follows from formulas (4) that $\kappa(s) = \kappa(q)$.

Differentiate the second equation in (6) with respect to q and get

$$\Big(t(q) - t(s)\frac{ds}{dq}, t(q)\Big) + (r(q) - r(s), r''(q)) = 0. \tag{9}$$

Note that $t(q) = -t(s)$ and $r''(q) = \kappa(q)\nu(q)$, where $\nu(q) = -\nu(s)$ because L_3 is parallel to L_4. Consequently, equation (9) implies

$$\frac{ds}{dq} + 1 - \kappa(s)L(s) = 0 \quad \text{for all } s \in S. \tag{10}$$

Compare (8) and (10) and get $\frac{ds}{dq} = \frac{dq}{ds}$. Thus, $(\frac{dq}{ds})^2 = 1$. Since $\frac{dq}{ds} > 0$, it follows that

$$\frac{ds}{dq} = \frac{dq}{ds} = 1 \quad \text{for all } s \in S. \tag{11}$$

Therefore, equation (8) implies

$$\kappa(s)L(s) = 2 \quad \text{for all } s \in S. \tag{12}$$

Let us derive from (12) *that D is a disc.*

Recall that s is the natural parameter on S, $L(s)$ is the width of D at the point s (that is the distance between two parallel supporting lines to S one of which passes through the point s) and $\kappa(s)$ is the curvature of S at the point s.

Lemma 2. *Assume that D is a strictly convex domain with a smooth boundary S. If equation* (12) *holds, then D is a disc.*

Proof. Denote by K the maximal disc inscribed in the strictly convex domain D, and by r the radius of K. If there are no points of S outside K, then D is a disc and we are done. If S contains points outside K, let $x \in S$ be such a point. Consider the line $\tilde{L}$ passing through the center of K and through the point $x \in S$, $x \notin K$. Let L' be the support line to S orthogonal to the line $\tilde{L}$ and tangent to S at a point x', $x' \notin K$. Denote the radius of curvature of S at the point x' by ρ. One has $\rho \le r$, because K is the maximal disc inscribed in D. The width L of D at the point x' in the direction of the line $\tilde{L}$ is greater than $2r$ because $x' \notin K$. One has $L > 2r$ and $L = 2\rho \le 2r$. This is a contradiction. It proves that $D = K$. Thus, D is a disc, and, consequently, the parameter p in formula (4) is equal to zero. Lemma 2 is proved. □

Thus, Theorem 1 is proved.

Remark 3. Let us give another proof that $p = 0$, where p is defined in formula (4). One has

$$L(s) = (r(q) - r(s), \nu(s)).$$

Differentiate this equation with respect to s and get

$$L'(s) = -(r(q) - r(s), \kappa(s)t(s)) + \Big(r'(q)\frac{dq}{ds} - r'(s), \nu(s)\Big), \tag{13}$$

where $t(s)$ is the unit vector tangential to S at the point s. Here the known formula $\nu(s)' = -\kappa(s)t(s)$ was used. The second term in equation (13) vanishes since $r'(s)$ and $r'(q)$ are orthogonal to ν. Thus, $L'(s) = -2\pi p\kappa(s)$. Since D is strictly convex, one has the inequality $\min_{s\in S}\kappa(s) \geq \kappa_0 > 0$, where $\kappa_0 > 0$ is a constant. The function $L(s)$ must be periodic, with the period equal to the arc length of S. The differential equation $L'(s) = -2\pi p\kappa(s)$ does not have periodic solutions unless $p = 0$. Therefore, $p = 0$.

The author considered other symmetry problems in [2, 4, 5].

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