STRESS-STRENGTH INTERFERENCE MODELS

IN

RELIABILITY

by

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CHAPTER 1

Introduction

1.1 Reliability Definition

Reliability is an important concept in the planning, design, and operation of systems. We always expect long-lasting performances of products in use, electric power which does not fail, and so on. These features are tied to reliability design. Reliability considerations are playing an increasing role in virtually all engineering disciplines. As demands increase for systems that perform better and cost less , there is a concomitant requirement to minimize the probability of failures. We need the deep insight into failures and their prevention which is to be gained by comparing and contrasting the reliability characteristics of systems of differing characteristics: dam structure design, electromechanical machinery, and plant structures, to name a few [46].

The term *reliability* may be applied to almost any object, which is the reason that the terms *system, device, equipment,* and *component* are all used in the definition. Each of these terms, however, has a somewhat different connotation. In a general probabilistic approach to reliability, it does not matter what the object of analysis is called. We will assume if there is no need to separate each subject of reliability analysis, the system can be considered as a set of interacting components working together as an integrated whole.

A system is said to fail when it ceases to perform its intended function. When there is total cessation of function-- an electric circuit breaks, a structure collapse-- the system has clearly failed.

On the other hand, reliability is defined positively, in terms of a system performing its intended function, and no distinction is made between types of failures.

1.2 Performance and Reliability

Much of engineering endeavor is concerned with designing and building products for improved performance. We want good performance as well as reliability. In the real world, there are always trade-offs between these two features.

Load is most often used in the mechanical sense of stress on a structure. We interpret it generally so that it may be the thermal load caused by high temperature, the electric load on a generator, and so on. Whatever the nature of the load on a system or its components may be, we can understand the system performance through accelerating load to test the system utilization life. Thus, by applying the load test, we can know the weak design of the whole system or approximately how the system can resist the external loads.

CHAPTER 2

Reliability Concepts

2.1 Introduction

Generally, reliability is defined as the probability that a system will perform properly for a specified period of time under a given set of operating condition [46]. Unreliability is in contrast with reliability; it is defined as the probability that the system is no longer functioning properly. Similarly, the treatment of operating conditions requires an understanding both of the loading to which the system is subjected and of the environment within which it must operate. Nevertheless, the most important variable to which we must relate reliability is time, in that loads can be changed from time to time and the system's resistant strength can become weaker as time goes on.

We first examine reliability as a function of time, and this leads to the definition of hazard rate, which is a very important concept in reliability work. Examining the time dependence of hazard rates allows us to gain insight into the study of failures. This charateristic is very useful in the nature of reliability. Similarly, the time dependence of failures can be viewed in terms of failure modes to differentiate between failures caused by different mechanisms.

2.2 Basic Formulations

For a given set of operating conditions, the reliability is defined as the probability that a system survives for some specified period of time. This may be expressed in terms of the random variable T, the time to failure. The probability density function (p.d.f.) has the following meaning

probability that failure takes place
at a time between t and t+dt

$$f(t)dt = P(t \le T \le t+dt) \tag{2.1}$$

From Eq. 2.1 it is possible to prove that the CDF (cumulative distribution function) has the following meaning

probability that failure takes place
at a time less than or equal to t

$$F(t) = P(T \le t) \tag{2.2}$$

Then the reliability is

probability that a system operates
without failure for a length of time t

$$R(t) = P(T > t) \tag{2.3}$$

Therefore, from Eqs. 2.2 and 2.3, we know that

$$R(t) = 1 - F(t) \tag{2.4}$$

It is traditional to describe the failure law in terms of the density function

$$f(t) = F'(t) \tag{2.5}$$

which must have the properties that

$$f(t) \geq 0$$

$$\int_0^\infty f(t)dt = 1$$

or equivalently

$$F(t) = \int_0^t f(x)dx \tag{2.6}$$

Then the reliability can be written as

$$R(t) = 1 - F(t) = 1 - \int_0^t f(x)dx \tag{2.7}$$

or

$$R(t) = \int_t^\infty f(x)dx \tag{2.8}$$

From the properties of the p.d.f., it is clear that F(t) is a monotone non-decreasing function of t with

$$F(0) = 0 \tag{2.9}$$

and

$$F(\infty) = 1 \tag{2.10}$$

Eq. 2.7 can be differentiated to give the p.d.f. of failure time in terms of the reliability

$$f(t) = -\frac{d}{dt} R(t) \tag{2.11}$$

We now define the hazard rate in terms of the reliability and the p.d.f. of time to failure as follows. Let $\lambda(t)dt$ be the probability that the system will fail at some time in $t < T < t+dt$, given that it has not yet failed at $T = t$. Thus it is the conditional probability

$$\lambda(t)dt = P(t < T < t+dt \mid T > t) \tag{2.12}$$

we have

$$P(t < T < t+dt \mid T > t) = \frac{P\{(T > t) \cap (T < t+dt)\}}{P(T > t)} \tag{2.13}$$

the numerator on the right-hand side is an alternative way of writing the p.d.f.; that is,

$$P\{(T > t) \cap (T < t+dt)\} \equiv P\{t < T < t+dt\} = f(t)dt \tag{2.14}$$

the denominator of Eq. 2.13 is just R(t), as seen on Eq. 2.3. Therefore, combining these equations, we obtain

$$\lambda(t) = \frac{f(t)}{R(t)} \tag{2.15}$$

This function is called the hazard rate or instantaneous failure rate or often simply failure rate.

A useful way to express the reliability and the p.d.f. is in terms of the hazard rate. We can change Eq. 2.15 to obtain the hazard rate in terms of the reliability,

$$\lambda(t) = -\frac{1}{R(t)} \frac{d}{dt} R(t) \qquad (2.16)$$

then multiplying by dt, we obtain

$$\lambda(t)dt = -\frac{dR(t)}{R(t)} \qquad (2.17)$$

We now have four different but interrelated functions for describing a statistical failure law for a component or a system . The relationship between them except hazard rate is shown in Fig. 2.1.

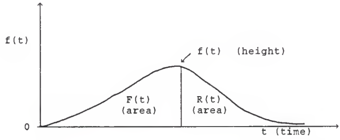


Figure 2.1 Relationship between the three performance functions

2.3 The Bathtub Curve

The behavior of $\lambda(t)$ with time may be quite revealing to an expressional pattern with respect to the causes of failure. It may have the general characteristics of a "bathtub" curve shown in Fig. 2.2; this curve is somewhat descriptive of human life times.

The short period of time on the left-hand side of Fig. 2.2 is a region of high but decreasing failure rates. This is referred to as the period of early failures or, in human populations infant mortality. Referring to mechanical units, we can say they are defective pieces of equipments prone to failure because they were not manufactured or constructed properly. We can call this phenomenon "burn-in".

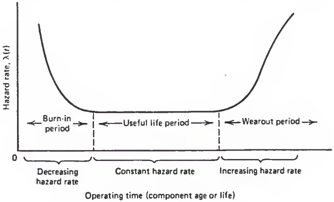


Figure 2.2 Bathtub Curve

The middle section of the bathtub curve contains the smallest and most nearly constant failure rates; this is referred to as the useful life period. Failures during this period of time are frequently referred to as "random failures".

On the right-hand side of the bathtub curve is a region of increasing failure rates; during this period of time aging failures are said to take place. For mechanical units, for example, there may be cumulative effects such as corrosion, fatigue, and diffusion of materials. We can say that this period is "wearout" period.

CHAPTER 3

Stress-Strength Interference Reliability Models

3.1 Introduction

We define stress and strength as follows [16] :

Stress s : that load which tends to produce a failure of a component, a device or a material. The term load may be defined as mechanical load, environment, temperature, electric current, etc.

Strength S : the ability of the component, a device or a material to accomplish its required mission satisfactorily without a failure when subject to the external loading and environment.

For many cases both stress and strength may be described as random variables. Strength may vary from component to component because of variations in the material properties due to variations in the production processes, etc. Therefore, when estimating the expected strength distribution of an equipment or a component all the important variabilities and their distributions must be considered and known (or assumed).

The techniques to predict the expected reliability from the variability distributions of stress and strength are presented in [16,17,40,46,62].

3.2 General Expression for Reliability

Let the density function for the stress s be denoted by $f_s(\cdot)$, and that for strength S by $f_S(\cdot)$ as shown in Figure 3.1 Then by definition,

$$R = P\ (S > s) = P\ (S - s > 0) \qquad (3.1)$$

We now present the argument developed by Kapur and Lamberson [40]. The shaded portion in Figure 3.1 shows the interference area, which is in some sense indicative of the probability of failure. The enlarged interference area is shown in Figure 3.2

The probability of a stress value lying in a small interval of width ds is equal to the area of the element ds; that is, up to differentials of a higher order,

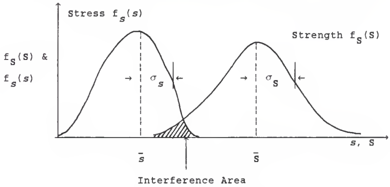


Figure 3.1 Stress-Strength Interference

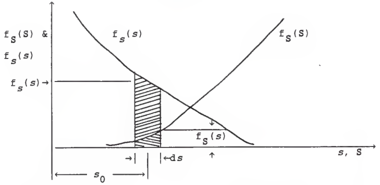


Figure 3.2 Computation of Reliability-enlarged portion of the interference area

$$P\left[s_0 - \frac{ds}{2} \le s \le s_0 + \frac{ds}{2}\right] = f_s(s_0)ds$$

The probability that the strength is greater than a certain stress s_o is given by

$$P(S > s_o) = \int_{s_o}^{\infty} f_S(S)dS$$

The probability of the stress value lying in the small interval ds and the strength S exceeding the stress given by s in this small interval ds under the assumption that the stress and the strength random variables are independent is given by

$$f_s(s_o)ds \cdot \int_{s_o}^{\infty} f_S(S)dS \tag{3.2}$$

Now the reliability of the component is the probability that the strength S is greater than the stress s for all possible values of the stress s and is given by

$$R = \int_0^\infty f_s(s) \left[\int_s^\infty f_S(S)\,dS \right] ds \qquad (3.3)$$

We can show the idea in Fig. 3.3 where the reliability is the volume over the shaded area

Reliability can also be computed on the basis that the stress is less than the strength. Again assume that the stress and the strength are independent variables. Using the same method as above, the reliability of the component for all the possible values of the strength S is

$$R = \int_0^\infty f_S(S) \left[\int_0^S f_s(s)\,ds \right] dS \qquad (3.4)$$

and we can show the idea as in Fig. 3.4

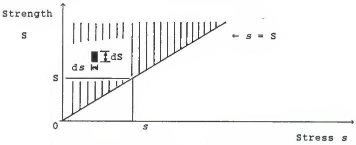


Figure 3.3 Reliability for fixed stress s

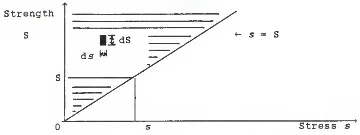


Figure 3.4 Reliability for fixed strength S

The same result can be obtained more simply by the geometrical argument.

Let S = strength, s = stress

$$P(S \le \tilde{S} \le S+dS,\ s \le \tilde{s} \le s+ds)$$

$$= f_S(S)dS \cdot f_s(s)ds$$

Then reliability

$$R = P(S \ge s)$$

$$= \int\int_{A_d} f_S(S)dS\ f_s(s)ds$$

where A_d stands for the shaded area in Fig. 3.3 and 3.4. Thus, we can write the reliability as

$$R = \int_0^{\infty}\int_s^{\infty} f_S(S)f_s(s)dS\ ds = \int_0^{\infty}\int_0^{S} f_s(s)f_S(S)ds\ dS$$

The unreliability is defined as

$$F = \text{probability of failure} = 1 - R = P(S \le s)$$

Substituting for R from Equation 3.3 yields

$$F = P(S \le s) = 1 - \int_0^{\infty} f_s(s) \left[\int_s^{\infty} f_S(S) dS \right] ds$$

$$= 1 - \int_0^{\infty} f_s(s) \left[1 - F_S(s) \right] ds$$

$$= \int_0^{\infty} F_S(s) \cdot f_s(s) ds \qquad (3.5)$$

Alternatively, using Equation 3.4 we have

$$F = P(S \le s) = 1 - \int_0^{\infty} f_S(S) \left[\int_0^{S} f_s(s) ds \right] dS$$

$$= 1 - \int_0^{\infty} f_S(S) \cdot F_s(S) \; dS$$

$$= \int_0^{\infty} [1 - F_s(S)] \cdot f_S(S) dS \qquad (3.6)$$

Now define $y = S - s$. Then y is called the interference random variable. Define the reliability as

$$R = P(y > 0) \qquad (3.7)$$

which means the component reliability has non-zero value only when the strength is greater than the stress. Assume that s and S are non-negative independent random variables, and that the p.d.f. of S is $f_S(S)$. Looking again at Eqs. 3.1 and 3.2, assume now that the strength S is fixed and that s has random magnitudes. Then from Eqs. 3.1 and 3.2 we can get the result

$$f_y(y)dy = f_S(S)dS \cdot \int_s^{\infty} f_s(s)ds \tag{3.8}$$

where the value of S is greater than the value of s in Eq. 3.8 The range of s can be from 0 to ∞, so

$$f_y(y) = \int_0^{\infty} f_S(S) \cdot f_s(s)ds \qquad S > s$$

$$= \int_0^{\infty} f_S(y+s) \cdot f_s(s)ds \qquad S > s \tag{3.9}$$

hence the reliability is given by y from 0 to infinity

$$R = \int_0^{\infty} f_y(y)dy$$

$$= \int_0^{\infty} \int_0^{\infty} f_S(y+s) \cdot f_s(s)dsdy \tag{3.10}$$

and the unreliability is

$$F = 1 - R = \int_{-\infty}^{0} \int_0^{\infty} f_S(y+s) \cdot f_s(s)dsdy \tag{3.11}$$

We illustrate the use of this formula for some particular distributions in the next section.

3.3 Reliability for Normally Distributed Strength and Stress

Assume that the probability density function for a normally distributed stress s is given by

$$f_s(s) = \frac{1}{\sigma_s\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{s-\mu_s}{\sigma_s}\right)^2\right],$$

$$-\infty < s < \infty \tag{3.12}$$

and the probability density function for a normally distributed strength S is given by

$$f_S(S) = \frac{1}{\sigma_S\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{S-\mu_S}{\sigma_S}\right)^2\right],$$

$$-\infty < S < \infty \tag{3.13}$$

Define y = S - s, the interference random variable, as before. It is known that the random variable y is normally distributed with a mean of

$$\mu_y = \mu_S - \mu_s \tag{3.14}$$

and a standard deviation of

$$\sigma_y = \sqrt{\sigma_S^2 + \sigma_s^2} \tag{3.15}$$

The reliability R can be expressed in terms of y as

$$R = P(y > 0)$$

$$= \int_0^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y - \mu_y}{\sigma_y}\right)^2\right] dy$$

To evaluate the integral, a change of variables is needed. Let $z = (y - \mu_y)/\sigma_y$, which is the standard normal variable. Then $\sigma_y dz = dy$. When $y = 0$, the lower limit of the integral is given by

$$z_0 = \frac{0 - \mu_y}{\sigma_y} = - \frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}} \qquad (3.16)$$

Therefore

$$R = \frac{1}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-z^2/2} dz$$

$$= 1 - \Phi\left[z_0 \right] \qquad (3.17)$$

3.3.1 Numerical Example

A component has been designed to withstand a certain level of stress which is normally distributed with a mean of 30,000 kPa (kPa stands for kilo-newton/m^2) and a standard deviation of 3,000 kPa. The strength of the component is normally distributed with a mean of 40,000 kPa and a standard deviation of 4,000 kPa. Calculate the reliability of the component.

We are given that

$$S \sim N(40{,}000,\ 4{,}000^2)\ \text{kPa}$$

$$s \sim N(30{,}000,\ 3{,}000^2)\ \text{kPa}$$

Then from Eq. 3.16 the lower limit of the integral for R is $z_0 = -2.0$, and hence $R = 1 - \Phi(-2) = 0.977$.

3.3.2 Numerical Example

A new component is to be designed; it will be subject to a tensile stress. There are variations in the load and the tensile stress is found to be normally distributed with a mean of 35,000 psi and standard deviation of 4,000 psi. The manufacturing operations create a residual compressive stress that is normally distributed with a mean of 10,000 psi and standard deviation of 1,500 psi. A strength analysis showed that the mean value of the strength is 50,000 psi. Now we want to know the maximum value of the standard deviation for the strength that will insure that the component reliability does not drop below 0.999. We are given that

$$s_t \sim N(35{,}000,\ 4{,}000^2)\ \text{psi}$$

$$s_c \sim N(10{,}000,\ 1{,}500^2)\ \text{psi}$$

where s_t is the tensile stress and s_c is the residual compressive stress. The mean effective stress s is obtained by

$$s = s_t - s_c = 35{,}000 - 10{,}000 = 25{,}000\ \text{psi}$$

with standard deviation

$$\sigma_s = \sqrt{\sigma^2_{s_t} + \sigma^2_{s_c}}$$

$$= 4{,}272 \text{ psi}$$

From the normal tables, the value of Z_0 associated with a reliability of 0.999 is $Z_0 = -3.1$. Setting

$$-3.1 = -\frac{50000-25000}{\sqrt{\sigma^2_S + (4272)^2}}$$

solving for σ_S, we obtain

$$\sigma_S = 6{,}840 \text{ psi}$$

3.4 Reliability for Lognormally Distributed Strength and Stress

The usual form of a lognormal density function is

$$f_y(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(\ln y - \mu)^2\right], \qquad y > 0 \qquad (3.18)$$

where y is the random variable. The parameters μ and σ are the mean and the standard deviation, respectively, of the variable ln y.

The first step is to develop the relationships for the lognormal distribution. Let x = ln y, then dx = (1/y)dy. From Eq. 3.18

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right], \qquad -\infty < x < \infty$$

Now consider the exponent of e in the expression

$$E(y) = E(e^x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^x \cdot \exp\left\{-\left(\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx = A \text{ (say)}$$

It has the form

$$x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 = x - \frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)$$

$$= \mu + \frac{\sigma^2}{2} - \frac{1}{2\sigma^2}\left[x - (\mu + \sigma^2)\right]^2$$

Therefore

$$E(y) = \exp\left[\mu + \frac{\sigma^2}{2}\right] \cdot \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[- \frac{\{x - (\mu+\sigma^2)\}^2}{2\sigma^2}\right] dx$$

$$= \exp\left[\mu + \frac{\sigma^2}{2} \right] \qquad (3.19)$$

To compute the variance of y we observe that

$$E(y^2) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[2x - \frac{1}{2\sigma^2}(x - \mu)^2\right]dx$$

Considering the exponent of e in the expression for $E(y^2)$,

$$2x - \frac{1}{2\sigma^2}(x - \mu)^2$$

$$= - \frac{1}{2\sigma^2}\left[x - (\mu + 2\sigma^2)\right]^2 + 2\mu + 2\sigma^2$$

which, when substituted back and simplified as before, yields,

$$E(y^2) = \exp\left[2(\mu + \sigma^2)\right]$$

Hence by definition of variance,

$$\mathrm{Var}(y) = E(y^2) - (E(y))^2 = B^2 \text{ (say)}$$

$$= \left[e^{(2\mu+\sigma^2)}\right] \cdot \left[e^{\sigma^2} - 1\right] \tag{3.20}$$

Now if $\hat{y}$ denotes the median of y, then we may write

$$0.5 = \int_0^{\hat{y}} \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[- \frac{1}{2\sigma^2} (\ln y - \mu)^2\right] dy$$

Using the transformation x = ln y,

$$0.5 = \int_{-\infty}^{\ln \hat{y}} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[- \frac{1}{2\sigma^2} (x - \mu)^2\right] dx$$

yields

$$\mu = \ln \hat{y} \tag{3.21}$$

or

$$\hat{y} = e^{\mu}$$

Returning now to the original problem in which S and s are lognormally distributed, we define the safety factor y = S/s, where y is also a random variable. We know

$$\hat{S} = e^{\mu_{\ln S}}$$

or

$$\mu_{\ln S} = \ln \hat{S}$$

and

$$\hat{s} = e^{\mu_{\ln s}}$$

or

$$\mu_{\ln s} = \ln \hat{s}$$

where $\hat{S}$ and $\hat{s}$ are the medians of S and s respectively.

Now

$$\ln y = \ln S - \ln s$$

$$E(\ln y) = E(\ln S) - E(\ln s)$$

$$\mu_{\ln y} = \mu_{\ln S} - \mu_{\ln s} \tag{3.22}$$

$$\ln \hat{y} = \ln \hat{S} - \ln \hat{s}$$

$$\hat{y} = \frac{\hat{S}}{\hat{s}}$$

We also know that for independently distributed normal variables

$$\sigma_{\ln y} = \sqrt{\sigma^2_{\ln S} + \sigma^2_{\ln s}} \tag{3.23}$$

the system is reliable if the safety factor y exceeds 1 and the probability of this is

$$R = P\left(\frac{S}{s} > 1 \right) = P(\ y > 1\ ) = \int_1^{\infty} f_y(y)dy$$

Let $Z = (\ \ln y - \mu_{\ln y}\ )/\sigma_{\ln y}$, i.e. Z is the standard normal variate. When $y = 1$, we get $Z = Z_0$ (say) where

$$Z_0 = \frac{\ln 1 - \mu_{\ln y}}{\sigma_{\ln y}} = - \frac{\ln \hat{S} - \ln \hat{s}}{\sqrt{\sigma^2_{\ln S} + \sigma^2_{\ln s}}}$$

The reliability can be computed as

$$R = \int_{-Z_0}^{\infty} \phi(Z)\,dz \tag{3.24}$$

where $\phi(Z)$ is the p.d.f. for the standard normal variate Z.

3.4.1 Numerical Example

Assume that the strength S and the stress s are lognormally distributed with the following parameters :

$$E(S) = 100{,}000 \text{ kPa}, \qquad \sigma_S = 10{,}000 \text{ kPa}$$

$$E(s) = 60{,}000 \text{ kPa}, \qquad \sigma_s = 20{,}000 \text{ kPa}$$

Compute the reliability.

Let

$$E(\ln S) = \mu_S \quad \text{and} \quad E(\ln s) = \mu_s$$

$$Var(\ln S) = \sigma_S^2 \quad \text{and} \quad Var(\ln s) = \sigma_s^2$$

For a generic lognormal variable where μ and σ^2 are the moments of ln y we know $E(y^2) = \exp\left[2(\mu + \sigma^2)\right]$, and from Eq. 3.20 we observe that

$$\frac{Var(y)}{[E(y)]^2} = \frac{B^2}{A^2} = e^{\sigma^2} - 1$$

which after rearranging, leads to

$$\sigma^2 = \ln\left[\frac{B^2}{A^2} + 1\right]$$

We now can compute the strength S

$$\sigma_S^2 = \ln\left[\frac{B^2}{A^2} + 1\right] = \ln 1.01 = 0.00995$$

and from Equation 3.19, we have

$$\mu_S = \ln E(S) - \frac{1}{2}\sigma_S^2 = 11.50795$$

similarly, for the stress we have

$$\sigma_s^2 = 0.10535 \qquad \mu_s = 10.94942$$

Therefore,

$$R = \int_{-Z_0}^{\infty} \phi(Z)\,dZ$$

where Z_0 is given by the equation :

$$Z_0 = -\frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}} = -\frac{11.50795-10.94942}{\sqrt{0.00995+0.10535}} = -1.64$$

From the normal table, we have R = 0.9495

3.5 Reliability for Exponentially Distributed Strength and Stress

In this case we have, for strength S,

$$f_S(S) = \lambda_S e^{-\lambda_S S}, \qquad 0 \le S < \infty$$

and, for stress s

$$f_s(s) = \lambda_s e^{-\lambda_s s}, \qquad 0 \le s < \infty$$

Using Equation 3.3, we have

$$R = \int_0^\infty f_s(s) \left[\int_s^\infty f_S(S) dS \right] ds$$

$$= \int_0^\infty \lambda_s e^{-\lambda_s s} \left[e^{-\lambda_S s} \right] ds$$

$$= \frac{\lambda_s}{\lambda_S + \lambda_s} \cdot \int_0^\infty (\lambda_S + \lambda_s) e^{-(\lambda_S + \lambda_s)s} ds$$

$$= \frac{\lambda_s}{\lambda_S + \lambda_s} \qquad (3.25)$$

3.6 Reliability for Normally (Exponentially) Distributed Strength and Exponentially (Normally) Distributed Stress

For a normally distributed strength the density function is

$$f_S(S) = \frac{1}{\sigma_S\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{S-\mu_S}{\sigma_S}\right)^2\right], \qquad -\infty < S < \infty$$

and the density function for an exponentially distributed stress is

$$f_s(s) = \lambda_s\, e^{-\lambda_s s}, \qquad s \geq 0$$

From Equation 3.4

$$R = \int_0^\infty f_S(S)\left[\int_0^S f_s(s)\,ds\right]dS$$

yields

$$R = \int_0^\infty \frac{1}{\sigma_S\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{S-\mu_S}{\sigma_S}\right)^2\right]\cdot\left[1 - e^{-\lambda_s S}\right]dS$$

$$= \frac{1}{\sigma_S\sqrt{2\pi}}\cdot\int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{S-\mu_S}{\sigma_S}\right)^2\right]dS$$

$$- \frac{1}{\sigma_S\sqrt{2\pi}}\cdot\int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{S-\mu_S}{\sigma_S}\right)^2\right]\cdot e^{-\lambda S}\,dS$$

$$= 1 - \Phi\left[-\frac{\mu_S}{\sigma_S}\right] - \frac{1}{\sigma_S\sqrt{2\pi}}\, T$$

$$\text{where } T = \int_0^{\infty} \exp\left[-\frac{1}{2\sigma_S^2}\left[(S-\mu_S+\lambda\sigma_S^2)^2 + 2\mu_S\sigma_S^2\lambda - \lambda^2\sigma_S^4\right]\right]dS$$

For convenience, we let $t = (S-\mu_S+\lambda\sigma_S^2)/\sigma_S$, then $\sigma_S dt = dS$. The reliability assumes the form

$$R = 1 - \Phi\left[-\frac{\mu_S}{\sigma_S}\right] - \frac{1}{\sqrt{2\pi}}\int_x^{\infty} \exp\left[-\frac{t^2}{2}\right]\cdot\exp\left[-\frac{1}{2}(2\mu_S\lambda-\lambda^2\sigma_S^2)\right]dt$$

$$\text{where } x = \frac{\mu_S-\lambda\sigma_S^2}{\sigma_S}$$

$$R = 1 - \Phi\left[-\frac{\mu_S}{\sigma_S}\right] - \exp\left[-\frac{1}{2}(2\mu_S\lambda_s-\lambda_s^2\sigma_S^2)\right]\left[1-\Phi\left[-\frac{\mu_S-\lambda_s\sigma_S^2}{\sigma_S}\right]\right] \tag{3.26}$$

When the distribution for the strength and the stress are interchanged, that is, when the strength has an exponential density function with parameter λ_S and the stress is normal with parameters μ_s and σ_s, Equation 3.3 can be used to obtain the following expression for the reliability :

$$R = \int_0^{\infty} f_s(s)\left[\int_s^{\infty} f_S(S)dS\right]ds$$

Therefore

$$R = \int_0^{\infty} \frac{1}{\sigma_s\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{s-\mu_s}{\sigma_s}\right)^2\right]\cdot\left[1-1+\exp(-\lambda_S s)\right]ds$$

in the method used above, let $t = (s-\mu_s+\lambda_S^2\sigma_s^2)/\sigma_s$, then $\sigma_s dt = ds$. We get

$$R = \Phi\left(-\frac{\mu_S}{\sigma_S}\right)+\exp\left[-\frac{1}{2}\left(2\mu_s\lambda_S - \lambda_S^2\sigma_s^2\right)\right]\left[1 - \Phi\left(-\frac{\mu_s-\lambda_S\sigma_s^2}{\sigma_s}\right)\right] \quad (3.27)$$

3.6.1 Numerical Example

The strength of a component is normally distributed with μ_S = 100 MPa (MPa stands for Mega-newton/m^2) and σ_S = 10 MPa. The stresses acting on the component follow the exponential distribution with mean value 50 MPa. Compute the reliability. Using Equation 3.26, we have

$$R = 1 - \Phi(-10) - \exp\left[-\frac{1}{2}\left(\frac{2(100)}{50} - \left(\frac{10}{50}\right)^2\right)\right]\left[1 - \Phi\left(-\frac{100-\frac{10^2}{50}}{10}\right)\right]$$

$$= 1 - 0.0 - e^{-1.98}\cdot[1 - 0.0]$$

$$= 0.86194$$

3.7 Reliability for Gamma Distributed Strength and Stress

The gamma density function for a random variable x is given by

$$f(x) = \frac{1}{\beta^{\alpha} \cdot \Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-(x/\beta)}; \qquad \alpha > 0,\ \beta > 0,\ 0 \le x < \infty$$

First consider the case when $\beta = 1$; we let $\alpha_S = m$ and $\alpha_s = n$, so that

$$f_S(S) = \frac{1}{\Gamma(m)} S^{m-1} \cdot e^{-S}, \qquad 0 \le S < \infty$$

and

$$f_s(s) = \frac{1}{\Gamma(n)} s^{n-1} \cdot e^{-s}, \qquad 0 \le s < \infty$$

Using Eq. 3.9 we have

$$f_y(y) = \int_0^{\infty} f_S(y + s) \cdot f_s(s)\, ds \qquad y \ge 0$$

In the same manner; let $\gamma = S - s$, we say γ is the excess value of $S - s$. Then

$$f_\gamma(\gamma) = \frac{1}{\Gamma(m)\Gamma(n)} \cdot \int_0^{\infty} (\gamma + s)^{m-1} \cdot e^{-(\gamma + s)} \cdot e^{-s}\, ds, \qquad \gamma \ge 0$$

For mathmatical convenience, let $v = s/\gamma$; then $dv = (1/\gamma)ds$, and

$$f_\gamma(\gamma) = \frac{1}{\Gamma(m)\Gamma(n)} \cdot \gamma^{m+n-1} \cdot e^{-\gamma} \cdot \int_0^{\infty} v^{n-1} (1 + v)^{m-1} \cdot e^{-2\gamma v}\, dv$$

Since

$$R = \int_0^{\infty} f_{\gamma}(\gamma)\ d\gamma$$

if we let $u = v/(1 + 2v)$, the reliability assumes the form

$$R = \frac{\Gamma(m + n)}{\Gamma(m)\Gamma(n)} \cdot \int_0^{1/2} (1 - u)^{m-1} \cdot u^{n-1} du$$

This integral can be recognized as the incomplete beta function; hence

$$R = \frac{\Gamma(m + n)}{\Gamma(m)\Gamma(n)} \cdot B_{1/2}(m,n) \qquad (3.28)$$

An example dealing with this case is shown in the next section.

3.7.1 Numerical Example - incomplete beta function

The incomplete beta function is defined as

$$B_x(m,n) = \int_0^x \frac{t^{m-1}(1-t)^{n-1}}{B(m,\ n)} dt$$

This is a constant whose value is determined by m and n and the integration limit x. We want to find, for given m and n values, the unique x value which makes the area(probability) equal to 0.5, as illustrated in Fig. 3.5. For the integral to be well defined, the values of m and n must be greater than zero. We use values from 0.5 to 5, in steps of 0.5, for both m and n, and summarize the results in Table 3.1. As we can see from Table 3.1, when the values of m and n are equal, the graph of the integrand found is symmetrical and the x value is 0.5 for all equal m and n values. In the situation where m > n, the graph is a left-skewed curve. The bigger the value of m, the higher the value of x is. On the other hand, in the situation where n > m, the graph is a right-skewed curve. The bigger the value of n, the lower the value of x is.

To solve for the desired mid-point x in the treatment above, we use a FORTRAN program with IMSL routine MDBETA. The program listing is in Appendix A.

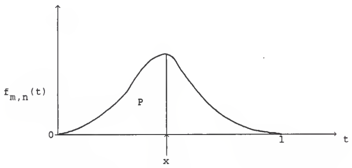


Figure 3.5 p.d.f. of Incomplete Beta Function

Table 3.1 median values of the Incomplete Beta Function

x value

m \ n	0.5	1.0	1.5	2.0	2.5
0.5	0.500	0.250	0.163	0.121	0.096
1.0	0.750	0.500	0.370	0.293	0.242
1.5	0.837	0.630	0.500	0.414	0.352
2.0	0.879	0.707	0.586	0.500	0.436
2.5	0.904	0.758	0.648	0.564	0.500
3.0	0.921	0.794	0.693	0.614	0.551
3.5	0.933	0.820	0.728	0.654	0.593
4.0	0.941	0.841	0.756	0.686	0.628
4.5	0.948	0.857	0.779	0.713	0.657
5.0	0.953	0.871	0.798	0.736	0.682

x value

m \ n	3.0	3.5	4.0	4.5	5.0
0.5	0.079	0.067	0.059	0.052	0.047
1.0	0.206	0.180	0.159	0.143	0.129
1.5	0.307	0.272	0.244	0.221	0.202
2.0	0.386	0.346	0.314	0.287	0.264
2.5	0.449	0.407	0.372	0.343	0.318
3.0	0.500	0.457	0.421	0.391	0.364
3.5	0.543	0.500	0.464	0.432	0.405
4.0	0.579	0.536	0.500	0.468	0.440
4.5	0.609	0.568	0.532	0.500	0.472
5.0	0.636	0.595	0.560	0.528	0.500

Now consider the case when $\beta \neq 1$. The strength and stress equations are

$$f_S(S) = \frac{1}{\beta_S^m\,\Gamma(m)} \cdot S^{m-1} \cdot e^{-(S/\beta_S)}; \qquad \beta_S > 0,\ m > 0,\ 0 \le S < \infty$$

and

$$f_s(s) = \frac{1}{\beta_s^n\,\Gamma(n)} \cdot s^{n-1} \cdot e^{-(s/\beta_s)}; \qquad \beta_s > 0,\ n > 0,\ 0 \le s < \infty$$

Using Equation 3.9 as before and the same integration techniques we have

$$R = \int_0^\infty f_y(y)\,dy$$

$$= \frac{r^n\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)} \cdot \int_0^\infty \frac{(1+v)^{m-1} v^{n-1}}{[1+(1+r)v]^{m+n}}\,dv$$

where $v = s/y$ and $r = \beta_S/\beta_s$. Now let $u = rv/(1+(1+r)v)$, then

$$R = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \cdot B_{r/(1+r)}(m,n) \qquad (3.29)$$

There are three special cases :

1. if $m = \alpha_S = 1$ and $n = \alpha_s = 1$, then S and s are exponentially distributed. The reliability from Eq. 3.29 reduces to

$$R = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \cdot \int_0^{r/(1+r)} du = \frac{r}{1+r} = \frac{\beta_s}{\beta_s + \beta_S}$$

2. if $m = \alpha_S = 1$ and $n = \alpha_s \neq 1$, then the strength S is exponentially distributed and the stress s is gamma distributed, and

$$R = \frac{\Gamma(n+1)}{\Gamma(1)\Gamma(n)} \int_0^{r/(1+r)} u^{n-1} du = \left(\frac{r}{1+r}\right)^n = \left(\frac{\beta_S}{\beta_s+\beta_S}\right)^n \qquad (3.30)$$

3. if $m = \alpha_S \neq 1$ and $n = \alpha_s = 1$, then the strength has a gamma distribution and the stress has an exponential distribution, and

$$R = \frac{\Gamma(m+1)}{\Gamma(m)\Gamma(1)} \cdot \int_0^{r/(1+r)} (1-u)^{m-1} du = 1 - \left(\frac{r}{1+r}\right)^m$$

$$= 1 - \left(\frac{\beta_S}{\beta_s+\beta_S}\right)^m \qquad (3.31)$$

3.8 Reliability for Weibull Distributed Strength and Stress

The probability density functions of the strength and the stress for the Weibull model are

$$f_S(S) = \frac{\beta_S}{\theta_S}\left[\frac{S - S_o}{\theta_S}\right]^{\beta_S - 1} \cdot \exp\left[-\left(\frac{S - S_o}{\theta_S}\right)^{\beta_S}\right], \qquad S_o \le S < \infty$$

and

$$f_s(s) = \frac{\beta_s}{\theta_s}\left[\frac{s - s_o}{\theta_s}\right]^{\beta_s - 1} \cdot \exp\left[-\left(\frac{s - s_o}{\theta_s}\right)^{\beta_s}\right], \qquad s_o \le s < \infty$$

respectively. The probability of failure given in Eq. 3.6 will be

$$F = P(S \le s)$$

$$= \int_S^{\infty} \exp\left[-\left(\frac{s - s_o}{\theta_s}\right)^{\beta_s}\right] \cdot \frac{\beta_S}{\theta_S}\left(\frac{S - S_o}{\theta_S}\right)^{\beta_S - 1} \cdot \exp\left[-\left(\frac{S - S_o}{\theta_S}\right)^{\beta_S}\right] dS$$

For mathematical convenience, let

$$y = \left(\frac{S - S_o}{\theta_S}\right)^{\beta_S}$$

so that

$$dy = \frac{\beta_S}{\theta_S}\left(\frac{S - S_o}{\theta_S}\right)^{\beta_S - 1} dS$$

Therefore

$$F = \int_0^{\infty} e^{-y} \cdot \exp\left\{ - \left[\frac{\Theta_S}{\Theta_s} y^{1/\beta_S} + \left(\frac{S_o - s_o}{\Theta_s} \right) \right]^{\beta_s} \right\} dy \qquad (3.32)$$

The values of the integral in Equation 3.32 have been computed by numerical integration methods [47,48] for different combinations of the parameters for strength and stress.

3.9 Chain Model

The chain model discussed in this section is a particular case of the weakest link model [46,49]. It is assumed that a "chain" is made up of identical "links" in the sense that breaking strengths of all the links in the chain have the same probability distribution. The chain breaks when its weakest link fails, and this occurence is assumed to be dependent only upon the variability of the link strengths based on a probability distribution of strengths.

Similarly, the stress to be applied to a given link is assumed to have its own probability distribution. The probability that the link does not break is then the probability that its strength exceeds the applied stress.

When the links are assembled to form a chain, it is assumed that a stress which is applied to the chain as a whole, is also applied equally to each of the links.

We first assume that the breaking strength, or simply "strength", of a link is known only in terms of a probability distribution of strengths. We know that the probability that the link strength lies between the values a and b is

$$F_S(b) - F_S(a) = \int_a^b f_S(S)\,dS \tag{3.33}$$

where $F_S(\cdot)$ is the strength CDF. Similarly, we assume stress to be defined by a probability density function and CDF $F_s(\cdot)$, such that

$$F_S(d) - F_S(c) = \int_c^d f_S(s)ds \tag{3.34}$$

is the probability that the stress lies between magnitudes c and d where d > c.

Now consider the process of forming a chain of n links selected from a population whose link strength probability density function is given by $f_S(S)$. This is the same as selecting a random sample $S_1, \cdots, S_n$ of link strengths from a population with probability density $f_S(S)$. The chain has strength equal to that of its weakest link; i.e., the strength $Y_{(1)}$ of an n-link chain is equal to the minimum of the S_i, (i = 1, ···, n).

The problem is to express the probability distribution of $Y_{(1)}$ in terms of $f_S(S)$. Let $f_n(y)$ denote the probability density function of $Y_{(1)}$; and let $Y_{(1)} = S_{(1)}$ (arbitrarily picked as the minimum of the S_i; (i = 1, ···, n). Then, as shown in Fig. 3.6, the S_i points fall into mutually exclusive cells, and we can use the multinominal distribution [6,pp 118] to express the probability that out of n points, one of them, say $S_{(1)}$, falls in the interval $(y, y+dy)$, and all other points $S_{(2)}, \cdots, S_{(n)}$ fall in a region to the right of y.

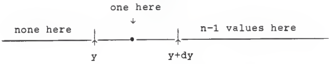


Figure 3.6 Scattering of n points

Thus,

$$P(y < Y_{(1)} < y+dy) = f_{Y_{(1)}}(y)dy = \frac{n!}{1!\cdot(n-1)!} f_S(y)dy[1-F_S(y)]^{n-1} \tag{3.35}$$

and therefore

$$f_{Y_{(1)}}(y) = n[1 - F_S(y)]^{n-1} f_S(y) \tag{3.36}$$

The strength distribution function of the n-link chain is given by

$$F_{Y_{(1)}}(y) = \int_0^y f_{Y_{(1)}}(\omega)d\omega \tag{3.37}$$

$$= \int_0^y n[1 - F_S(\omega)]^{n-1} f_S(\omega)d\omega$$

We use a transformation of variables; let $u = F_S(\omega)$ then $du = f_S(\omega)d\omega$. When $\omega = 0$ then $u = 0$, and when $\omega = y$ then $u = F_S(y)$, that Eq. 3.37 becomes

$$F_{Y_{(1)}}(y) = n\int_0^{F_S(y)} (1-u)^{n-1}du \tag{3.38}$$

Thus

$$F_{Y_{(1)}}(y) = \left[-(1 - u)^{n}\right]_{0}^{F_S(y)} = 1 - \left[1 - F_S(y)\right]^{n} \qquad (3.39)$$

Now for any stress s with probability density $f_S(s)$, the probability that the chain strength $Y_{(1)}$ exceeds the stress s applied to the chain is, from Equation 3.3

$$R_{Y_{(1)}} = P(Y_{(1)} > s) = \int_0^{\infty} f_S(s) \cdot [1 - F_{Y_{(1)}}(s)] ds \qquad (3.40)$$

or, using Equation 3.39

$$R_{Y_{(1)}} = \int_0^{\infty} f_S(s) \cdot [1 - F_S(s)]^{n} ds \qquad (3.41)$$

where $R_{Y_{(1)}}$ is the reliability of the n-link chain

3.10 Graphical Approach for Empirically Determined Stress and Strength Distributions

This technique is used to determine the reliability of a component from experimental data; it may be applied to any distribution. The transform is based on Eq. 3.3 and we will let G stand for the probability that when there is a known stress s and the strength S is greater than s

$$G(s) = P(S > s) = \int_s^{\infty} f_S(S)dS$$

$$= 1 - \int_0^s f_S(S)dS \tag{3.42}$$

$$= 1 - F_S(s)$$

Similarly, let H stand for the cumulative probability of stress

$$H(s) = \int_0^s f_s(u)du = F_s(s) \tag{3.43}$$

Eq. 3.43 is rewritten to the form

$$dH = f_s(s)ds \tag{3.44}$$

Obviously when s ranges from 0 to infinity, H takes value from 0 to 1. By substituting Eqs. 3.42 and 3.44 in Eq. 3.3 we get reliability

$$R = \int_0^1 G dH \tag{3.45}$$

Equation 3.45 suggests that the area under a G vs. H plot would represent the reliability of the component. Based on strength and stress data, we can determine for various values of s, the values of $F_S(s)$ and $F_s(s)$, and hence those of G and H. Plotting these values of G and H and measuring the area graphically is all that is needed to determine the component reliability.

3.10.1 Numerical Example

The stress applied to a component is exponentially distributed, where it is assumed that the stress cannot be less than 10,000 psi. The mean life of the component is 20,000 hour. Hence, the density function for the stress may be written as the shifted exponential density

$$f_s(s) = \begin{cases} 0, & 0 \le s < 10{,}000 \\ \dfrac{1}{10{,}000} \exp\left[-\dfrac{(s-10{,}000)}{10{,}000}\right], & s \ge 10{,}000 \end{cases}$$

The strength of the component is assumed to follow a Weibull distribution; the material used is such that the strength is never less than 15,000 psi. The strength distribution is assumed to have the parameters

$$S_o = 15{,}000, \quad \theta = 20{,}000, \quad \text{and} \quad \beta = 2$$

Hence the strength p.d.f. is given by

$$f_S(S) = \frac{2(S - 15{,}000)}{(20{,}000-15000)^2} \exp\left[-\left(\frac{S - 15{,}000}{20{,}000-15{,}000}\right)^2\right]$$

$$= \frac{2(S - 15{,}000)}{(5{,}000)^2} \exp\left[-\frac{(S - 15{,}000)^2}{(5{,}000)^2}\right], \qquad S \geq 15{,}000$$

The cumulative distribution functions for the stress and the strength are given by

$$F_s(s) = 1 - \exp\left[-\frac{(s - 10{,}000)}{10{,}000}\right]$$

$$F_S(S) = 1 - \exp\left[-\frac{(S - 15{,}000)^2}{(5{,}000)^2}\right]$$

Hence

$$G(s) = \int_s^\infty f_S(S)dS = 1 - F_S(s) = \exp\left[-\frac{(s - 15{,}000)^2}{(5{,}000)^2}\right]$$

and

$$H(s) = \int_0^s f_s(s)ds = F_s(s) = 1 - \exp\left[-\frac{(s - 10{,}000)}{10{,}000}\right]$$

The values of H and G are computed for various values of s as shown Table 3.2.

Table 3.2 H and G values

Stress *s*	H	G	Stress *s*	H	G
10000	0.0000	1.0000	23800	0.7484	0.0452
12000	0.1813	1.0000	24000	0.7534	0.0392
14000	0.3297	1.0000	24200	0.7583	0.0339
15000	0.3935	1.0000	24400	0.7631	0.0292
15400	0.4173	0.9936	24800	0.7724	0.0215
15600	0.4288	0.9857	25000	0.7769	0.0183
15800	0.4401	0.9747	25200	0.7813	0.0156
16000	0.4512	0.9608	25400	0.7856	0.0132
16600	0.4832	0.9027	25800	0.7940	0.0094
17000	0.5034	0.8521	26500	0.8080	0.0051
17200	0.5133	0.8240	27500	0.8262	0.0019
17600	0.5323	0.7631	28500	0.8428	0.0009
18000	0.5507	0.6977	29500	0.8577	0.0002
18400	0.5683	0.6298	30000	0.8647	0.0001
18800	0.5852	0.5612	32000	0.8892	0.0001
19200	0.6015	0.4938	34000	0.9093	0.0000
19600	0.6171	0.4290	36000	0.9257	0.0000
20000	0.6321	0.3679	38000	0.9392	0.0000
20400	0.6466	0.3115	40000	0.9502	0.0000
20800	0.6604	0.2604	42000	0.9592	0.0000
21200	0.6737	0.2149	44000	0.9666	0.0000
21600	0.6865	0.1751	46000	0.9727	0.0000
22000	0.6988	0.1409	48000	0.9776	0.0000
22400	0.7106	0.1119	50000	0.9817	0.0000
22800	0.7220	0.0877	52000	0.9850	0.0000
23000	0.7275	0.0773	54000	0.9877	0.0000
23200	0.7329	0.0679	56000	0.9900	0.0000
23400	0.7382	0.0595	58000	0.9918	0.0000
23600	0.7433	0.0519	60000	0.9933	0.0000

A plot of G vs. H is shown in Figure 3.7. The area under the curve measures 0.609 [40,pp 150], which is therefore the estimated reliability of the item.

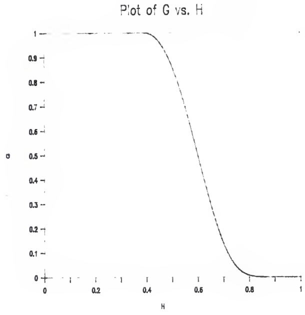


Figure 3.7 Plot of G vs. H

CHAPTER 4

Time Dependent Stress-Strength Models

4.1 Introduction

Stress-strength interference models, as introduced in Chapter 3, are good for a single stress application and these stress-strength models are independent of time. In real life, however, this may not necessarily be true. The component strength may change with time and a component may experience repeated application of stresses. In other words, the stress or load may follow a random pattern with respect to time t.

Examples of time-dependent reliability models are weakening caused by aging or cumulative damage. As better estimates of distributions become available from a performance history, these models provide a means for reassessing reliability.

As discussed in Chapter 2, the bathtub curve is the paradigm hazard function. We can see from Fig. 2.2 that; in the early "burn-in" region there is a decreasing failure rate, in the "chance failure" region there is a nearly constant failure rate, and in the "wearout" region there is an increasing failure rate. We will discuss how these models are applied in the stress-strength models.

Further, we assumed that the system strength is a variable [46] described by a p.d.f., $f_S(S)$. Such statictical variability is to be expected because of variations in the properties of materials and in dimensional tolerances and from innumerable other variables in the manufacturing and construction processes. In these cases, there may be an initial burn-in period of decreasing failure rate.

If the strength of the system is not independent of time, then we can take into account the wear effects that cause failure rates to increase with time, such as degradation of strength, which is often divided into three categories. If strength varies only with time, it is referred to as aging. If the strength of a system decreases with the number of times that it has been loaded, cyclic damage is said to occur. If the strength decrease depends both on the number of times that loading takes place and on the loading magnitudes, the phenomena are referred to as cumulative damage. We will concentrate our discussion on the aging effect.

4.2 Failure Rates and Repetitive Stress

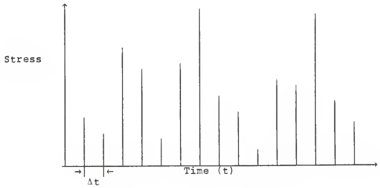
In the preceding chapter it was assumed that a system would fail under a single stress. In this section we examine the reliability of a system under repeated stress of random magnitude. We restrict our attention to a known strength that is independent of time .

Suppose that a system is subjected to repeated stresses, as indicated by Fig. 4.1a and b. The two graphs differ in that the stresses in Fig. 4.1a occur at fixed intervals, whereas those in Fig. 4.1b occur at random intervals. However, we are now more interested in the distribution of magnitudes rather than in their spacing over time. We assume that the stress magnitudes are random and independent; further, we will collect data to find out what is the probability distribution according to how many counts in each different stress, and what kind of distribution it would seem to be.

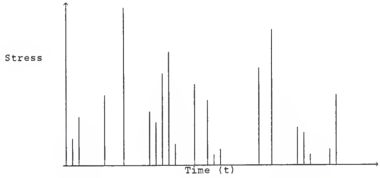
For a system with fixed, time-independent strength S, the reliability R(S) for any single stress occurance is independent of the reliabilities from the other occurances. That is,

$$R(S) = \int_0^S f_s(s)ds \tag{4.1}$$

and the probability of surviving n such stresses is



(a) Periodic stress, interval Δt



(b) stress at random intervals

Figure 4.1 Repetitive stresses of random magnitudes

$$R_n(S) = [\ R(S)\ ]^n \tag{4.2}$$

To convert this expression to $R(t|S)$, the reliability as a function of time, we must determine how frequently the stresses occur. Two cases are considered, periodic stress, and stress at random Poisson-distributed time intervals.

4.2.1 Periodic Stress

For stress at fixed time intervals we use the identity $\exp[\ln G] = G$ to convert the form of Eq. 4.2 to

$$R_n(S) = [R(S)]^n = \exp[\ln R(S)^n] \tag{4.3}$$

$$= \exp[n \ln R(S)] \tag{4.4}$$

If the probability of failure during any one stress is small, then $1 - R(S) = F(S) \ll 1$, and we may expand the natural logarithm on the right-hand side of Equation 4.4 as

$$\ln R(S) = \ln[1 - F(S)] \cong - F(S) \tag{4.5}$$

Thus a approximation to Eq. 4.3 is

$$R_n(S) = \exp[-n \cdot F(S)] \tag{4.6}$$

To convert the independent variable from count n to time t, we must know the interval Δt at which the stresses take place. With Δt known, we can say that at time t there have already been

$$n = \frac{t}{\Delta t} \tag{4.7}$$

stresses. Thus, combining Eqs. 4.6 and 4.7, we find the time dependence of the reliability, for given strength S, to be

$$R(t|S) = \exp\left[- \frac{F(S)}{\Delta t} t \right] \tag{4.8}$$

or simply

$$R(t|S) = e^{-h(S)t}, \tag{4.9}$$

where the strength-dependent failure rate is given by

$$h(S) = F(S)/\Delta t \tag{4.10}$$

Periodic phenomena are often discussed in terms of the return period T(S) for a stress that exceeds the strength S, defined by

$$T(S) = \frac{\Delta t}{1 - R(S)} \tag{4.11}$$

Equation 4.11 is exactly the reciprocal of Equation 4.10. The reliability given by Equation 4.8 may be written as

$$R(t|S) = e^{-t/T(S)} \tag{4.12}$$

T(S) is used to represent the frequency at which a stress greater than strength S may be expected to recur. It is usually applied to natural stresses on a calendar-year basis.

Numerical Example

Historically, a design rule for structures subjected to flooding has been to design for a flood with a return period of twice the design life. If this criterion is used, what is the probability of failure during the design life ?

Let T be the design life. Then T(S) = 2T and

$$R(t) = e^{-t/2T}$$

The probability of failure during design life is

$$1 - R(T) = 1 - e^{-T/2T} = 1 - e^{-1/2} = 0.393$$

4.2.2 Stress at Random Intervals

We now consider the other case that of non-periodic stress. In random stress the time until the next stress occurs is independent of when the last stress occured. In this situation the Poisson distribution is applicable. The random events are now taken to be peaks in the stresses, such as indicated in Fig. 4.1 b.

The probability of there being n stresses during time t is given by the Poisson formula

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n\,!} \tag{4.13}$$

where λ is the mean frequency of the stress. Now we take Equation 4.1 as the conditional probability that the system will survive, given n stresses and fixed strength S. Thus the reliability for given strength S is obtained from summing over n

$$R(t|S) = \sum_{n=0}^{\infty} R_n(S)\cdot P_n(t) \tag{4.14}$$

combining Equation 4.13 and 4.14 with Eq. 4.1, we have

$$R(t|S) = \sum_{n=0}^{\infty} [R(S)\lambda t]^n \frac{e^{-\lambda t}}{n\,!} \tag{4.15}$$

Noting, however, that the exponential may be expanded as

$$e^{R(S)\lambda t} = \sum_{n=0}^{\infty} \frac{[R(S)\lambda t]^n}{n\,!}, \tag{4.16}$$

so Equation 4.15 can be written as

$$R(t|S) = e^{-\lambda t} \cdot e^{R(S)\lambda t}$$

$$= e^{-\lambda t(1-R(S))}$$

$$= e^{-\lambda tF(S)} \tag{4.17}$$

$$= e^{-h(S)t}, \quad \text{(say)} \tag{4.18}$$

where constant failure rate is given by

$$h(S) = \lambda \cdot F(S) \tag{4.19}$$

Just as for the periodic stress mentioned before, we have once again obtained an expression for a time-independent failure rate for fixed system strength. The close relation between Eq. 4.10 and Eq. 4.19 for stress at random intervals is apparent. If we define τ as the mean time between stresses, we have for periodic stress $\tau = \Delta t$. Similarly, if the stress is a Poisson process, the mean time between stress may be shown to be $\tau = 1/\lambda$. Thus in either case,

$$h(S) = F(S)/\tau \tag{4.20}$$

This expression is thus valid for stresses at totally correlated time intervals(i.e., periodic) as well as at totally uncorrelated time intervals (Poisson). It is understandable that empirical data often yield constant failure rate for intermediate cases in which the stress intervals are partially correlated.

The observed increase in failures with decreased strength is clear from Equation 4.10 and 4.19. In both we have

$$h(S) \propto F(S) = \int_{S}^{\infty} f_S(s)ds \tag{4.21}$$

Numerical Example [46,pp 203]

A telecommunications leasing firm finds that during the one-year warrantee period, six percent of its telephones are returned at least once because they have been dropped and damaged. An extensive program earlier indicated that in only 20% of the drops should telephones be damaged. Assuming that the dropping of telephones in normal use is a Poisson process, (a) what is the MTBD (mean time between drops)? (b) Determine the probabilities that the telephone will not be dropped, will be dropped once, and will be dropped more than once during a year of service. (c) If the telephones are redesigned so that only 4% of drops cause damage, what fraction of the phones will be returned with dropping damage at least once during the first year of service ?

(a) The fraction of telephones not returned is $R = e^{-\lambda t F(S)}$

We know that $F(S) = 20\% = 0.2$, $t = 1$ (year), so

$$0.94 = e^{-\lambda \cdot 1 \cdot 0.2}$$

$$\lambda = \frac{1}{0.2} \ln\left(\frac{1}{0.94}\right) = 0.3094/\text{year}$$

$$\text{MTBD} = 1/\lambda = 3.23 \text{ year}$$

(b) From Equation 3.44 we have

$$P_0(0) = e^{-\lambda \cdot 1} = e^{-0.3094} = 0.734 \qquad \text{(no drops)}$$

$$P_1(0) = \lambda \cdot 1 \cdot e^{-\lambda \cdot 1} = 0.3094e^{-0.3094} = 0.227 \qquad \text{(one drop)}$$

$$1 - P_0(0) - P_1(0) = 1 - 0.734 - 0.227 = 0.039$$

(more than one drop)

(c) For the improved design $R = e^{-\lambda t F(S)} = e^{-0.3094(0.04)1}$ $= 0.9877$. Therefore the fraction of the phones returned at least once is

$$1 - 0.9877 = 1.23\%$$

4.3 Burn-in

The results of the previous section assume that the system strength S has a fixed value. To examine burn-in, we now relax this restriction and assume that the strength is a random variable described by a p.d.f, $f_S(S)$. This probability distribution may be viewed in two different ways. For mass-produced items it may be represent the variability in capacity within the batch of manufactured items. For single or few-of-a-kind systems, such as large structures or industrial plants, the p.d.f. may represent the designer's uncertainty about the as-built strength of the system. In either case we retain, for now, the assumption that the strength does not change with time.

The reliability $R(t|S)$ is just a conditional probability, given the strength. Therefore, we may obtain the expected value of the reliability R(t) by averaging over strength S

$$R(t) = \int_{-\infty}^{\infty} f_S(S) R(t|S) dS \tag{4.22}$$

Now suppose that we employ the constant failure rate model given by Eq. 4.18 for $R(t|S)$. We have

$$R(t) = \int_{-\infty}^{\infty} f_S(S) \cdot e^{-\lambda(S)t} dS \tag{4.23}$$

Let us consider two cases. In the first case we assume that the variation in strength is small, given by a normal p.d.f. with a small standard deviation. We also assume that the variation of the failure rate over the range of the strength is so small that it can be ignored. Hence, Eq.4.23 simply reduces to Eq. 4.18. The second case is slightly different; some fraction, say p_d, of the system under consideration are flawed in a serious way; these flaws will cause early or burn-in failures.

Before describing the probability density that systems are flawed, we will introduce the Dirac delta function.

Dirac delta function

If the normal distribution is used to describe a random variable x, the mean μ is the measure of the average value of x and the standard deviation σ is a measure of the dispersion of x about μ. Suppose that we consider a series of measurements of a quantity with increasing precision. The p.d.f. for the measurements might look similar to Fig. 4.2.

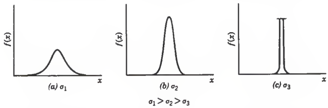


Figure 4.2 Normal distributions with different values of variance.

As the precision is increased—decreasing the uncertainty—the value of σ decreases. In the limit where there is no uncertainty (i.e. σ 0), x is no longer a random variable, for we know that $x = \mu$.

The Dirac delta function is used to treat this situation. It may be defined as

$$\delta(x - \mu) = \lim_{\sigma\ 0} \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left[- \frac{1}{2\sigma^2} (x - \mu)^2 \right] \tag{4.24}$$

Two most important properties immediately follow from this definition :

$$\delta(x - \mu) = \begin{cases} x, & x = \mu, \\ 0, & x \neq \mu, \end{cases} \tag{4.25}$$

and

$$\int_{\mu-\varepsilon}^{\mu+\varepsilon} \delta(x - \mu) dx = 1, \qquad \varepsilon > 0. \tag{4.26}$$

Specifically, even though $\delta(0)$ is infinite, the area under the curve is equal to one.

The primary use of the Dirac delta function is to simplify integrals in which one of the variables has a fixed value.

Now back to the burn-in, we write the p.d.f. of strength in terms of Dirac delta functions as

$$f_S(S) = (1 - p_d)\delta(S - S_o) + p_d \cdot \delta(S - S_d), \qquad (4.27)$$

where $p_d < 1$ is the probability that the system is defective. The first term on the right-hand side corresponds to the probability that the system will be a properly built system with specified design strength S_o. The second term corresponds to the probability that the system will be defective and have a reduced strength $S_d < S_o$. By using the Dirac delta function, we are assuming that in the first term the strength variability of the properly built systems can be ignored and in the second term the situation might arise, for example, if a critical component were to be left out of a small fraction of the systems in assembly with some probability p_d.

To see the effect on the failure rate, we first substitute Eq. 4.27 into Eq. 4.22 :

$$R(t) = \int_{-\infty}^{\infty} \left[(1 - p_d)\delta(S - S_o) + p_d\ \delta(S - S_d)\right] \cdot R(t|S)dS$$

$$= (1 - p_d) \cdot \int_{-\infty}^{\infty} \delta(S - S_o)R(t|S)dS + p_d \cdot \int_{-\infty}^{\infty} \delta(S - S_d)R(t|S)dS$$

$$= (1 - p_d) \cdot R(t|S_o) + p_d \cdot R(t|S_d)$$

$$= (1 - p_d) e^{-h(s_o)t} + p_d \cdot e^{-h(s_d)t} \tag{4.28}$$

Since the failure rate increases with decreased strength, we know that $h(s_d) > h(s_o)$. From the Chapter 2, definition of $h(t)$

$$h(t) = \frac{f(t)}{R(t)} = -\frac{R'(t)}{R(t)} \tag{4.29}$$

after evaluating the derivative of Eq. 4.28, we obtain

$$h(t) = h(s_o) \left\{ \frac{1 + \frac{p_d}{1 - p_d} \cdot \frac{h(s_d)}{h(s_o)} \cdot \exp\left[-(h(s_d) - h(s_o))t\right]}{1 + \frac{p_d}{1 - p_d} \cdot \exp\left[-(h(s_d) - h(s_o))t\right]} \right\} \tag{4.30}$$

The burn-in effect may be seen more explicitly by considering a system whose probability of defective is small, $p_d \ll 1$, but the defect greatly increased the failure rate, $h(s_d) \gg h(s_o)$. In this case the equation for $h(t)$ reduces to

$$h(t) \approx h(s_o) \left[1 + \frac{p_d h(s_d)}{h(s_o)} e^{-h(s_d)t} \right] \tag{4.31}$$

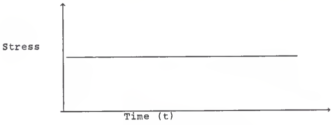
Thus the failure rate decreases from an initial value of $h(s_o) + p_d h(s_d)$ at time zero to the value $h(s_o)$ of the unflawed system after the defective units have failed.

4.4 Wearout

As we know from the burn-in phenomenon, the decreasing failure rates of burn-in are due to the variance of strength of a system. If the strength of a system is steady and has no variance, there is no burn-in effect. In general, wearout from aging may be viewed as a determministic phenomenon that would be present even if both stress and strength were known exactly. Suppose that a system has a strength that is a known function of time, $S_o = S_o(t)$, and that at any time there is no uncertainty in its value. If there is a constant stress s, such as in Fig. 4.3, the system will fail at time t_f for which

$$S_0(t_f) = s_0 \tag{4.32}$$

as illustrated in Fig. 4.4. The reliability for this system is then



constant stress

Figure 4.3 Pattern of stress variation

$$R(t) = \begin{cases} 1, & t < t_f \\ 0, & t > t_f \end{cases} \tag{4.33}$$

Generally, neither stress nor strength is known exactly, and the probability density functions $f_S(S)$ and $f_s(s)$ give rise to a p.d.f. of times to failure $f(t)$. The corresponding $R(t)$ is then characterized by a failure rate that increases with time, provided only that the strength is a decreasing function of time. We will assume in the following model that the strength is a known function of time in which there is no variability, whereas the stress is treated as a random variable.

We now consider a system whose strength is known with certainty, but its stress is repetitive and of random magnitude, as in Fig. 4.1a.

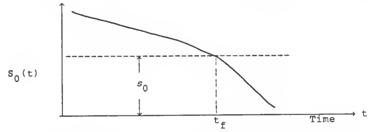


Figure 4.4 Strength vs. time for a system under constant stress

Suppose that we let S_n be the strength at the time of the nth stress. Then the probability of surviving the nth stress is just $R(S_n)$, given by Eq. 4.1. Since the magnitudes of the successive stresses are independent of one another, we may write the probability of surviving the first n stresses as

$$R_n = R(S_1)R(S_2)R(S_3) \cdots\cdots\cdot R(S_n) \tag{4.34}$$

Then, taking the exponential of $\ln R_n$, we obtain

$$R_n = \exp\left[\sum_{k=1}^{n} \ln R(S_k)\right] \tag{4.35}$$

Assuming that the probability of failure for any one stress is small, $F(S_n) = 1 - R(S_n) < 1$, we then obtain

$$R_n \cong \exp\left[-\sum_{k=1}^{n} F(S_k)\right] \tag{4.36}$$

To illustrate that the failure rate increases with time, there is a special model proposed by Lewis [54,pp 207] where it is assumed that the S_n decreases with n so that $F(S_n)$ increases linearly with the stress application according to the rule

$$F(S_n) \approx F_o(1 + \varepsilon n), \quad \varepsilon \ll 1 \tag{4.37}$$

According to Lewis, we can obtain the sum of $F(S_n)$

$$\sum_{k=1}^{n} F(S_k) = n + \frac{\varepsilon n^2}{2} \tag{4.38}$$

Finally, if we assume that the stresses appear with a mean time between stress application of τ, we may change variables to write the result in terms of time :

$$t = n \cdot \tau \tag{4.39}$$

Therefore, Eqs. 4.36 through 4.39 yield

$$R(t) = \exp\left[- F_o\left(1 + \frac{\varepsilon t}{2\tau}\right)\frac{t}{\tau}\right] \tag{4.40}$$

Using Eq. 4.29, we see that for the model the failure rate increases with time t

$$h(t) = \frac{F_o}{\Delta t}\left[1 + \frac{\varepsilon t}{\tau}\right] \tag{4.41}$$

In this situation, it is just like in Fig. 2.2 "wearout" period; the hazard rate is increasing with time.

CHAPTER 5

Extreme-Value Distributions

5.1 Introduction

A salient feature from Fig. 3.1 is the important fact that the probability of failure depends strongly on the lower tail of the strength distribution and on the upper tail of the stress distribution. The normal distribution and exponential distribution are useful in representing these tails when there are many contributions, no one of which is dominant. Still, there are many situations that the tails are not described well by the normal or exponential distribution, when the stress or strength is not determined by either the sum or the product of many relatively small contributions. In contrast, it may be the extreme of many contributions that governs the stress or the strength [54,pp 185]. For example, it is not the sum of the accelerations but rather the extreme value that determines the primary earthquake loading on a structure. Extreme-value distributions have proved to be very useful in the analysis of reliability problems of this nature [34].

We will briefly introduce the maximum extreme-value distribution for the treatment of stresses, and the minimum extreme-value distribution for strength determination. We then proceed to the standard asymptotic extreme-value distributions for large numbers of random variables which are useful in treating a variety of reliability problems.

5.2 Distribution of the *i*th Order Statistic

Consider the *i*th order statistic [10], $X_{(i)}$, which has arisen from a probability density $f_X(x)$ and a distribution function $F_X(x)$. It is assumed that n observations have been recorded, and that one needs to find the probability density function of $X_{(i)}$, say $f_{X_{(i)}}(x)$.

Let E denote the event that the *i*th ordered observation $X_{(i)}$ lies between x and x+dx. This implies that i-1 observations occur before x, and n-i observations after x+dx. Then we can see this concept better in Fig. 5.1.

$$P\{E\} = P(x \le X_{(i)} \le x+dx) = f_{X_{(i)}}(x)dx$$

$$= \frac{n!}{(i-1)!1!(n-i)!}\left[F_X(x)\right]^{i-1} f_X(x)dx\left[1 - F_X(x)\right]^{n-i} \qquad (5.1)$$

The corresponding density function for $X_{(i)}$ is

$$f_{X_{(i)}}(x) = i\binom{n}{i}\left[F_X(x)\right]^{i-1}\left[1 - F_X(x)\right]^{n-i} f_X(x) \qquad (5.2)$$

Specifically, if i = 1, $f_{X_{(1)}}(x)$ is the probability density function of the first (smallest) order statistic

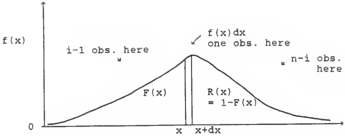


Figure 5.1 Basic reliability concept

$$f_{X_{(1)}}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x) \tag{5.3}$$

and, if $i = n$, $f_{X_{(n)}}(x)$ is the probability density function of the last (largest) order statistic

$$f_{X_{(n)}}(x) = n\left[F_X(x)\right]^{n-1} f_X(x) \tag{5.4}$$

The distribution function of $X_{(1)}$ and $X_{(n)}$ can be obtained as follows

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} \ge x)$$

and we know

$$\begin{aligned} P(X_{(1)} \ge x) &= P(X_1 \ge x,\ X_2 \ge x,\ \ldots\ldots,\ X_n \ge x) \\ &= P(X_1 \ge x)\ P(X_2 \ge x)\ \cdots P(X_n \ge x) \\ &= \left[1 - F_X(x)\right]^n \end{aligned} \tag{5.5}$$

therefore,

$$F_{X_{(1)}}(x) = 1 - \left[1 - F_X(x)\right]^n \tag{5.6}$$

Next consider $F_{X_{(n)}}(x)$,

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(X_1 \le x,\ X_2 \le x,\ \ldots,\ X_n \le x)$$

$$= P(X_1 \le x)\ P(X_2 \le x)\ \cdots P(X_n \le x)$$

$$F_{X_{(n)}}(x) = \left[F_X(x)\right]^n \tag{5.7}$$

5.2.1 Numerical Example - Smallest Value

We can refer to section 3.9, the chain model, as the standard type of smallest value problem. Suppose that the original p.d.f. is exponential, then

$$f_X(x) = e^{-\lambda x}$$

$$F_X(x) = 1 - e^{-\lambda x}$$

Then it follows that

$$F_{X_{(1)}}(x) = 1 - \left[1 - F_X(x)\right]^n$$

$$= 1 - e^{-\lambda n x}$$

and

$$f_{X_{(1)}}(x) = n\lambda \cdot e^{-n\lambda x}$$

5.2.2 Numerical Example - Largest Value

We still use the exponential distribution as the original p.d.f., then

$$F_{X_{(n)}}(x) = \left[F_X(x)\right]^n$$

$$= \left[1 - e^{-\lambda x}\right]^n$$

and

$$f_{X_{(n)}}(x) = n\lambda \cdot (1 - e^{-\lambda x})^{n-1} e^{-\lambda x}$$

5.3 Asymptotic Extreme-Value Distributions

The extreme-value distributions discussed in the preceeding section serve to illustrate, in a simple way, the effect of maximum extreme values and minimum extreme values. In practice, however, the use of Eqs. 5.6 and 5.7 for the CDFs may become cumbersome. Often n, the number of variables, is very large and the assumption that all the X_n are identically distributed may not be valid.

There are three classes of asymptotic extreme-value distributions [59], the CDFs for which are given in Table 5.1. They may be shown to arise when n, the number of variables over which the extreme is taken, becomes large, with only a few restrictions on the forms of the original distributions. The distributions differ both in the domain of the extreme-value variable and in the form of the upper or lower tail of the original distributions.

The application of the extreme-value distributions can be seen in [3,15,24,25,28,31,34,52].

Table 5.1 Extreme-Value Distributions

Distributions of the largest value x	
Type I	
$F_X(x) = \exp\left[- e^{-(x-u)/\theta}\right]$	$-\infty \le x \le \infty$ $\theta > 0$
Type II	
$F_X(x) = \exp\left[- \left(\frac{x}{\theta}\right)^{-m}\right]$	$x \ge 0$ $\theta > 0,\ m > 0$
Type III	
$F_X(x) = \exp\left[- \left(\frac{u-x}{\theta}\right)^{m}\right]$	$x \le u$ $\theta > 0,\ m > 0$

Distributions of the smallest value x	
Type I	
$F_X(x) = 1 - \exp\left[- e^{(x-u)/\theta}\right]$	$-\infty \le x \le \infty$
Type II	
$F_X(x) = 1 - \exp\left[- \left(\frac{-x}{\theta}\right)^{-m}\right]$	$-\infty \le x \le 0$ $\theta > 0,\ m > 0$
Type III	
$F_X(x) = 1 - \exp\left[- \left(\frac{x-u}{\theta}\right)^{m}\right]$	$u \le x < \infty$ $\theta > 0,\ m > 0$

Chapter 6

Conclusion

6.1 Introduction

The purpose of investigating mechanical interference reliability theory is to develop some basic concepts based on the stress-strength models. The cases discussed in this report all focused on the application of real working components. A component may be endowed with a certain strength and, at the same time, bear outside world stresses against the strength. By knowing how the stress-strength interference affects the reliability, we may be able to improve the component structure to get higher reliability.

A useful concept used in the interference theory is the hazard rate. When dealing with a real-life situation, the hazard rate concept can explain how mechanical reliability reflects the bathtub curve formation. Further, we can expand this idea to the extreme cases, which focus on the minimum-value strength and the maximum-value stress, to understand how the reliability will change, subject to some critical conditions.

In the feature of the complex design of modern systems, highly dependable performance is always the prerequisite for all system design. The stress-strength interference model can be applied to the accelerated testing which involves deliberate increase in stress in order to shorten test time or to detect the weak point of the whole system. Through fully understanding the stress-strength interaction, system failures can be decreased noticably.

6.2 Conclusion

From the previous study, the following conclusions can be drawn :

1. Interference is a useful tool for realizing the effects of stress and strength interaction.
2. Decreasing the original variability of the strength of the system will prolong the useful life of the system.
3. The time wearout effect on components is the necessary factor to understand the component useful life.
4. When the member of individual components in the system becomes very large, we can use the extreme-value distribution.
5. By applying the extreme-value distribution, we can detect the weakest point of the whole system strength and the strongest point of possible outside stress. Through understanding the relationship between stress and strength, the better design can be possibly made.
6. The optimal reliability depends on :

   i) deciding the stress probability density function

ii) deciding the strength probability density function

7. The way to improve the mechanical reliability

   i) decrease the maximum-value for stress

   ii) increase the minimum-value for strength

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APPENDIX A

IMSL program for computing incomplete beta function

```
C
C  This is the program to compute the incomplete
C  beta function by calling IMSL subroutine
C
      REAL A, B, P, X
      NUM=10
C
C  The iteration work
C
      DO 100  N=1, NUM
      A=N*0.5
      WRITE(6,90)A
90    FORMAT(3X,'A VALUE IS : ',F4.1)
      DO 200  J=1, NUM
      B=J*0.5
      P=0.5
C
C  Calling the IMSL subroutine
C
      CALL MDBETI(P,A,B,X,IER)
      WRITE(6,95)B,X
95    FORMAT(3X,'B VALUE IS : ',F4.1,10X,'X : ',2X,F5.3)
200   CONTINUE
100   CONTINUE
      STOP
      END
```

STRESS-STRENGTH INTERFERENCE MODELS

IN

RELIABILITY

by

AN-MIN LEE

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AN ABSTRACT OF A REPORT

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ABSTRACT

The technique of stress-strength interference model applied in mechanical reliability is often used in realizing how the system interacts with outside pressure. By building up the ideal model, the basic reliability concept can be established. The bathtub curve is not necessarily true in any real world situation; nevertheless, it can be proven to be accepted in the mechanical stress-strength interference model. The extreme case is suitably applicable when the sample size is very large or the sample occurrence is rare. The extreme-value distribution will enlarge the usage of the interference model to a practival method. Knowing how the stress is loading on the system, a effective design for the system to resist the most possible impulse can be possibly achieved.

This report is a review of the literature related to mechanical stress-strength interference theory. The literature is reviewed from early 1940's to 1987.